J. COMPUTATIONAL ANALYSIS AND APPLICATIONS, VOL. 15, NO.3, 481-497, 2013, COPYRIGHT 2013 EUDOXUS PRESS, LLC

Representations for Drazin inverse of block matrix*

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Abstract

In this paper we offer new representations for Drazin inverse of block matrix, which recover some representations from current literature on this subject.

2000 *Mathematics Subject Classification*: 15A09
Key words: Drazin inverse; block matrix; representation.

1 Introduction

Let A be a square complex matrix. By $\mathrm{rank}(A)$ we denote the rank of matrix A. The index of matrix A, denoted by $\mathrm{ind}(A)$, is the smallest nonnegative integer k such that $\mathrm{rank}(A^{k+1}) = \mathrm{rank}(A^k)$. For every matrix $A \in \mathbb{C}^{n\times n}$, such that $\mathrm{ind}(A) = k$, there exists the unique matrix $A^d \in \mathbb{C}^{n\times n}$, which satisfies following relations:

$$A^{k+1}A^d = A^k,\ A^dAA^d = A^d,\ AA^d = A^dA.$$

Matrix A^d is called the Drazin inverse of matrix A (see [1]). In the case $\mathrm{ind}(A) = 1$, the Drazin inverse of A is called the group inverse of A, denoted by $A^\#$ or A^g. The case $\mathrm{ind}(A) = 0$ is valid if and only if A is nonsingular, so in that case A^d reduces to A^{-1}. Throughout this paper we suppose that $A^0 = I$, where I is identity matrix, and $\sum_{i=1}^{k-j} * = 0$, for $k \le j$.

The theory of Drazin inverse of square matrix has numerous applications, such as in singular differential equations and singular difference equations,

*supported by Grant No. 174007 of the Ministry of Science and Technological Development, Republic of Serbia

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Markov chains and iterative methods (see [2, 4, 5, 6, 8, 9]). An application of the Drazin inverse of a 2×2 block matrix can be found in [2, 3, 7].
In 1979 Campbell and Meyer[4] posed the problem of finding an explicit representation for the Drazin inverse of 2×2 complex matrix

$$M = \begin{bmatrix} A & B \\ C & D \end{bmatrix}, \tag{1.1}$$

in terms of its blocks, where A and D are square matrices, not necessarily of the same size. Until now, there has been no formula for M^d without any side conditions for blocks of matrix M. However, many papers studied special cases of this open problem and offered a formula for M^d under some specific conditions for blocks of M. Here we list some of them:

(i) $B = 0$ (or $C = 0$) (see [10, 11]);

(ii) $BC = 0$, $BD = 0$ and $DC = 0$ (see [6]);

(iii) $BC = 0$, $DC = 0$ (or $BD = 0$) and D is nilpotent (see [7]);

(iv) $BC = 0$ and $DC = 0$ (see [12]);

(v) $CB = 0$ and $AB = 0$ (or $CA = 0$) (see [12, 13]);

(vi) $BCA = 0$, $BCB = 0$, $DCA = 0$ and $DCB = 0$ (see [14]);

(vii) $ABC = 0$, $CBC = 0$, $ABD = 0$ and $CBD = 0$ (see [14]);

(viii) $BCA = 0$, $BCB = 0$, $ABD = 0$ and $CBD = 0$ (see [15]);

(ix) $BCA = 0$, $DCA = 0$, $CBC = 0$, and $CBD = 0$ (see [15]);

(x) $BCA = 0$, $BD = 0$ and $DC = 0$ (or BC is nilpotent) (see [16]);

(xi) $BCA = 0$, $DC = 0$ and D is nilpotent (see [16]);

(xii) $ABC = 0$, $DC = 0$ and $BD = 0$ (or BC is nilpotent, or D is nilpotent) (see [17]);

(xiii) $BCA = 0$ and $BD = 0$ (see [18]);

(xiv) $ABC = 0$ and $DC = 0$ (or $BD = 0$) (see [18, 19]).

In this paper we derive representations for M^d which recover representations from previous list.

2 Key lemmas

In order to prove our main results, we first state some lemmas.

Lemma 2.1 [14] *Let $P, Q \in \mathbb{C}^{n\times n}$ be such that* $\operatorname{ind}(P) = r$ *and* $\operatorname{ind}(Q) = s$. *If $PQP = 0$ and $PQ^2 = 0$ then*

$$(P+Q)^d = Y_1 + Y_2 + \left(Y_1(P^d)^2 + (Q^d)^2 Y_2 - Q^d(P^d)^2 - (Q^d)^2 P^d\right) PQ,$$

where

$$Y_1 = \sum_{i=0}^{s-1} Q^\pi Q^i (P^d)^{i+1}, \; Y_2 = \sum_{i=0}^{r-1} (Q^d)^{i+1} P^i P^\pi. \tag{2.1}$$

Lemma 2.2 [14] *Let $P, Q \in \mathbb{C}^{n\times n}$ be such that* $\operatorname{ind}(P) = r$ *and* $\operatorname{ind}(Q) = s$. *If $QPQ = 0$ and $P^2Q = 0$ then*

$$(P+Q)^d = Y_1 + Y_2 + PQ\left(Y_1(P^d)^2 + (Q^d)^2 Y_2 - Q^d(P^d)^2 - (Q^d)^2 P^d\right),$$

where Y_1 and Y_2 are defined by (2.1).

Lemma 2.3 [20] *Let $M \in \mathbb{C}^{n\times n}$ be such that $M = \begin{bmatrix} 0 & B \\ C & 0 \end{bmatrix}$, $B \in \mathbb{C}^{p\times(n-p)}$, $C \in \mathbb{C}^{(n-p)\times p}$. Then*

$$M^d = \begin{bmatrix} 0 & B(CB)^d \\ (CB)^d C & 0 \end{bmatrix}.$$

Deng and Wei [21] gave representations for the Drazin inverse of upper anti-triangular block matrix under some specific conditions. Here we state these results and some additional facts, which we will be useful to prove our results. Consider the block matrix of a form (1.1), where $D = 0$:

$$M = \begin{bmatrix} A & B \\ C & 0 \end{bmatrix}. \tag{2.2}$$

Lemma 2.4 [21] *Let $M \in \mathbb{C}^{n\times n}$ be matrix of a form* (2.2). *If $ABC = 0$, then*

$$M^d = \begin{bmatrix} \Phi A & \Phi B \\ C\Phi & C\Phi^2 AB \end{bmatrix},$$

where

$$\Phi = (A^2 + BC)^d = \sum_{i=0}^{t_1-1} (BC)^\pi (BC)^i (A^d)^{2i+2} + \sum_{i=0}^{\nu_1-1} ((BC)^d)^{i+1} A^{2i} A^\pi \tag{2.3}$$

and $t_1 = \operatorname{ind}(BC)$, $\nu_1 = \operatorname{ind}(A^2)$.

Remark 1 *Let M be matrix of a form* (2.2). *If conditions of* Lemma 2.4 *are satisfied, we have that:*

$$M^{2k+1} = \begin{bmatrix} (A^2+BC)^k A & (A^2+BC)^k B \\ C(A^2+BC)^k & C(A^2+BC)^{k-1}AB \end{bmatrix}, \text{ for } k \geq 1$$

and

$$M^{2k} = \begin{bmatrix} (A^2+BC)^k & (A^2+BC)^{k-1}AB \\ C(A^2+BC)^{k-1}A & C(A^2+BC)^{k-1}B \end{bmatrix}, \text{ for } k \geq 1.$$

Notice that $(A^2+BC)^k = \sum_{j=0}^{k}(BC)^{k-j}A^{2j}$, *for* $k \geq 0$. *Also,* $(A^2+BC)^\pi = A^\pi - BC\Phi = (BC)^\pi - \Phi A^2$. *We can check that*

$$\Phi^k = \sum_{i=0}^{t_1-1}(BC)^\pi (BC)^i (A^d)^{2i+2k} + \sum_{i=0}^{\nu_1-1}((BC)^d)^{i+k}A^{2i}A^\pi - \sum_{i=1}^{k-1}((BC)^d)^{k-i}(A^d)^{2i},$$

for $k \geq 1$. *Therefore we have*

$$(M^d)^{2k+1} = \begin{bmatrix} \Phi^{k+1}A & \Phi^{k+1}B \\ C\Phi^{k+1} & C\Phi^{k+2}AB \end{bmatrix}, \text{ for } k \geq 0$$

and

$$(M^d)^{2k} = \begin{bmatrix} \Phi^k & \Phi^{k+1}AB \\ C\Phi^{k+1}A & C(\Phi^{k+1}B \end{bmatrix}, \text{ for } k \geq 1.$$

Lemma 2.5 [21] *Let $M \in \mathbb{C}^{n\times n}$ be as in* (2.2). *If $BCA = 0$, then*

$$M^d = \begin{bmatrix} A\Omega & \Omega B \\ C\Omega & CA\Omega^2 B \end{bmatrix},$$

where

$$\Omega = (A^2+BC)^d = \sum_{i=0}^{t_1-1}(A^d)^{2i+2}(BC)^i(BC)^\pi + \sum_{i=0}^{\nu_1-1} A^\pi A^{2i}((BC)^d)^{i+1} \quad (2.4)$$

and $t_1 = \mathrm{ind}(BC)$, $\nu_1 = \mathrm{ind}(A^2)$.

Remark 2 *Let M be matrix of a form* (2.2)*. If conditions of* Lemma 2.5 *hold, we have that:*

$$M^{2k+1} = \begin{bmatrix} A(A^2+BC)^k & (A^2+BC)^kB \\ C(A^2+BC)^k & CA(A^2+BC)^{k-1}B \end{bmatrix}, \text{ for } k \geq 1$$

and

$$M^{2k} = \begin{bmatrix} (A^2+BC)^k & A(A^2+BC)^{k-1}B \\ CA(A^2+BC)^{k-1} & C(A^2+BC)^{k-1}B \end{bmatrix}, \text{ for } k \geq 1.$$

Clearly, $(A^2+BC)^k = \sum_{j=0}^{k} A^{2j}(BC)^{k-j}$*, for* $k \geq 0$*. Also* $(A^2+BC)^\pi = A^\pi - \Omega BC = (BC)^\pi - A^2\Omega$*. Furthermore, we have that*

$$\Omega^k = \sum_{i=0}^{t_1-1}(A^d)^{2i+2k}(BC)^i(BC)^\pi + \sum_{i=0}^{\nu_1-1} A^\pi A^{2i}((BC)^d)^{i+k} - \sum_{i=1}^{k-1}(A^d)^{2i}((BC)^d)^{k-i},$$

for $k \geq 1$*. Hence we get that*

$$(M^d)^{2k+1} = \begin{bmatrix} A\Omega^{k+1} & \Omega^{k+1}B \\ C\Omega^{k+1} & CA\Omega^{k+2}B \end{bmatrix}, \text{ for } k \geq 0$$

and

$$(M^d)^{2k} = \begin{bmatrix} \Omega^k & A\Omega^{k+1}B \\ CA\Omega^{k+1} & C\Omega^{k+1}B \end{bmatrix}, \text{ for } k \geq 1.$$

In following two lemmas we present two new representations for Drazin inverse of lower anti-triangular block matrix. Consider the block matrix of a form (1.1) such that $A = 0$:

$$M = \begin{bmatrix} 0 & B \\ C & D \end{bmatrix}. \tag{2.5}$$

Lemma 2.6 *Let $M \in \mathbb{C}^{n\times n}$ be matrix of a form* (2.5)*. If $DCB = 0$, then*

$$M^d = \begin{bmatrix} B\Psi^2DC & B\Psi \\ \Psi C & \Psi D \end{bmatrix},$$

where

$$\Psi = (D^2+CB)^d = \sum_{i=0}^{t_2-1}(CB)^\pi(CB)^i(D^d)^{2i+2} + \sum_{i=0}^{\nu_2-1}((CB)^d)^{i+1}D^{2i}D^\pi \tag{2.6}$$

and $t_2 = \mathrm{ind}(CB)$, $\nu_2 = \mathrm{ind}(D^2)$.

Proof. First, notice that from $DCB = 0$ we have that matrices D^2 and CB satisfy the conditions of Lemma 2.1. Hence we get

$$(D^2 + CB)^d = \sum_{i=0}^{t_2-1}(CB)^\pi (CB)^i (D^d)^{2i+2} + \sum_{i=0}^{\nu_2-1}((CB)^d)^{i+1} D^{2i} D^\pi.$$

Consider the splitting of matrix M

$$M = \begin{bmatrix} 0 & B \\ C & D \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & D \end{bmatrix} + \begin{bmatrix} 0 & B \\ C & 0 \end{bmatrix} := P + Q.$$

Since $DCB = 0$ we have that $PQ^2 = 0$. Also, we have $PQP = 0$. Therefore matrices P and Q satisfy the conditions of Lemma 2.1 and

$$(P+Q)^d = Y_1 + Y_2 + \Big(Y_1(P^d)^2 + (Q^d)^2 Y_2 - Q^d (P^d)^2 - (Q^d)^2 P^d\Big) PQ, \quad (2.7)$$

where Y_1, Y_2 are as in (2.1). Clearly,

$$Q^{2k} = \begin{bmatrix} (BC)^k & 0 \\ 0 & (CB)^k \end{bmatrix}, \; Q^{2k+1} = \begin{bmatrix} 0 & B(CB)^k \\ (CB)^k C & 0 \end{bmatrix}, \text{for } k \geq 0.$$

Furthermore, by Lemma 2.3 we have

$$(Q^d)^{2k} = \begin{bmatrix} B((CB)^d)^{k+1} & 0 \\ 0 & ((CB)^d)^k \end{bmatrix}, \text{for } k \geq 1,$$

$$(Q^d)^{2k+1} = \begin{bmatrix} 0 & B((CB)^d)^{k+1} \\ ((CB)^d)^{k+1} C & 0 \end{bmatrix}, \text{for } k \geq 0.$$

After computing, we get

$$Y_1 = \begin{bmatrix} 0 & B\sum_{i=0}^{t_2-1}(CB)^\pi (CB)^i (D^d)^{2i+2} \\ 0 & \sum_{i=0}^{t_2-1}(CB)^\pi (CB)^i (D^d)^{2i+1} \end{bmatrix}, \quad (2.8)$$

$$Y_2 = \begin{bmatrix} 0 & B\sum_{i=0}^{\nu_2-1}((CB)^d)^{i+1} D^{2i} D^\pi \\ (CB)^d C & \sum_{i=0}^{\nu_2-1}((CB)^d)^{i+1} D^{2i+1} D^\pi \end{bmatrix}. \quad (2.9)$$

After substituting (2.8) and (2.9) into (2.7) we get that the statement of the lemma is valid. □

Remark 3 *Let M be matrix of a form* (2.5) *such that $DCB = 0$. Then*

$$M^{2k+1} = \begin{bmatrix} B(D^2+CB)^{k-1}DC & B(D^2+CB)^k \\ (D^2+CB)^k C & (D^2+CB)^k D \end{bmatrix}, \ \textit{for } k \geq 1$$

and

$$M^{2k} = \begin{bmatrix} B(D^2+CB)^{k-1}C & B(D^2+CB)^{k-1}D \\ (D^2+CB)^{k-1}DC & (D^2+CB)^k \end{bmatrix}, \ \textit{for } k \geq 1.$$

It can be checked easily that $(D^2+CB)^k = \sum_{j=0}^{k} (CB)^{k-j} D^{2j}$*, for $k \geq 0$, and* $(D^2+CB)^\pi = D^\pi - CB\Psi = (CB)^\pi - \Psi D^2$*. Also, we have that*

$$\Psi^k = \sum_{i=0}^{t_2-1} (CB)^\pi (CB)^i (D^d)^{2i+2k} + \sum_{i=0}^{\nu_2-1} ((CB)^d)^{i+k} D^{2i} D^\pi - \sum_{i=1}^{k-1} ((CB)^d)^{k-i} (D^d)^{2i},$$

for $k \geq 1$. Therefore we get

$$(M^d)^{2k+1} = \begin{bmatrix} B\Psi^{k+2}DC & B\Psi^{k+1} \\ \Psi^{k+1}C & \Psi^{k+1}D \end{bmatrix}, \ \textit{for } k \geq 0$$

and

$$(M^d)^{2k} = \begin{bmatrix} B\Psi^{k+1}C & B\Psi^{k+1}D \\ \Psi^{k+1}DC & \Psi^k \end{bmatrix}, \ \textit{for } k \geq 1.$$

Using the similar method as in the proof of Lemma 2.6 we can get the following result.

Lemma 2.7 *Let $M \in \mathbb{C}^{n\times n}$ be as in* (2.5)*. If $CBD = 0$, then*

$$M^d = \begin{bmatrix} BD\Gamma^2 C & B\Gamma \\ \Gamma C & D\Gamma \end{bmatrix},$$

where

$$\Gamma = \sum_{i=0}^{t_2-1} (D^d)^{2i+2} (CB)^i (CB)^\pi + \sum_{i=0}^{\nu_2-1} D^\pi D^{2i} ((CB)^d)^{i+1} \tag{2.10}$$

and $t_2 = \operatorname{ind}(CB)$*,* $\nu_2 = \operatorname{ind}(D^2)$*.*

Proof. Since $CBD = 0$, using Lemma 2.1 we get (2.10). Now, if we split matrix M as

$$M = \begin{bmatrix} 0 & B \\ C & D \end{bmatrix} = \begin{bmatrix} 0 & B \\ C & 0 \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 0 & D \end{bmatrix} := P + Q,$$

we have that $QPQ = 0$ and $P^2Q = 0$. Hence, the conditions of Lemma 2.2 are satisfied. After applying Lemma 2.2 and Lemma 2.3 we complete the proof.□

Remark 4 *Let M be as in* (2.5) *and let $CBD = 0$. Then*

$$M^{2k+1} = \begin{bmatrix} BD(D^2+CB)^{k-1}C & B(D^2+CB)^k \\ (D^2+CB)^kC & D(D^2+CB)^k \end{bmatrix}, \text{ for } k \geq 1$$

and

$$M^{2k} = \begin{bmatrix} B(D^2+CB)^{k-1}C & BD(D^2+CB)^{k-1} \\ (D^2+CB)^{k-1}C & (D^2+CB)^k \end{bmatrix}, \text{ for } k \geq 1.$$

Clearly $(D^2 + CB)^k = \sum_{j=0}^{k} D^{2j}(CB)^{k-j}$, *for* $k \geq 0$, *and* $(D^2 + CB)^\pi = D^\pi - \Gamma CB = (CB)^\pi - D^2\Gamma$. *In addition, we can get that*

$$\Gamma^k = \sum_{i=0}^{t_2-1}(D^d)^{2i+2k}(CB)^i(CB)^\pi + \sum_{i=0}^{\nu_2-1} D^\pi D^{2i}((CB)^d)^{i+k} - \sum_{i=1}^{k-1}(D^d)^{2i}((CB)^d)^{k-i},$$

for $k \geq 1$. *Also, we can get that*

$$(M^d)^{2k+1} = \begin{bmatrix} BD\Gamma^{k+2}C & B\Gamma^{k+1} \\ \Gamma^{k+1}C & D\Gamma^{k+1} \end{bmatrix}, \text{ for } k \geq 0$$

and

$$(M^d)^{2k} = \begin{bmatrix} B\Gamma^{k+1}C & BD\Gamma^{k+1} \\ D\Gamma^{k+1}C & \Gamma^k \end{bmatrix}, \text{ for } k \geq 1.$$

3 Representations

Consider the block matrix M of a form (1.1). Djordjević and Stanimirović [6] gave explicit representation for M^d under conditions $BC = 0$, $BD = 0$ and $DC = 0$. This result was extended to a case $BC = 0$, $DC = 0$ (see [12]). As another generalization of these results, Yang and Liu [14] gave the representation for M^d under conditions $BCA = 0$, $BCB = 0$, $DCA = 0$ and $DCB = 0$. In the next theorem we derive an explicit representation for M^d under conditions $BCA = 0$, $DCA = 0$ and $DCB = 0$. Therefore we can see that the condition $BCB = 0$ from [14] is superfluous.

Theorem 3.1 *Let M be matrix of a form* (1.1) *such that $BCA = 0$, $DCA = 0$ and $DCB = 0$. Then*

$$M^d = \begin{bmatrix} A^d + \Sigma_0 C & B\Psi + A\Sigma_0 \\ \begin{array}{c}\Psi C + CA\Sigma_1 C + C(A^d)^2 \\ -CA^d(B\Psi^2 D + AB\Psi^2)C\end{array} & D^d + C\Sigma_0 \end{bmatrix},$$

where

$$\Sigma_k = \left(V_1\Psi^k + (A^d)^{2k}V_2\right) D + A\left(V_1\Psi^k + (A^d)^{2k}V_2\right), \textit{for } k = 0, 1, \tag{3.1}$$

$$V_1 = \sum_{i=0}^{\nu_1 - 1} A^\pi A^{2i} B\Psi^{i+2}, \tag{3.2}$$

$$V_2 = \sum_{i=0}^{\mu_1 - 1} (A^d)^{2i+4} B(D^2 + CB)^i D^\pi - \sum_{i=0}^{\mu_1} (A^d)^{2i+2} B(CB)^i \Psi, \tag{3.3}$$

$\nu_1 = \mathrm{ind}(A^2)$, $\mu_1 = \mathrm{ind}(D^2 + CB)$ *and Ψ is defined by* (2.6).

Proof. Consider the splitting of matrix M

$$M = \begin{bmatrix} A & B \\ C & D \end{bmatrix} = \begin{bmatrix} 0 & B \\ C & D \end{bmatrix} + \begin{bmatrix} A & 0 \\ 0 & 0 \end{bmatrix} := P + Q.$$

Since $BCA = 0$ and $DCA = 0$ we get $P^2Q = 0$ and $QPQ = 0$. Hence matrices P and Q satisfy the conditions of Lemma 2.2 and

$$(P+Q)^d = Y_1 + Y_2 + PQY_1(P^d)^2 + PQ^dY_2 - PQQ^d(P^d)^2 - PQ^dP^d, \tag{3.4}$$

where Y_1 and Y_2 are as in (2.1). By the assumption of the theorem $DCB = 0$ we have that matrix P satisfy the conditions of Lemma 2.6. After applying Lemma 2.6 and using Remark 3, we get

$$Y_1 = \begin{bmatrix} (V_1D + AV_1)C & A^\pi B\Psi + A(V_1D + AV_1) \\ \Psi C & \Psi D \end{bmatrix}, \tag{3.5}$$

$$Y_2 = \begin{bmatrix} A^d + (V_2D + AV_2)C & B\Psi - A^\pi B\Psi + A(V_2D + AV_2) \\ 0 & 0 \end{bmatrix}, \tag{3.6}$$

where V_1 and V_2 are defined by (3.2) and (3.3), respectively. After substituting (3.5) and (3.6) into (3.4) and computing all elements of (3.4) we obtain the result. □
As a direct corollary of the previous theorem we get the following result.

Corollary 3.1 *Let M be as in* (1.1). *If $DCB = 0$ and $CA = 0$ then*

$$M^d = \begin{bmatrix} A^d + \Sigma_0 C & B\Psi + A\Sigma_0 \\ \Psi C & \Psi D \end{bmatrix},$$

where Σ_0 is defined by (3.1) *and Ψ is given in* (2.6).

Notice that Corollary 3.1, therefore and Theorem 3.1 is also a generalization of representation for M^d under conditions $CB = 0$ and $CA = 0$ which is given in [13].

The next result is a corollary of Theorem 3.1. Also, we can get the following result using the splitting $M = \begin{bmatrix} 0 & 0 \\ 0 & D \end{bmatrix} + \begin{bmatrix} A & B \\ C & 0 \end{bmatrix} := P + Q$ and applying Lemma 2.1 and Lemma 2.5.

Corollary 3.2 *Let M be matrix of a form* (1.1). *If $BCA = 0$ and $DC = 0$ then*

$$M^d = \begin{bmatrix} A\Omega & \Omega B + RD \\ C\Omega & D^d + CR \end{bmatrix},$$

where

$$\begin{aligned}
R &= (R_1 + R_2)D + A(R_1 + R_2), \\
R_1 &= \sum_{i=0}^{\mu_2 - 1} A^\pi (A^2 + BC)^i B (D^d)^{2i+4} - \sum_{i=0}^{\mu_2} \Omega (BC)^i B (D^d)^{2i+2}, \\
R_2 &= \sum_{i=0}^{\nu_2 - 1} \Omega^{i+2} B D^{2i} D^\pi,
\end{aligned}$$

$\nu_2 = \mathrm{ind}(D^2)$, $\mu_2 = \mathrm{ind}(A^2 + BC)$ and Ω is defined by (2.4).

We remark that Corollary 3.2, hence and Theorem 3.1 is also extension of results from [16], where beside conditions $BCA = 0$ and $DC = 0$ additional condition $BD = 0$ (or D is nilpotent) is required.

Castro–González et al. (see [16]) gave explicit representation for M^d under conditions $BCA = 0$, $BD = 0$ and BC is nilpotent (or $DC = 0$).

This result was extended to a case when $BCA = 0$ and $BD = 0$ (see [18]). The following theorem is extension of these results.

Theorem 3.2 *Let M be matrix of a form* (1.1) *such that $BCA = 0$, $ABD = 0$ and $CBD = 0$. Then*

$$M^d = \begin{bmatrix} A\Omega + B(F_1 + F_2) & \begin{array}{c}\Omega B + BD(F_1\Omega + (D^d)^2 F_2)B \\ +B(D^d)^2 - BD^d(CA + DC)\Omega^2 B\end{array} \\ C\Omega + D(F_1 + F_2) & D^d + (F_1 + F_2)B \end{bmatrix}, \quad (3.7)$$

where

$$F_1 = \sum_{i=0}^{\nu_2-1} D^\pi D^{2i}(CA + DC)\Omega^{i+2},$$

$$F_2 = \sum_{i=0}^{\mu_2-1} (D^d)^{2i+4}(CA + DC)(A^2 + BC)^i (BC)^\pi - \sum_{i=0}^{\mu_2} (D^d)^{2i+2}(CA + DC)A^{2i}\Omega,$$

$\nu_2 = \mathrm{ind}(D^2)$, $\mu_2 = \mathrm{ind}(A^2 + BC)$ and Ω is defined by (2.4).

Proof. If we split matrix M as

$$M = \begin{bmatrix} A & B \\ C & 0 \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 0 & D \end{bmatrix} := P + Q.$$

we have that $QPQ = 0$ and $P^2Q = 0$. Hence, matrices P and Q satisfy the conditions of Lemma 2.2. Since $BCA = 0$, matrix P satisfies conditions of Lemma 2.5. Using the similar method as in the proof of Theorem 3.1, after applying Lemma 2.2, Lemma 2.5 and using Remark 2, we get that (3.7) holds. □

Notice that Theorem 3.2 is also generalization of representation from [15] where additional condition $BCB = 0$ is required.

In [15] a formula for M^d is given under conditions $BCA = 0$, $DCA = 0$, $CBD = 0$ and $CBC = 0$. In the next theorem we offer a representation for M^d under conditions $BCA = 0$, $DCA = 0$ and $CBD = 0$, without additional condition $CBC = 0$.

Theorem 3.3 *Let M be as in* (1.1)*. If $BCA = 0$, $DCA = 0$ and $CBD = 0$ then*

$$M^d = \begin{bmatrix} A^d + (G_1 + G_2)C & B\Gamma + A(G_1 + G_2) \\ \begin{array}{c}\Gamma C + CA(G_1\Gamma + (A^d)^2 G_2)C \\ +C(A^d)^2 - CA^d(AB + BD)\Gamma^2 C\end{array} & D\Gamma + C(G_1 + G_2) \end{bmatrix},$$

where

$$G_1=\sum_{i=0}^{\nu_1-1}A^{\pi}A^{2i}(AB+BD)\Gamma^{i+2}, \tag{3.8}$$

$$G_2=\sum_{i=0}^{\mu_1-1}(A^d)^{2i+4}(AB+BD)(D^2+CB)^i(CB)^{\pi}-\sum_{i=0}^{\mu_1}(A^d)^{2i+2}(AB+BD)D^{2i}\Gamma, \tag{3.9}$$

$\nu_1=\mathrm{ind}(A^2)$, $\mu_1=\mathrm{ind}(D^2+CB)$ *and* Γ *is given in* (2.10).

Proof. Using the splitting of matrix M

$$M=\begin{bmatrix}0 & B\\ C & D\end{bmatrix}+\begin{bmatrix}A & 0\\ 0 & 0\end{bmatrix}:=P+Q,$$

we get that conditions of Lemma 2.2 are satisfied. Also, we have that matrix P satisfies the conditions of Lemma 2.7. Using these lemmas and Remark 4, similarly as in the proof of Theorem 3.1, we get that the statement of the theorem is valid. □

Corollary 3.3 *Let* M *be matrix of a form* (1.1). *If* $CBD=0$ *and* $CA=0$, *then*

$$M^d=\begin{bmatrix}A^d+(G_1+G_2)C & B\Gamma+A(G_1+G_2)\\ \Gamma C & D\Gamma\end{bmatrix},$$

where Γ, G_1 *and* G_2 *are defined by* (2.10), (3.8) *and* (3.9) *respectively.*

We can see that Theorem 3.3 and Corollary 3.3 are also extensions of representation for M^d under conditions $CB=0$ and $CA=0$ (see [13]).

In [12] a representation for M^d is offered under conditions $AB=0$ and $CB=0$. This result was extended in [14], where a formula for M^d is given under conditions $ABC=0$, $ABD=0$, $CBD=0$ and $CBC=0$. In our following result we derive the representation for M^d under conditions $ABC=0$, $ABD=0$ and $CBD=0$, without additional condition $CBC=0$.

Theorem 3.4 *Let* M *be matrix of a form* (1.1). *If* $ABC=0$, $ABD=0$ *and* $CBD=0$. *Then*

$$M^d=\begin{bmatrix}A^d+B\Theta_0 & \begin{matrix}B\Gamma+B\Theta_1AB+(A^d)^2B\\ -B(\Gamma^2CA+D\Gamma^2C)A^dB\end{matrix}\\ \Gamma C+\Theta_0A & D^d+\Theta_0B\end{bmatrix}, \tag{3.10}$$

where

$$\Theta_k = \left(K_1(A^d)^{2k} + \Gamma^k K_2\right) A + D\left(K_1(A^d)^{2k} + \Gamma^k K_2\right), \ \textit{for } k = 0, 1, \quad (3.11)$$

$$K_1 = \sum_{i=0}^{\mu_1 - 1} D^\pi (D^2 + CB)^i C (A^d)^{2i+4} - \sum_{i=0}^{\mu_1} \Gamma (CB)^i C (A^d)^{2i+2}, \quad (3.12)$$

$$K_2 = \sum_{i=0}^{\nu_1 - 1} \Gamma^{i+2} C A^{2i} A^\pi, \quad (3.13)$$

$\nu_1 = \operatorname{ind}(A^2)$, $\mu_1 = \operatorname{ind}(D^2 + CB)$ *and* Γ *is defined by* (2.10).

Proof. We can split matrix M as $M = P + Q$, where

$$P = \begin{bmatrix} A & 0 \\ 0 & 0 \end{bmatrix}, \ Q = \begin{bmatrix} 0 & B \\ C & D \end{bmatrix}.$$

According to assumptions of the theorem, we have that $PQP = 0$ and $PQ^2 = 0$. Hence we can apply Lemma 2.1 and we have

$$(P + Q)^d = Y_1 + Y_2 + \left(Y_1 (P^d)^2 + (Q^d)^2 Y_2 - Q^d (P^d)^2 - (Q^d)^2 P^d\right) PQ, \quad (3.14)$$

where Y_1 and Y_2 are defined by (2.1). Since $CBD = 0$, matrix Q satisfies condition of Lemma 2.7. After applying Lemma 2.7 and facts from Remark 4 we get

$$Y_1 = \begin{bmatrix} A^d + B(K_1 A + D K_1) & 0 \\ \Gamma C - \Gamma C A^\pi + (K_1 A + D K_1) A & 0 \end{bmatrix}, \quad (3.15)$$

$$Y_2 = \begin{bmatrix} B(K_2 A + D K_2) & B\Gamma \\ \Gamma C A^\pi + (K_2 A + D K_2) A & D\Gamma \end{bmatrix}, \quad (3.16)$$

where K_1 and K_2 are given in (3.12) and (3.13), respectively. Now, by substituting (3.16) and (3.15) into (3.14) we get that (3.10) holds. $\square$

Notice that Theorem 3.4 is also an extension of a case when $ABC = 0$ and $BD = 0$ (see [19]).

The following result is direct corollary of Theorem 3.4.

Corollary 3.4 *Let* M *be given by* (1.1). *If* $CBD = 0$ *and* $AB = 0$ *then*

$$M^d = \begin{bmatrix} A^d + B\Theta_0 & B\Gamma \\ \Gamma C + \Theta_0 A & D\Gamma \end{bmatrix},$$

where Γ *and* Θ_0 *are defined by* (2.10) *and* (3.11) *respectively.*

As another extension of a result from [12], where formula for M^d is given under conditions $AB = 0$ and $CB = 0$, we offer the following theorem and its corollary.

Theorem 3.5 *Let M be matrix of a form* (1.1). *If $ABC = 0$, $ABD = 0$ and $DCB = 0$ then*

$$M^d = \begin{bmatrix} A^d + B(N_1+N_2) & \begin{matrix} B\Psi + B(N_1(A^d)^2 + \Psi N_2)AB \\ +(A^d)^2B - B\Psi^2(CA+DC)A^dB \end{matrix} \\ \Psi C + (N_1+N_2)A & \Psi D + (N_1+N_2)B \end{bmatrix}, \tag{3.17}$$

where

$$N_1 = \sum_{i=0}^{\mu_1-1}(CB)^\pi(D^2+CB)^i(CA+DC)(A^d)^{2i+4} - \sum_{i=0}^{\mu_1}\Psi D^{2i}(CA+DC)(A^d)^{2i+2}, \tag{3.18}$$

$$N_2 = \sum_{i=0}^{\nu_1-1}\Psi^{i+2}(CA+DC)A^{2i}A^\pi, \tag{3.19}$$

$\nu_1 = \mathrm{ind}(A^2)$, $\mu_1 = \mathrm{ind}(D^2+CB)$ *and Ψ is defined by* (2.6).

Proof. Using the splitting

$$M = \begin{bmatrix} A & 0 \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} 0 & B \\ C & D \end{bmatrix} := P + Q,$$

we get that matrices P and Q satisfy the conditions of Lemma 2.1. Furthermore, matrix Q satisfies the conditions of Lemma 2.6. After applying these lemmas, using Remark 3 and computing, we get that (3.17) holds. □

Next corollary follows immediately from Theorem 3.5.

Corollary 3.5 *Let M be given by* (1.1). *If $DCB = 0$ and $AB = 0$ then*

$$M^d = \begin{bmatrix} A^d + B(N_1+N_2) & B\Psi \\ \Psi C + (N_1+N_2)A & \Psi D \end{bmatrix},$$

where Ψ, N_1 and N_2 are defined by (2.6), (3.18) *and* (3.19), *respectively.*

Cvetković and Milovanović (see [17]) offered a representation for M^d under conditions $ABC = 0$, $DC = 0$, with third condition $BD = 0$ (or BC is nilpotent, or D is nilpotent). Cvetković - Ilić (see [18]) extended this result and gave a formula for M^d under conditions $ABC = 0$ and $DC = 0$, without any additional condition. In our next result we replace second condition $DC = 0$ from [18] with two weaker conditions. Therefore, we can get results from [17, 18] as direct corollaries.

Theorem 3.6 *Let M be matrix of a form* (1.1)*, such that $ABC = 0$, $DCA = 0$ and $DCB = 0$. Then*

$$M^d = \begin{bmatrix} \Phi A + (U_1 + U_2)C & \Phi B + (U_1 + U_2)D \\ \begin{array}{c} C\Phi + C(U_1(D^d)^2 + \Phi U_2)DC \\ +(D^d)^2C - C\Phi^2(AB + BD)D^dC \end{array} & D^d + C(U_1 + U_2) \end{bmatrix},$$

where

$$U_1 = \sum_{i=0}^{\mu_2 - 1} (BC)^\pi (A^2 + BC)^i (AB + BD)(D^d)^{2i+4} - \sum_{i=0}^{\mu_2} \Phi A^{2i}(AB + BD)(D^d)^{2i+2}$$

$$U_2 = \sum_{i=0}^{\nu_2 - 1} \Phi^{i+2}(AB + BD)D^{2i}D^\pi,$$

$\nu_2 = \mathrm{ind}(D^2)$*,* $\mu_2 = \mathrm{ind}(A^2 + BC)$ *and* Φ *is defined by* (2.3)*.*

Proof. If we split matrix M as

$$M = \begin{bmatrix} 0 & 0 \\ 0 & D \end{bmatrix} + \begin{bmatrix} A & B \\ C & 0 \end{bmatrix} := P + Q,$$

we have $PQP = 0$ and $PQ^2 = 0$. Also, matrix P satisfies conditions of Lemma 2.4. After applying Lemma 2.1, Lemma 2.4, Remark 1 and computing we get that the statement of the theorem is valid. $\square$

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and the inequality (see Section 3.6.6 of [22]):

$$e^x \leq 1 + x + \frac{x^2}{2} + \frac{x^3}{2(3-x)}, \quad (0 \leq x < 3),$$

it follows that

$$\sum_{j=1}^{N} \frac{1}{j! n^{\alpha j}} < \frac{3}{n^\alpha}, \quad \sum_{j=1}^{N} \frac{2^{(j+1)}}{j!} \leq 16.$$

Therefore,

$$|\Xi_2| \leq \left(\frac{3}{n^\alpha} + \frac{16}{n} e^{-n(n^{1-\alpha} - \frac{3}{2})} \right) \|f\|_N.$$

To estimate Ξ_3, we use the result (see P. 72-73 of [23]):

$$\left| \int_x^{\frac{k}{n}} \left(f^{(N)}(t) - f^{(N)}(x) \right) \frac{(\frac{k}{n} - t)^{N-1}}{(N-1)!} \mathrm{d}t \right| \leq \begin{cases} \omega\left(f^{(N)}, \frac{1}{n^\alpha}\right) \frac{1}{n^{\alpha N} N!}, & |\frac{k}{n} - x| \leq \frac{1}{n^\alpha}, \\ \|f^{(N)}\| \frac{2^{(N+1)}}{N!}, & |\frac{k}{n} - x| > \frac{1}{n^\alpha} \end{cases}$$

and deduce that

$$\begin{aligned} |\Xi_3| &\leq \omega\left(f^{(N)}, \frac{1}{n^\alpha}\right) \frac{1}{n^{\alpha N} N!} \sum_{k:|\frac{k}{n} - x| \leq \frac{1}{n^\alpha}} \Phi(nx - k) + \|f^{(N)}\| \frac{2^{(N+1)}}{N!} \sum_{k:|\frac{k}{n} - x| > \frac{1}{n^\alpha}} \Phi(nx - k) \\ &\leq \omega\left(f^{(N)}, \frac{1}{n^\alpha}\right) \frac{1}{n^{\alpha N} N!} + \frac{2^{(N+2)} \|f^{(N)}\|}{nN!} e^{-n(n^{1-\alpha} - \frac{3}{2})}. \end{aligned}$$

Combining the estimates of Ξ_1, Ξ_2 and Ξ_3 leads to

$$\begin{aligned} |F_n(f,x) - f(x)| &\leq 4e^{-\frac{n}{2}} \|f\| + \left(\frac{3}{n^\alpha} + \frac{16}{n} e^{-n(n^{1-\alpha} - \frac{3}{2})} \right) \|f\|_N \\ &+ \omega\left(f^{(N)}, \frac{1}{n^\alpha}\right) \frac{1}{n^{\alpha N} N!} + \frac{2^{(N+2)} \|f^{(N)}\|}{nN!} e^{-n(n^{1-\alpha} - \frac{3}{2})}. \end{aligned}$$

This finishes the proof of Theorem 8. □

Remark 2. For $f \in C([-1,1]^2)$, we can establish the same result as Theorem 6.

Remark 3. For $f \in C^N([-1,1]^2)$, a similar result to Theorem 8 can be established.

Remark 4. In fact, we can establish corresponding results in $C([-1,1]^d)$ and $C^N([-1,1]^d)(d > 2, d \in \mathbb{N})$.

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J. COMPUTATIONAL ANALYSIS AND APPLICATIONS, VOL. 15, NO.6, 1057-1068, 2013, COPYRIGHT 2013 EUDOXUS PRESS, LLC

ORTHOGONAL STABILITY OF AN ADDITIVE FUNCTIONAL EQUATION IN BANACH MODULES OVER A C^*-ALGEBRA

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Abstract. Using fixed point method, we prove the Hyers-Ulam stability of the following additive functional equation

$$\sum_{i=1}^{m} f\left(ma_i + \sum_{j=1, j\neq i}^{m} a_j\right) + f\left(\sum_{i=1}^{m} a_i\right) = 2f\left(\sum_{i=1}^{m} a_i\right)$$

in Banach modules over a unital C^*-algebra and in non-Archimedean Banach modules over a unital C^*-algebra.

1. Introduction and preliminaries

Assume that X is a real inner product space and $f : X \to \mathbb{R}$ is a solution of the orthogonal Cauchy functional equation $f(x + y) = f(x) + f(y), \langle x, y\rangle = 0$. By the Pythagorean theorem $f(x) = \|x\|^2$ is a solution of the conditional equation. Of course, this function does not satisfy the additivity equation everywhere. Thus orthogonal Cauchy equation is not equivalent to the classic Cauchy equation on the whole inner product space.

G. Pinsker [53] characterized orthogonally additive functionals on an inner product space when the orthogonality is the ordinary one in such spaces. K. Sundaresan [65] generalized this result to arbitrary Banach spaces equipped with the Birkhoff-James orthogonality. The orthogonal Cauchy functional equation $f(x+y) = f(x)+f(y)$, $x \perp y$, in which $\perp$ is an abstract orthogonality relation, was first investigated by S. Gudder and D. Strawther [30]. They defined $\perp$ by a system consisting of five axioms and described the general semi-continuous real-valued solution of conditional Cauchy functional equation. In 1985, J. Rätz [60] introduced a new definition of orthogonality by using more restrictive axioms than of S. Gudder and D. Strawther. Moreover, he investigated the structure of orthogonally additive mappings. J. Rätz and Gy. Szabó [61] investigated the problem in a rather more general framework.

Let us recall the orthogonality in the sense of J. Rätz; cf. [60].

Suppose X is a real vector space (algebraic module) with $\dim X \geq 2$ and $\perp$ is a binary relation on X with the following properties:

(O_1) totality of $\perp$ for zero: $x \perp 0, 0 \perp x$ for all $x \in X$;

(O_2) independence: if $x, y \in X - \{0\}, x \perp y$, then x, y are linearly independent;

(O_3) homogeneity: if $x, y \in X, x \perp y$, then $\alpha x \perp \beta y$ for all $\alpha, \beta \in \mathbb{R}$;

(O_4) the Thalesian property: if P is a 2-dimensional subspace of $X, x \in P$ and $\lambda \in \mathbb{R}_+$, which is the set of nonnegative real numbers, then there exists $y_0 \in P$ such that $x \perp y_0$ and $x + y_0 \perp \lambda x - y_0$.

The pair $(X, \perp)$ is called an orthogonality space (module). By an orthogonality normed space (normed module) we mean an orthogonality space (module) having a normed (normed

2010 *Mathematics Subject Classification.* Primary 39B55, 46S10, 47H10, 39B52, 47S10, 30G06, 46H25, 46L05, 12J25.

Key words and phrases. Hyers-Ulam stability, orthogonally Cauchy-Jensen additive functional equation, fixed point, non-Archimedean Banach module over C^*-algebra, orthogonality space.

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module) structure. Assume that if A is a C^*-algebra and X is a module over A and if $x, y \in X, x \perp y$, then $ax \perp by$ for all $a, b \in A$.

Some interesting examples are

(i) The trivial orthogonality on a vector space X defined by (O_1), and for non-zero elements $x, y \in X$, $x \perp y$ if and only if x, y are linearly independent.

(ii) The ordinary orthogonality on an inner product space $(X, \langle ., . \rangle)$ given by $x \perp y$ if and only if $\langle x, y \rangle = 0$.

(iii) The Birkhoff-James orthogonality on a normed space $(X, \|.\|)$ defined by $x \perp y$ if and only if $\|x + \lambda y\| \geq \|x\|$ for all $\lambda \in \mathbb{R}$.

The relation $\perp$ is called symmetric if $x \perp y$ implies that $y \perp x$ for all $x, y \in X$. Clearly examples (i) and (ii) are symmetric but example (iii) is not. It is remarkable to note, however, that a real normed space of dimension greater than 2 is an inner product space if and only if the Birkhoff-James orthogonality is symmetric. There are several orthogonality notions on a real normed space such as Birkhoff-James, Boussouis, Singer, Carlsson, unitary-Boussouis, Roberts, Phythagorean, isosceles and Diminnie (see [1]–[3], [5, 14, 35, 36, 44]).

The stability problem of functional equations originated from the following question of Ulam [67]: *Under what condition does there is an additive mapping near an approximately additive mapping?* In 1941, Hyers [32] gave a partial affirmative answer to the question of Ulam in the context of Banach spaces. In 1978, Th.M. Rassias [55] extended the theorem of Hyers by considering the unbounded Cauchy difference $\|f(x+y) - f(x) - f(y)\| \leq \varepsilon(\|x\|^p + \|y\|^p)$, $(\varepsilon > 0, p \in [0, 1))$. During the last decades several stability problems of functional equations have been investigated in the spirit of Hyers-Ulam-Rassias. The reader is referred to [11, 33, 37, 59] and references therein for detailed information on stability of functional equations.

R. Ger and J. Sikorska [29] investigated the orthogonal stability of the Cauchy functional equation $f(x + y) = f(x) + f(y)$, namely, they showed that if f is a mapping from an orthogonality space X into a real Banach space Y and $\|f(x + y) - f(x) - f(y)\| \leq \varepsilon$ for all $x, y \in X$ with $x \perp y$ and some $\varepsilon > 0$, then there exists exactly one orthogonally additive mapping $g : X \to Y$ such that $\|f(x) - g(x)\| \leq \frac{16}{3}\varepsilon$ for all $x \in X$.

The first author treating the stability of the quadratic equation was F. Skof [64] by proving that if f is a mapping from a normed space X into a Banach space Y satisfying $\|f(x + y) + f(x - y) - 2f(x) - 2f(y)\| \leq \varepsilon$ for some $\varepsilon > 0$, then there is a unique quadratic mapping $g : X \to Y$ such that $\|f(x) - g(x)\| \leq \frac{\varepsilon}{2}$. P.W. Cholewa [8] extended the Skof's theorem by replacing X by an abelian group G. The Skof's result was later generalized by S. Czerwik [9] in the spirit of Hyers-Ulam-Rassias. The stability problem of functional equations has been extensively investigated by some mathematicians (see [6, 7, 10, 51], [16]–[18], [40], [56]–[58], [63]).

The orthogonally quadratic equation

$$f(x + y) + f(x - y) = 2f(x) + 2f(y), \ x \perp y$$

was first investigated by F. Vajzović [68] when X is a Hilbert space, Y is the scalar field, f is continuous and $\perp$ means the Hilbert space orthogonality. Later, H. Drljević [15], M. Fochi [28], and Gy. Szabó [66] generalized this result.

In 1897, Hensel [31] introduced a normed space which does not have the Archimedean property. It turned out that non-Archimedean spaces have many nice applications (see [12, 39, 41, 43]).

Definition 1.1. By a *non-Archimedean field* we mean a field $\mathbb{K}$ equipped with a function (valuation) $|\cdot| : \mathbb{K} \to [0, \infty)$ such that for all $r, s \in \mathbb{K}$, the following conditions hold:

(1) $|r| = 0$ if and only if $r = 0$; (2) $|rs| = |r||s|$; (3) $|r + s| \leq max\{|r|, |s|\}$.

Definition 1.2. Let X be a vector space over a scalar field $\mathbb{K}$ with a non-Archimedean non-trivial valuation $|\cdot|$. A function $||\cdot|| : X \to R$ is a *non-Archimedean norm* (valuation) if it satisfies the following conditions:

(1) $||x|| = 0$ if and only if $x = 0$; (2) $||rx|| = |r|||x||$ ($r \in \mathbb{K}$, $x \in X$); (3) The strong triangle inequality (ultrametric); namely, $||x + y|| \leq max\{||x||, ||y||\}, \quad x, y \in X$. Then $(X, ||.||)$ is called a non-Archimedean space.

Assume that if A is a C^*-algebra and X is a module over A, which is a non-Archimedean space, and if $x, y \in X, x \perp y$, then $ax \perp by$ for all $a, b \in A$. Then $(X, ||.||)$ is called an *orthogonality non-Archimedean module.*

Due to the fact that

$$||x_n - x_m|| \leq max\{||x_{j+1} - x_j|| : m \leq j \leq n-1\} \quad (n > m).$$

Definition 1.3. A sequence $\{x_n\}$ is *Cauchy* if and only if $\{x_{n+1} - x_n\}$ converges to zero in a non-Archimedean space. By a complete non-Archimedean space we mean one in which every Cauchy sequence is convergent.

Let X be a set. A function $d : X \times X \to [0, \infty]$ is called a *generalized metric* on X if d satisfies

(1) $d(x, y) = 0$ if and only if $x = y$; (2) $d(x, y) = d(y, x)$ for all $x, y \in X$; (3) $d(x, z) \leq d(x, y) + d(y, z)$ for all $x, y, z \in X$.

We recall a fundamental result in fixed point theory.

Theorem 1.4. [13] *Let (X, d) be a complete generalized metric space and let $J : X \to X$ be a strictly contractive mapping with Lipschitz constant $\alpha < 1$. Then for each given element $x \in X$, either $d(J^n x, J^{n+1}x) = \infty$ for all nonnegative integers n or there exists a positive integer n_0 such that*

(1) $d(J^n x, J^{n+1}x) < \infty, \quad \forall n \geq n_0$; (2) the sequence $\{J^n x\}$ converges to a fixed point y^* of J; (3) y^* is the unique fixed point of J in the set $Y = \{y \in X \mid d(J^{n_0}x, y) < \infty\}$; (4) $d(y, y^*) \leq \frac{1}{1-\alpha} d(y, Jy)$ for all $y \in Y$.

In 1996, G. Isac and Th.M. Rassias [34] were the first to provide applications of stability theory of functional equations for the proof of new fixed point theorems with applications. By using fixed point methods, the stability problems of several functional equations have been extensively investigated by a number of authors (see [4],[19]–[27],[45]–[52], [54]).

This paper is organized as follows: In Section 2, we prove the Hyers-Ulam stability of the orthogonally additive functional equation in Banach modules over a unital C^*-algebra. In Section 3, we prove the Hyers-Ulam stability of the orthogonally additive functional equation in non-Archimedean Banach modules over a unital C^*-algebra.

2. Stability of the orthogonally additive functional equation in Banach modules over a C^*-algebra

Throughout this section, assume that A is a unital C^*-algebra with unit e and unitary group $U(A) := \{u \in A \mid u^*u = uu^* = e\}$, $(X, \perp)$ is an orthogonality normed module over A and $(Y, \|.\|_Y)$ is a Banach module over A.

In this section, applying some ideas from [29, 33], we deal with the stability problem for the orthogonally additive functional equation

$$\sum_{i=1}^{m} f\left(mx_i + \sum_{j=1, j\neq i}^{m} x_j\right) + f\left(\sum_{i=1}^{m} x_i\right) = 2f\left(\sum_{i=1}^{m} x_i\right)$$

for all $x_1, \cdots, x_m \in X$ with $x_i \perp x_j$ for all $i \neq j$.

Theorem 2.1. *Let $\varphi : X^m \to [0, \infty)$ be a function such that there exists an $\alpha < 1$ with*

$$\varphi(x_1, x_2, \cdots, x_m) \leq m\alpha\varphi\left(\frac{x}{m}, \frac{x_2}{m}, \cdots, \frac{x_m}{m}\right) \tag{2.1}$$

for all $x_1, \cdots, x_m \in X$ with $x_i \perp x_j$ for all $i \neq j$. Let $f : X \to Y$ be a mapping satisfying

$$\left\| \sum_{i=1}^{m} f\left(mux_i + \sum_{j=1, j\neq i}^{m} ux_j\right) + f\left(\sum_{i=1}^{m} ux_i\right) - 2uf\left(\sum_{i=1}^{m} x_i\right) \right\|_Y \leq \varphi(x_1, \cdots, x_n) \tag{2.2}$$

for all $u \in U(A)$ and all $x_1, \cdots, x_m \in X$ with $x_i \perp x_j$ for all $i \neq j$. If for each $x \in X$ the mapping $f(tx)$ is continuous in $t \in \mathbb{R}$, then there exists a unique orthogonally additive and A-linear mapping $L : X \to Y$ such that

$$\|f(x) - L(x)\|_Y \leq \frac{1}{m - m\alpha} \psi(x) \tag{2.3}$$

for all $x \in X$, where $\psi(x) = \varphi(x, 0, \cdots, 0)$.

Proof. Putting $x_1 = x$ and $x_2 = \cdots = x_m = 0$ and $u = e$ in (2.2), since $x \perp 0$, we get

$$\left\| f(x) - \frac{f(mx)}{m} \right\|_Y \leq \frac{\psi(x)}{m} \tag{2.4}$$

for all $x \in X$. Consider the set $S := \{h : X \to Y\}$ and introduce the generalized metric on S:

$$d(g, h) = \inf \left\{\mu \in \mathbb{R}_+ : \|g(x) - h(x)\|_Y \leq \mu\psi(x), \ \forall x \in X\right\},$$

where, as usual, $\inf \phi = +\infty$. It is easy to show that (S, d) is complete (see [42]). Now we consider the linear mapping $J : S \to S$ such that

$$Jg(x) := \frac{1}{m} g(mx)$$

for all $x \in X$. Let $g, h \in S$ be given such that $d(g, h) = \varepsilon$. Then $\|g(x) - h(x)\|_Y \leq \varepsilon\psi(x)$ for all $x \in X$. Hence

$$\|Jg(x) - Jh(x)\|_Y = \left\| \frac{g(mx)}{m} - \frac{h(mx)}{m} \right\|_Y \leq \frac{\psi(mx)}{m} \leq \frac{m\alpha\psi(x)}{m} \leq \alpha\psi(x)$$

for all $x \in X$. So $d(g, h) = \varepsilon$ implies that $d(Jg, Jh) \leq \alpha\varepsilon$. This means that $d(Jg, Jh) \leq \alpha d(g, h)$ for all $g, h \in S$. It follows from (2.4) that

$$d(f, Jf) \leq \frac{1}{m}.$$

By Theorem 1.4, there exists a mapping $L : X \to Y$ satisfying the following:

(1) L is a fixed point of J, i.e.,

$$L(mx) = mL(x) \tag{2.5}$$

for all $x \in X$. The mapping L is a unique fixed point of J in the set $M = \{g \in S : d(h, g) < \infty\}$. This implies that L is a unique mapping satisfying (2.5) such that there exists a $\mu \in (0, \infty)$ satisfying $\|f(x) - L(x)\|_Y \leq \mu\psi(x)$ for all $x \in X$;

(2) $d(J^k f, L) \to 0$ as $k \to \infty$. This implies the equality

$$\lim_{k\to\infty} \frac{1}{m^k} f\left(m^k x\right) = L(x)$$

for all $x \in X$;

(3) $d(f, L) \leq \frac{1}{1-\alpha} d(f, Jf)$, which implies the inequality

$$d(f, L) \leq \frac{1}{m - m\alpha}.$$

This implies that (2.3) holds true. Let $u = e$ in (2.2). It follows from (2.1) and (2.2) that

$$\begin{aligned}
&\left\| \sum_{i=1}^{m} L\left(mx_i + \sum_{j=1, j\neq i}^{m} x_j\right) + L\left(\sum_{i=1}^{m} x_i\right) - 2L\left(\sum_{i=1}^{m} x_i\right)\right\|_Y \\
&\quad = \lim_{k\to\infty} \frac{1}{m^k} \left\| \sum_{i=1}^{m} f\left(m^k\left(mx_i + \sum_{j=1, j\neq i}^{m} x_j\right)\right) + f\left(\sum_{i=1}^{m} m^k x_i\right) - 2f\left(\sum_{i=1}^{m} m^k x_i\right)\right\|_Y \\
&\quad \leq \lim_{k\to\infty} \frac{\varphi(m^k x_1, m^k x_2, \cdots, m^k x_m)}{m^k} \\
&\quad \leq \lim_{k\to\infty} \frac{m^k \alpha^n \varphi(x_1, \cdots, x_m)}{m^k} = 0
\end{aligned}$$

for all $x_1, \cdots, x_m \in X$ with $x_i \perp x_j$ for all $i \neq j$. So

$$\sum_{i=1}^{m} L\left(mx_i + \sum_{j=1, j\neq i}^{m} x_j\right) + L\left(\sum_{i=1}^{m} x_i\right) - 2L\left(\sum_{i=1}^{m} x_i\right) = 0$$

for all $x_1, \cdots, x_n \in X$ with $x_1 \perp x_j$ for all $i \neq j$. Hence $L : X \to Y$ is an orthogonally additive mapping. Let $x_2 = \cdots = x_n = 0$ in (2.2). It follows from (2.1) and (2.2) that

$$\begin{aligned}
\|L(mux) - muL(x)\|_Y &= \lim_{k\to\infty} \frac{\|f(m^{k+1}ux) - mf(m^k ux)\|_Y}{m^k} \\
&= m \lim_{k\to\infty} \left\| \frac{f(m^{k+1}ux)}{m^{k+1}} - \frac{f(m^k ux)}{m^k}\right\|_Y \\
&\leq \lim_{k\to\infty} \frac{\psi(m^k x)}{m^k} \leq \lim_{k\to\infty} \frac{m^k \alpha^n \psi(x)}{m^k} \\
&= \lim_{k\to\infty} \alpha^n \psi(x) = 0
\end{aligned}$$

for all $x \in X$ and all $u \in U(A)$. So

$$muL\left(\frac{x}{m}\right) - L(ux) = 0$$

for all $x \in X$ and all $u \in U(A)$. Hence

$$L(ux) = muL\left(\frac{x}{m}\right) = uL(x) \tag{2.6}$$

for all $u \in U(A)$ and all $x \in X$.

By the same reasoning as in the proof of [55, Theorem], we can show that $L : X \to Y$ is $\mathbb{R}$-linear, since the mapping $f(tx)$ is continuous in $t \in \mathbb{R}$ for each $x \in X$ and $L : X \to Y$ is additive.

Since L is $\mathbb{R}$-linear and each $a \in A$ is a finite linear combination of unitary elements (see [38, Theorem 4.1.7]), i.e., $a = \sum_{j=1}^{m} \lambda_j u_j$ ($\lambda_j \in \mathbb{C}$, $u_j \in U(A)$), it follows from (2.6) that

$$\begin{aligned}
L(ax) &= L\left(\sum_{j=1}^{m} \lambda_j u_j x\right) = L\left(\sum_{j=1}^{m} |\lambda_j| \cdot \frac{\lambda_j}{|\lambda_j|} u_j x\right) = \sum_{j=1}^{m} |\lambda_j| L\left(\frac{\lambda_j}{|\lambda_j|} u_j x\right) \\
&= \sum_{j=1}^{m} |\lambda_j| \cdot \frac{\lambda_j}{|\lambda_j|} u_j L(x) = \sum_{j=1}^{m} \lambda_j u_j L(x) = aL(x)
\end{aligned}$$

for all $x \in X$. It is obvious that $\frac{\lambda_j}{|\lambda_j|} u_j \in U(A)$. Thus $L : X \to Y$ is a unique orthogonally additive and A-linear mapping satisfying (2.3). $\square$

Corollary 2.2. *Let θ be a positive real number and p a real number with $0 < p < 1$. Let $f : X \to Y$ be a mapping satisfying*

$$\left\| \sum_{i=1}^{m} f\left(mux_i + \sum_{j=1, j\neq i}^{m} ux_j \right) + f\left(\sum_{i=1}^{m} ux_i \right) - 2uf\left(\sum_{i=1}^{m} x_i \right) \right\|_Y \leq \theta \left(\sum_{i=1}^{m} \|x_i\|^p \right) \tag{2.7}$$

for all $u \in U(A)$ and all $x_1, \cdots, x_m \in X$ with $x_i \perp x_j$ for all $i \neq j$. If for each $x \in X$ the mapping $f(tx)$ is continuous in $t \in \mathbb{R}$, then there exists a unique orthogonally additive and A-linear mapping $L : X \to Y$ such that

$$\|f(x) - L(x)\|_Y \leq \frac{\theta \|x\|^p}{m - m^p}$$

for all $x \in X$.

Proof. The proof follows from Theorem 2.1 by taking

$$\varphi(x_1, x_2, \cdots, x_n) = \theta \left(\sum_{i=1}^{n} \|x_i\|^p \right)$$

for all $x_1, \cdots, x_m \in X$ with $x_i \perp x_j$ for all $i \neq j$. Then we can choose $\alpha = m^{p-1}$ and we get the desired result. $\square$

Theorem 2.3. *Let $f : X \to Y$ be a mapping satisfying* (2.2) *for which there exists a function $\varphi : X^m \to [0, \infty)$ such that*

$$\varphi\left(\frac{x_1}{m}, \frac{x_2}{m}, \cdots, \frac{x_m}{m} \right) \leq \frac{\alpha \varphi(x_1, x_2, \cdots, x_m)}{m}$$

for all $x_1, \cdots, x_m \in X$ with $x_i \perp x_j$ for all $i \neq j$. If for each $x \in X$ the mapping $f(tx)$ is continuous in $t \in \mathbb{R}$, then there exists a unique orthogonally additive and A-linear mapping $L : X \to Y$ such that

$$\|f(x) - L(x)\|_Y \leq \frac{\alpha \psi(x)}{m - m\alpha} \tag{2.8}$$

for all $x \in X$.

Proof. Let (S, d) be the generalized metric space defined in the proof of Theorem 2.1. Now we consider the linear mapping $J : S \to S$ such that

$$Jg(x) := mg\left(\frac{x}{m} \right)$$

for all $x \in X$. Let $g, h \in S$ be given such that $d(g, h) = \varepsilon$. Then $\|g(x) - h(x)\|_Y \leq \varepsilon \psi(x)$ for all $x \in X$. Hence

$$\|Jg(x) - Jh(x)\|_Y = \left\| mg\left(\frac{x}{m} \right) - mh\left(\frac{x}{m} \right) \right\|_Y \leq m\psi\left(\frac{x}{m} \right) \leq \frac{m\alpha\psi(x)}{m} \leq \alpha\psi(x)$$

for all $x \in X$. So $d(g, h) = \varepsilon$ implies that $d(Jg, Jh) \leq \alpha\varepsilon$. This means that $d(Jg, Jh) \leq \alpha d(g, h)$ for all $g, h \in S$. It follows from (2.4) that

$$\left\| mf\left(\frac{x}{m} \right) - f(x) \right\|_Y \leq \psi\left(\frac{x}{m} \right) \leq \frac{\alpha}{m} \psi(x).$$

Therefore

$$d(f, Jf) \leq \frac{\alpha}{m}.$$

By Theorem 1.4, there exists a mapping $L : X \to Y$ satisfying the following:

(1) L is a fixed point of J, i.e.,

$$L\left(\frac{x}{m}\right) = \frac{1}{m}L(x) \tag{2.9}$$

for all $x \in X$. The mapping L is a unique fixed point of J in the set $M = \{g \in S : d(h, g) < \infty\}$. This implies that L is a unique mapping satisfying (2.9) such that there exists a $\mu \in (0, \infty)$ satisfying $\|f(x) - L(x)\|_Y \le \mu\psi(x)$ for all $x \in X$;

(2) $d(J^k f, L) \to 0$ as $k \to \infty$. This implies the equality

$$\lim_{k\to\infty} m^k f\left(\frac{x}{m^k}\right) = L(x)$$

for all $x \in X$;

(3) $d(f, L) \le \frac{1}{1-\alpha}d(f, Jf)$, which implies the inequality

$$d(f, L) \le \frac{\alpha}{m - m\alpha}.$$

This implies that (2.8) holds true.

The rest of the proof is similar to the proof of Theorem 2.1. □

Corollary 2.4. *Let θ be a positive real number and p a real number with $p > 1$. Let $f : X \to Y$ be a mapping satisfying (2.7). If for each $x \in X$ the mapping $f(tx)$ is continuous in $t \in \mathbb{R}$, then there exists a unique orthogonally additive and A-linear mapping $L : X \to Y$ such that*

$$\|f(x) - L(x)\|_Y \le \frac{\theta\|x\|^p}{m^p - m}$$

for all $x \in X$.

Proof. The proof follows from Theorem 2.3 by taking

$$\varphi(x_1, x_2, \cdots, x_n) = \theta\left(\sum_{i=1}^{m}\|x_i\|^p\right)$$

for all $x_1, \cdots, x_m \in X$ with $x_i \perp x_j$ for all $i \neq j$. Then we can choose $\alpha = m^{1-p}$ and we get the desired result. □

3. Stability of the orthogonally additive functional equation in non-Archimedean Banach modules over a C^*-algebra

Throughout this section, assume that A is a unital C^*-algebra with unit e and unitary group $U(A) := \{u \in A \mid u^*u = uu^* = e\}$, $(X, \perp)$ is an orthogonality non-Archimedean normed module over A and $(Y, \|.\|_Y)$ is a non-Archimedean Banach module over A. Assume that $|m| \neq 1$.

In this section, applying some ideas from [29, 33], we deal with the stability problem for the orthogonally Jensen functional equation.

Theorem 3.1. *Let $\varphi : X^m \to [0, \infty)$ be a function such that there exists an $\alpha < 1$ with*

$$\varphi(x_1, x_2, \cdots, x_m) \le |m|\alpha\varphi\left(\frac{x}{m}, \frac{x_2}{m}, \cdots, \frac{x_m}{m}\right) \tag{3.1}$$

for all $x_1, \cdots, x_m \in X$ with $x_i \perp x_j$ for all $i \neq j$. Let $f : X \to Y$ be a mapping satisfying (2.2). If for each $x \in X$ the mapping $f(tx)$ is continuous in $t \in \mathbb{R}$, then there exists a unique orthogonally additive and A-linear mapping $L : X \to Y$ such that

$$\|f(x) - L(x)\|_Y \le \frac{\psi(x)}{|m| - |m|\alpha} \tag{3.2}$$

for all $x \in X$.

Proof. It follows from (2.4) that

$$\left\| f(x) - \frac{f(mx)}{m} \right\|_Y \leq \frac{\psi(x)}{|m|} \tag{3.3}$$

for all $x \in X$. Let (S, d) be the generalized metric space defined in the proof of Theorem 2.1. Now we consider the linear mapping $J : S \to S$ such that

$$Jg(x) := \frac{g(mx)}{m}$$

for all $x \in X$. It follows from (3.3) that $d(f, Jf) \leq |m|$. By Theorem 1.4, there exists a mapping $L : X \to Y$ satisfying the following:

(1) $d(J^k f, L) \to 0$ as $k \to \infty$. This implies the equality

$$\lim_{k\to\infty} \frac{1}{m^k} f\left(m^k x\right) = L(x)$$

for all $x \in X$;

(2) $d(f, L) \leq \frac{1}{1-\alpha} d(f, Jf)$, which implies the inequality

$$d(f, L) \leq \frac{1}{|m| - |m|\alpha}.$$

This implies that (3.2) holds true. It follows from (3.1) and (2.2) that

$$\begin{aligned}
&\left\| \sum_{i=1}^m L\left(mux_i + \sum_{j=1, j\neq i}^m ux_j \right) + L\left(\sum_{i=1}^m ux_i \right) - 2uL\left(\sum_{i=1}^m x_i \right) \right\|_Y \\
&= \lim_{k\to\infty} \frac{1}{|m|^k} \left\| \sum_{i=1}^m f\left(m^k \left(mux_i + \sum_{j=1, j\neq i}^m ux_j \right) \right) \right. \\
&\quad \left. + f\left(\sum_{i=1}^m m^k ux_i \right) - 2uf\left(\sum_{i=1}^m m^k x_i \right) \right\|_Y \\
&\leq \lim_{k\to\infty} \frac{\varphi(m^k x_1, m^k x_2, \cdots, m^k x_m)}{|m|^k} \\
&\leq \lim_{k\to\infty} \frac{|m|^k \alpha^n \varphi(x_1, \cdots, x_m)}{|m|^k} = 0
\end{aligned}$$

for all $u \in U(A)$ and all $x_1, \cdots, x_m \in X$ with $x_i \perp x_j$ for all $i \neq j$. So

$$\sum_{i=1}^m L\left(mux_i + \sum_{j=1, j\neq i}^m ux_j \right) + L\left(\sum_{i=1}^m ux_i \right) = 2uL\left(\sum_{i=1}^m x_i \right)$$

for all $u \in U(A)$ and all $x_1, \cdots, x_n \in X$ with $x_i \perp x_j$ for all $i \neq j$. Hence $L : X \to Y$ is an orthogonally additive mapping.

The rest of the proof is similar to the proof of Theorem 2.1. □

Corollary 3.2. *Let θ be a positive real number and p a real number with $p > 1$. Let $f : X \to Y$ be a mapping satisfying (2.7). If for each $x \in X$ the mapping $f(tx)$ is continuous in $t \in \mathbb{R}$, then there exists a unique orthogonally additive and A-linear mapping $L : X \to Y$ such that*

$$\|f(x) - L(x)\|_Y \leq \frac{\theta \|x\|^p}{|m| - |m|^{p+1}}$$

for all $x \in X$.

Proof. The proof follows from Theorem 3.1 by taking

$$\varphi(x_1, x_2, \cdots, x_n) = \theta\left(\sum_{i=1}^{n} \|x_i\|^p\right)$$

for all $x_1, \cdots, x_m \in X$ with $x_i \perp x_j$ for all $i \neq j$. Then we can choose $\alpha = |m|^{p-1}$ and we get the desired result. □

Theorem 3.3. *Let $f : X \to Y$ be a mapping satisfying (2.2) and for which there exists a function $\varphi : X^m \to [0, \infty)$ such that*

$$\varphi\left(\frac{x_1}{m}, \frac{x_2}{m}, \cdots, \frac{x_m}{m}\right) \leq \frac{\alpha\varphi\left(x_1, x_2, \cdots, x_m\right)}{|m|}$$

for all $x_1, \cdots, x_m \in X$ with $x_i \perp x_j$ for all $i \neq j$. If for each $x \in X$ the mapping $f(tx)$ is continuous in $t \in \mathbb{R}$, then there exists a unique orthogonally additive and A-linear mapping $L : X \to Y$ such that

$$\|f(x) - L(x)\|_Y \leq \frac{\alpha\psi(x)}{|m| - |m|\alpha} \tag{3.4}$$

for all $x \in X$.

Proof. Let (S, d) be the generalized metric space defined in the proof of Theorem 2.1.

Now we consider the linear mapping $J : S \to S$ such that

$$Jg(x) := mg\left(\frac{x}{m}\right)$$

for all $x \in X$. It follows from (2.4) that $d(f, Jf) \leq \frac{\alpha}{|m|}$. The rest of the proof is similar to the proofs of Theorems 2.1 and 3.1. □

Corollary 3.4. *Let θ be a positive real number and p a real number with $0 < p < 1$. Let $f : X \to Y$ be a mapping satisfying (2.7). If for each $x \in X$ the mapping $f(tx)$ is continuous in $t \in \mathbb{R}$, then there exists a unique orthogonally Jensen and A-linear mapping $L : X \to Y$ such that*

$$\|f(x) - L(x)\|_Y \leq \frac{|m|\theta\|x\|^p}{|m|^{p+1} - |m|}$$

for all $x \in X$.

Proof. The proof follows from Theorem 3.3 by taking

$$\varphi(x_1, x_2, \cdots, x_n) = \theta\left(\sum_{i=1}^{n} \|x_i\|^p\right)$$

for all $x_1, \cdots, x_m \in X$ with $x_i \perp x_j$ for all $i \neq j$. Then we can choose $\alpha = |m|^{1-p}$ and we get the desired result. □

Acknowledgments

C. Park was supported by Basic Science Research Program through the National Research Foundation of Korea funded by the Ministry of Education, Science and Technology (NRF-2012R1A1A2004299). D. Y. Shin was supported by Basic Science Research Program through the National Research Foundation of Korea funded by the Ministry of Education, Science and Technology (NRF-2010-0021792).

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J. COMPUTATIONAL ANALYSIS AND APPLICATIONS, VOL. 15, NO.6, 1069-1084, 2013, COPYRIGHT 2013 EUDOXUS PRESS, LLC

SOME CHARACTERIZATIONS AND CONVERGENCE PROPERTIES OF THE CHOQUET INTEGRAL WITH RESPECT TO A FUZZY MEASURE OF FUZZY COMPLEX VALUED FUNCTIONS

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Abstract. In this paper, we consider Choquet integrals with respect to a fuzzy measure and fuzzy complex valued functions. We define the Choquet integral with respect to a fuzzy measure of a fuzzy complex valued functions and investigate their characterizations. Furthermore, we discuss some convergence properties of the Choquet integral with respect to a fuzzy measure of an integrably bounded fuzzy complex valued measurable function.

§1. Introduction

Choquet integrals, introduced in [8,9,10], has emerged as an interesting extension of the Lebesgue integral. Puri and Ralescu [11] have been studied Lebesgue integral with respect to a classical measure of closed set-valued measurable functions. In the papers [4-7], we defined interval-valued Choquet integrals and have studied some convergence theorems for Choquet integrals with respect to a fuzzy measure of interval-valued measurable functions under some sufficient conditions. Zhang, Guo and Liu [14] restudied Choquet integrals with respect to a fuzzy measure of closed set-valued measurable functions.

Burkley [1-3] introduced the concept of fuzzy complex numbers, the differentiability and integrability of fuzzy complex valued functions on a complex plane $\mathbb{C}$. Wang and Li [11] have researched generalized Lebesgue integrals with respect to a complex valued fuzzy measure of fuzzy complex valued functions.

2000 AMS Subject Classification: 28E10, 03E72, 26E50
keywords and phrases : Choquet integrals, fuzzy measures, fuzzy complex numbers, fuzzy complex valued functions

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In this paper, we define the Choquet integral with respect to a fuzzy measure of a fuzzy complex valued function and discuss their properties. In particular, we prove some convergence theorems for the Choquet integrals of a fuzzy complex valued function. In section 2, we list the definitions and various properties of fuzzy measures and Choquet integrals. In section 3, we introduce fuzzy complex numbers and fuzzy complex valued functions. We define Choquet integrals with respect to a fuzzy measure of a fuzzy complex valued functions and discuss some of their some characterizations. In section 4, we discuss some convergence properties of the Choquet integrals of integrably bounded fuzzy complex valued functions. In section 5, we give a brief summery results and some conclusions.

§2. Definitions and Preliminaries

Throughout this paper, we assume that $(X, \Im(X))$ is a measurable space and denote $\mathbb{R}^+ = [0, \infty)$ and $\bar{\mathbb{R}}^+ = [0, \infty]$. We list the definitions of fuzzy measures and Choquet integrals(see [4-12]).

Definition 2.1. (1) *A set function $\mu : \Im(X) \longrightarrow \bar{\mathbb{R}}^+$ is called a fuzzy measure if (i) $\mu(\emptyset) = 0$ and (ii) $\mu(A) \leq \mu(B)$ whenever $A, B \in \Im(X)$ and $A \subset B$.*

(2) *If $\mu(X) < \infty$, μ is said to be finite.*

(3) *A set function μ is said to be lower semi-continuous if for each increasing sequence $\{A_n\}$ in $\Im(X)$,*

$$\mu(\cup_{n=1}^{\infty} A_n) = \lim_{n\to\infty} \mu(A_n).$$

(4) *A set function μ is said to be lower semi-continuous if for each decreasing sequence$\{A_n\}$ in $\Im(X)$ with $\mu(A_1) < \infty$,*

$$\mu(\cap_{n=1}^{\infty} A_n) = \lim_{n\to\infty} \mu(A_n).$$

(5) *If μ is both lower semi-continuous and upper semi-continuous, it is said to be semi-continuous.*

We remark that fuzzy measures are known to be the generalization of classical measures where additivity is replaced by the weaker condition of monotonicity and that fuzzy measures are not assumed to be semi-continuous. We introduce the Choquet integral proposed by M. Sugeno(see [8]) as follows.

Definition 2.2. *(1) The Choquet integral with respect to a fuzzy measure μ of a measurable function $f : X \longrightarrow \mathbb{R}^+$ on $A \in \Im(X)$ is defined by*

$$(C)\int_A f d\mu = \int_0^{\infty} \mu(\{x|f(x) > r\} \cap A)dr$$

where the integral on the right-hand side is the Lebesgue integral.

(2) A measurable function f is said to be C-integrable if the Choquet integral of f on X can be defined and its value is finite.

Instead of $(C)\int_X f d\mu$, we will write $(C)\int f d\mu$. We consider the (decreasing) distribution function $G_f(r) = \mu(\{x|f(x) > r\})$ of a measurable function f for any $r \in \mathbb{R}^+ = [0,\infty)$.

Definition 2.3. *Let μ be a fuzzy measure on $\Im(X)$ and f a measurable function. We say that f and g are comonotonic, in symbol, $f \sim g$ if $f(x) < f(x') \Longrightarrow g(x) \leq g(x')$ for all $x, x' \in X$.*

Now we introduce the following basic properties of the comonotonicity and the Choquet integral.

Theorem 2.4. [8-10, 12]) *Let f, g, and h be measurable functions. Then we have*
(1) $f \sim f$,
(2) $f \sim g \Longrightarrow g \sim f$,
(3) $f \sim a$ *for all* $a \in \mathbb{R}^+$,
(4) $f \sim g$ *and* $g \sim h \Longrightarrow f \sim g + h$.

Theorem 2.5. [8-10, 12]) *Let f and g be C-integrable functions. Then we have*
(1) *if* $f \leq g$, *then* $(C)\int f d\mu \leq (C)\int g d\mu$,
(2) *if* $E_1 \subset E_2$ *and* $E_1, E_2 \in \Im(X)$, *then* $(C)\int_{E_1} f d\mu \leq (C)\int_{E_2} f d\mu$,
(3) *if* $f \sim g$ *and* $a, b \in \mathbb{R}^+$, *then*

$$(C)\int (af + bg) d\mu = a(C)\int f d\mu + b(C)\int g d\mu,$$

(4) *if we define* $(f \vee g)(x) = f(x) \vee g(x)$ *and* $(f \wedge g)(x) = f(x) \wedge g(x)$ *for all* $x \in X$, *then*

$$(C)\int f \vee g d\mu \geq (C)\int f d\mu \vee (C)\int g d\mu$$

and

$$(C)\int f \wedge g d\mu \leq (C)\int f d\mu \wedge (C)\int g d\mu$$

Throughout this paper, $I(\mathbb{R}^+)$ is the class of all closed intervals in $\mathbb{R}^+$, that is,

$$I(\mathbb{R}^+) = \{[a^-, a^+] | a^-, a^+ \in \mathbb{R}^+ \text{ and } a^- \leq a^+\}.$$

For any $a \in \mathbb{R}^+$, we define $a = [a, a]$. Obviously, $a \in I(\mathbb{R}^+)$(see[4-7]).

Definition 2.6. *If $\bar{a} = [a^-, a^+], \bar{b} = [b^-, b^+] \in I(\mathbb{R}^+)$ and $c \in \mathbb{R}^+$, then we define the following operations:*
(1) $\bar{a} + \bar{b} = [a^- + b^-, a^+ + b^+]$.
(2) $k\bar{a} = [ca^-, ca^+]$.
(3) $\bar{a}\bar{b} = [a^-b^-, a^+b^+]$.
(4) $\bar{a} \wedge \bar{b} = [a^- \wedge b^-, a^+ \wedge b^+]$.
(5) $\bar{a} \vee \bar{b} = [a^- \vee b^-, a^+ \vee b^+]$.
(6) $\bar{a} \leq \bar{b}$ if and only if $a^- \leq b^-$ and $a^+ \leq b^+$.
(7) $\bar{a} < \bar{b}$ if and only if $\bar{a} \leq \bar{b}$ and $\bar{a} \neq \bar{b}$.
(8) $\bar{a} \subset \bar{b}$ if and only if $b^- \leq a^-$ and $a^+ \leq b^+$.

Definition 2.7. *If $\bar{a} = [a_k^-, a_k^+] \in I(\mathbb{R}^+)$ for $k = 1, 2, \cdots$, then we define the following operations:*
(1) $\wedge_{k=1}^{\infty} \bar{a}_k = [\wedge_{k=1}^{\infty} a_k^-, \wedge_{k=1}^{\infty} a_k^+]$.
(2) $\vee_{k=1}^{\infty} \bar{a}_k = [\vee_{k=1}^{\infty} a_k^-, \vee_{k=1}^{\infty} a_k^+]$.

Theorem 2.8. *For $\bar{a}, \bar{b}, \bar{c} \in I(\mathbb{R}^+)$, we have*
(1) idempotent law: $\bar{a} \wedge \bar{a} = \bar{a}, \bar{a} \vee \bar{a} = \bar{a}$,
(2) commutative law: $\bar{a} \wedge \bar{b} = \bar{b} \wedge \bar{a}, \bar{a} \vee \bar{b} = \bar{b} \vee \bar{a}$,
(3) associative law: $(\bar{a} \wedge \bar{b}) \wedge \bar{c} = \bar{a} \wedge (\bar{b} \wedge \bar{c})$,
(4) absorption law: $\bar{a} \wedge (\bar{a} \vee \bar{b}) = \bar{a} \vee (\bar{a} \wedge \bar{b}) = \bar{a}$,
(5) distributive law: $\bar{a} \wedge (\bar{b} \vee \bar{c}) = (\bar{a} \wedge \bar{b}) \vee (\bar{a} \wedge \bar{c}), \bar{a} \vee (\bar{b} \wedge \bar{c}) = (\bar{a} \vee \bar{b}) \wedge (\bar{a} \vee \bar{c})$.

W note that $(I(\mathbb{R}^+), d_H)$ is a metric space, where a mapping $d_H : I(\mathbb{R}^+) \times I(\mathbb{R}^+) \longrightarrow \bar{\mathbb{R}}^+$ is the Hausdorff metric defined by

$$d_H(A, B) = \max\{\sup_{x \in A} \inf_{y \in B} |x - y|, \sup_{y \in B} \inf_{x \in A} |x - y|\}$$

for all $A, B \in I(\mathbb{R}^+)$. By the definition of the Hausdorff metric, it is easy to see that for any $\bar{a} = [a^-, a^+], \bar{b} = [b^-, b^+] \in I(\mathbb{R}^+)$, we have

$$d_H(\bar{a}, \bar{b}) = \max\{|a^- - b^-|, |a^+ - b^+|\}.$$

Note that for a sequence of closed intervals $\{\bar{a}_n\}$ converges to $\bar{a}$, in symbols $d_H - \lim_{n\to\infty} \bar{a}_n = \bar{a}$ if $\lim_{n\to\infty} d_H(\bar{a}_n, \bar{a}) = 0$ and that $d_H - \lim_{n\to\infty} \bar{a}_n = \bar{a}$ if and only if $\lim_{n\to\infty} a_n^- = a^-$ and $\lim_{n\to\infty} a_n^+ = a^+$. In the following definition, we introduce fuzzy numbers and some operations on them which are used in the next sections.

Definition 2.9. *A fuzzy set $\widetilde{u}$ on $\mathbb{R}^+$ is called a fuzzy number if it satisfies the following conditions;*
(i) (normality) $\widetilde{u}(x) = 1$ for some $x \in \mathbb{R}^+$,

(ii) (fuzzy convexity) for every $\lambda \in (0,1]$,

$$\widetilde{u}_\lambda = \{x \in \mathbb{R}^+ |\ \widetilde{u}(x) \geq \lambda\} \in I(\mathbb{R}^+),$$

where $\widetilde{u}_\lambda$ is the level set of $\widetilde{u}$.

Let $FN(\mathbb{R}^+)$ denote the class of all fuzzy numbers. We define the following basic operations on $FN(\mathbb{R}^+)$(see[8,9,12]); for every $\widetilde{u}, \widetilde{v} \in FN(\mathbb{R}^+)$ and $k \in \mathbb{R}^+$,
$(\widetilde{u} + \widetilde{v})_\lambda = \widetilde{u}_\lambda + \widetilde{v}_\lambda$,
$(k\widetilde{u})_\lambda = k\widetilde{u}_\lambda$,
$(\widetilde{u}\widetilde{v})_\lambda = \widetilde{u}_\lambda \widetilde{v}_\lambda$,
$\widetilde{u} \leq \widetilde{v}$ if and only if $\widetilde{u}_\lambda \leq \widetilde{v}_\lambda$, for all $\lambda \in (0,1]$,
$\widetilde{u} < \widetilde{v}$ if and only if $\widetilde{u} \leq \widetilde{v}$ and $\widetilde{u} \neq \widetilde{v}$,
$\widetilde{u} \subset \widetilde{v}$ if and only if $\widetilde{u}_\lambda \subset \widetilde{v}_\lambda$, for all $\lambda \in (0,1]$.

§3. Choquet integrals of fuzzy complex fuzzy functions

In this section, we consider a fuzzy number and fuzzy complex numbers(see[1-3,13]).

Definition 3.1. *Let $\widetilde{a}, \widetilde{b} \in FN(\mathbb{R}^+)$. We define a double ordered fuzzy numbers $(\widetilde{a}, \widetilde{b})$ as follows:*

$$(\widetilde{a}, \widetilde{b}) : \mathbb{C}^+ \longrightarrow [0,1]$$

$$z = x + yi \longmapsto (\widetilde{a}, \widetilde{b})(z) = \widetilde{a}(x) \wedge \widetilde{y}(y),$$

where $\mathbb{C}^+ = \{x + yi | x, y \in \mathbb{R}^+\}$. Then the mapping $(\widetilde{a}, \widetilde{b})$ determines a fuzzy complex number, where $\widetilde{a}$ and $\widetilde{b}$ is called a real part and an imaginary part of $(\widetilde{a}, \widetilde{b})$, respectively.

We note that if we put $C = (\widetilde{a}, \widetilde{b})$, then $\widetilde{a} = ReC$ and $\widetilde{b} = ImC$. Let $FCN(\mathbb{C}^+)$ be the class of all fuzzy complex numbers on $\mathbb{C}^+$, writing

$$C \equiv \widetilde{a} + \widetilde{b}i.$$

Note that if $c = a+bi$ is a nonnegative complex number, then its membership function is

$$c(z) = \begin{cases} 1 & \text{if } x = a, y = b \\ 0 & \text{otherwise} \end{cases}$$

where $z = x+yi \in \mathbb{C}^+$. Clearly, $c \in FCN(\mathbb{C}^+)$, that is, a fuzzy complex number is also a generalization of an ordinary complex number. We recall that if $C_1, C_2 \in FCN(\mathbb{C}^+)$ and we define

$$C_1 * C_2 = (ReC_1 * ReC_2, ImC_1 * ImC_2)$$

for an operation $* \in \{+, -, \times, \wedge, \vee\}$, then clearly we have $C_1 * C_2 \in FCN(\mathbb{C}^+)$.

Definition 3.2. *Let $C_1, C_2 \in FCN(\mathbb{C}^+)$. Then we define the following order and equality operations:*

(1) $C_1 \leq C_2$ if and only if $ReC_1 \leq ReC_2$ and $ImC_1 \leq ImC_2$.

(2) $C_1 < C_2$ if and only if $C_1 \leq C_2$ and $C_1 \neq C_2$.

(3) $C_1 = C_2$ if and only if $C_1 \leq C_2$ and $C_2 \leq C_1$.

(4) $C_1 \subset C_2$ if and only if $ReC_1 \subset ReC_2$ and $ImC_1 \subset ImC_2$.

From Definition 3.2, it is easy to see that if we define λ-cut set $C_\lambda = \{z = x + yi \in \mathbb{C}^+ | (ReC)(x) \geq \lambda \text{ and } (ImC)(y) \geq \lambda\}$, then it is a closed rectangle region in $\mathbb{C}^+$. Now, we consider fuzzy complex valued functions as follows(see [13]).

Definition 3.3. *If a mapping $\widetilde{f} : \mathbb{C}^+ \longrightarrow FCN(\mathbb{C}^+)$ is defined by*

$$z = x + yi \longmapsto \widetilde{f}(z) = (Re\widetilde{f}, Im\widetilde{f})(z) = Re\widetilde{f}(x) \wedge Im\widetilde{f}(y),$$

then $\widetilde{f}$ is called a fuzzy complex valued function on $\mathbb{C}^+$.

We note that for any $\lambda \in (0, 1]$, let

$$\widetilde{f}_\lambda(z) \equiv (\widetilde{f}(z))_\lambda = ((Re\widetilde{f}(x))_\lambda, (Im\widetilde{f}(y))_\lambda), \text{ for all } z = x + yi \in \mathbb{C}^+,$$

where $(Re\widetilde{f})_\lambda \equiv [(Re\widetilde{f})^-_\lambda, (Re\widetilde{f})^+_\lambda]$ and $(Im\widetilde{f})_\lambda \equiv [(Im\widetilde{f})^-_\lambda, (Im\widetilde{f})^+_\lambda]$ for all $\lambda \in (0, 1]$ and that $\widetilde{f}$ is said to be measurable if for any $\lambda \in (0, 1]$, $(Re\widetilde{f})_\lambda$ and $(Im\widetilde{f})_\lambda$ are measurable. We introduce Choqeut integral of interval-valued measurable functions as follows(see [4-7,14]).

Definition 3.4. ([4-7, 14]) *Let $(\mathbb{R}^+, \Im(\mathbb{R}^+))$ be a measurable space. A closed set-valued function $F : X \longrightarrow I(\mathbb{R}^+)$ is said to be measurable if for any open set $O \subset \mathbb{R}^+$,*

$$F^{-1}(O) = \{x \in \mathbb{R}^+ | F(x) \cap O \neq \emptyset\} \in \Im(\mathbb{R}^+).$$

Definition 3.5. ([4-7, 14]) *(1) Let F be a closed set-valued function and $A \in \Im(\mathbb{R}^+)$. The Choquet integral of F on A is defined by*

$$(C)\int_A F d\mu = \left\{ (C)\int_A f d\mu \mid f \in S_c(F) \right\},$$

where $S_c(F)$ is the family of measurable selections of F.

(2) A closed set-valued functions F is said to be integrable if $(C)\int F d\mu \neq \emptyset$.

(3) A closed set-valued function F is said to be integrably bounded if there exists a integrable function g such that

$$\parallel F(x) \parallel = sup_{r \in F(x)} |r| \leq g(x) \quad \textit{for all } x \in \mathbb{R}^+.$$

Theorem 3.6. ([14 Theorem 3.16(iii)]) *Let μ be a semi-continuous fuzzy measure. If $F = [f^-, f^+] : \mathbb{R}^+ \longrightarrow I(\mathbb{R}^+)$ is an integrably bounded interval-valued measurable function, then*

$$(C)\int F d\mu = \left[(C)\int f^- d\mu, (C)\int f^+ d\mu\right].$$

Theorem 3.7. *([13])If $\widetilde{f}_1$ and $\widetilde{f}_2$ are fuzzy complex valued measurable functions, then $\widetilde{f}_1 \pm \widetilde{f}_2$ and $\widetilde{f}_1 \cdot \widetilde{f}_2$ are fuzzy complex valued measurable functions, where $\widetilde{f}_1 \pm \widetilde{f}_2 = (Re\widetilde{f}_1 \pm Re\widetilde{f}_2, Im\widetilde{f}_1 \pm Im\widetilde{f}_2)$ and $\widetilde{f}_1 \cdot \widetilde{f}_2 = (Re\widetilde{f}_1 \cdot Re\widetilde{f}_2, Im\widetilde{f}_1 \cdot Im\widetilde{f}_2)$.*

Now, we define the Choquet integral with respect to a fuzzy measure of a fuzzy complex valued function as follows.

Definition 3.8. *Let μ be a semi-continuous fuzzy measure on $(\mathbb{R}^+, \Im(\mathbb{R}^+))$ and $\widetilde{f} = (Re\widetilde{f}, Im\widetilde{f})$ a fuzzy complex valued measurable function.*

(1) For every $A, B \in \Im(\mathbb{R}^+)$, the Choquet integral with respect to μ to $\widetilde{f}$ on $A \times B$ is defined by

$$\left((C)\int_{A\times B} \widetilde{f} d\mu\right)_\lambda = \left((C)\int_A (Re\widetilde{f})_\lambda d\mu, (C)\int_B (Im\widetilde{f})_\lambda d\mu\right)$$

for all $\lambda \in (0,1]$.

(2) If there exists $\widetilde{u} \in FCN(\mathbb{C}^+)$ such that $(\widetilde{u})_\lambda = \left((C)\int_{A\times B} \widetilde{f} d\mu\right)_\lambda$ for all $\lambda \in (0,1]$, then $\widetilde{f}$ is said to be integrable on $A \times B$.

(3) $\widetilde{f}$ is said to be integrably bounded if for any $\lambda \in (0,1]$, $(Re\widetilde{f})_\lambda$ and $(Im\widetilde{f})_\lambda$ are integrably bounded.

Instead of $(C)\int_{\mathbb{R}^+\times\mathbb{R}^+} \widetilde{f} d\mu$, we will write $(C)\int \widetilde{f} d\mu$. If we set $A \times B = \mathbb{R}^+ \times \mathbb{R}^+$, then we denote

$$\left((C)\int \widetilde{f} d\mu\right)_\lambda = \left((C)\int (Re\widetilde{f})_\lambda d\mu, (C)\int (Im\widetilde{f})_\lambda d\mu\right).$$

In order to prove the existence of the Choquet integral of $\widetilde{f}$, we need the Choquet integral of a fuzzy complex valued measurable function to satisfy the following lemma.

Lemma 3.9 ([7,10]). *Let $\{[a_\lambda, b_\lambda] | \lambda \in (0,1]\}$ be a family of nonempty closed intervals in $I(\mathbb{R}^+)$. If (i) for all $0 < \lambda_1 \le \lambda_2$, $[a_{\lambda_1}, b_{\lambda_1}] \supset [a_{\lambda_2}, b_{\lambda_2}]$ and (ii) for any increasing sequence $\{\lambda_k\}$ in $(0,1]$ converging to λ, $[a_\lambda, b_\lambda] = \cap_{k=1}^\infty [a_{\lambda_k}, b_{\lambda_k}]$. Then there exists a unique fuzzy number $\widetilde{u} \in FN(\mathbb{R}^+)$ such that the family $[a_\lambda, b_\lambda]$ represents the λ-level sets of a fuzzy number $\widetilde{u}$.*

Conversely, if $[a_\lambda, b_\lambda]$ are the λ-level sets of a fuzzy number $\widetilde{u} \in FN(\mathbb{R}^+)$, then the conditions (i) and (ii) are satisfied.

From Theorem 3.6 and Definition 3.8, we obtain the following theorem.

Theorem 3.10. *Let μ be a semi-continuous fuzzy measure on $\Im(\mathbb{R}^+)$. If an integrably bounded fuzzy complex valued measurable function $\widetilde{f} = (Re\widetilde{f}, Im\widetilde{f})$ is measurable, then for any $\lambda \in (0, 1]$,*

$$(C)\int (Re\widetilde{f})_\lambda d\mu = \left[(C)\int (Re\widetilde{f})_\lambda^- d\mu, (C)\int (Re\widetilde{f})_\lambda^+ d\mu\right]$$

and

$$(C)\int (Im\widetilde{f})_\lambda d\mu = \left[(C)\int (Im\widetilde{f})_\lambda^- d\mu, (C)\int (Im\widetilde{f})_\lambda^+ d\mu\right].$$

Lemma 3.11. *If $\{\lambda_k\}$ is an increasing sequence in $(0, 1]$ converging to λ and μ is lower semi-continuous, then we have*

$$\lim_{n\to\infty} \mu(\{x|(Re\widetilde{f})_{\lambda_n}^-(x) > \alpha\}) = \mu(\{x|(Re\widetilde{f})_\lambda^-(x) > \alpha\}),$$

$$\lim_{n\to\infty} \mu(\{x|(Re\widetilde{f})_{\lambda_n}^+(x) > \alpha\}) = \mu(\{x|(Re\widetilde{f})_\lambda^+(x) > \alpha\}),$$

$$\lim_{n\to\infty} \mu(\{x|(Im\widetilde{f})_{\lambda_n}^-(x) > \alpha\}) = \mu(\{x|(Im\widetilde{f})_\lambda^-(x) > \alpha\}),$$

and

$$\lim_{n\to\infty} \mu(\{x|(Im\widetilde{f})_{\lambda_n}^+(x) > \alpha\}) = \mu(\{x|(Im\widetilde{f})_\lambda^+(x) > \alpha\}.$$

Under same condition for $\{\lambda_k\}$ in Lemma 3.11, we have

$$\lim_{n\to\infty} \mu(\{x|(Re\widetilde{f})_{\lambda_n}^-(x) > \alpha\}) = \mu(\cap_{n=1}^\infty \{x|(Re\widetilde{f})_{\lambda_n}^-(x) > \alpha\}),$$

$$\lim_{n\to\infty} \mu(\{x|(Re\widetilde{f})_{\lambda_n}^+(x) > \alpha\}) = \mu(\cap_{n=1}^\infty \{x|(Re\widetilde{f})_{\lambda_n}^+(x) > \alpha\}),$$

$$\lim_{n\to\infty} \mu(\{x|(Im\widetilde{f})_{\lambda_n}^-(x) > \alpha\}) = \mu(\cap_{n=1}^\infty \{x|(Im\widetilde{f})_{\lambda_n}^-(x) > \alpha\}),$$

and

$$\lim_{n\to\infty} \mu(\{x|(Im\widetilde{f})_{\lambda_n}^+(x) > \alpha\}) = \mu(\cap_{n=1}^\infty \{x|(Im\widetilde{f})_{\lambda_n}^+(x) > \alpha\}.$$

Thus, by Lemma 3.11, we can obtain the following theorem.

Theorem 3.12. *Let μ be a semi-continuous fuzzy measure. If a fuzzy complex valued function $\widetilde{f}$ is integrably bounded and $\{\lambda_k\}$ is an increasing sequence in $(0,1]$ converging to λ, then we have*
(i) for any $0 < \lambda_1 \leq \lambda_2 \leq 1$,

$$\left((C)\int \widetilde{f}d\mu\right)_{\lambda_1} \supset \left((C)\int \widetilde{f}d\mu\right)_{\lambda_2},$$

and (ii) for any increasing sequence $\{\lambda_k\}$ in $(0,1]$ converging to λ,

$$\left((C)\int Re\widetilde{f}d\mu\right)_{\lambda} = \cap_{k=1}^{\infty}\left((C)\int Re\widetilde{f}d\mu\right)_{\lambda_k}$$

and

$$\left((C)\int Im\widetilde{f}d\mu\right)_{\lambda} = \cap_{k=1}^{\infty}\left((C)\int Im\widetilde{f}d\mu\right)_{\lambda_k}.$$

Proof. (i) Note that $(Re\widetilde{f})_{\lambda_1} = [(Re\widetilde{f})^-_{\lambda_1}, (Re\widetilde{f})^+_{\lambda_1}] \subset (Re\widetilde{f})_{\lambda_2} = [(Re\widetilde{f})^-_{\lambda_2}, (Re\widetilde{f})^+_{\lambda_2}]$ implies

$$(Re\widetilde{f})^-_{\lambda_1} \leq (Re\widetilde{f})^-_{\lambda_2} \text{ and } (Re\widetilde{f})^+_{\lambda_1} \leq (Re\widetilde{f})^+_{\lambda_2}$$

and that $(Im\widetilde{f})_{\lambda_1} = [(Im\widetilde{f})^-_{\lambda_1}, (Im\widetilde{f})^+_{\lambda_1}] \subset (Im\widetilde{f})_{\lambda_2} = [(Im\widetilde{f})^-_{\lambda_2}, (Im\widetilde{f})^+_{\lambda_2}]$ implies

$$(Im\widetilde{f})^-_{\lambda_1} \leq (Im\widetilde{f})^-_{\lambda_2} \text{ and } (Im\widetilde{f})^+_{\lambda_1} \leq (Im\widetilde{f})^+_{\lambda_2}.$$

Thus, by Theorem 2.4(1) and Definition 2.5 (8) and Theorem 3.10, we obtain the followings:

$$\begin{aligned}
&\left((C)\int Re\widetilde{f}d\mu\right)_{\lambda_1} = (C)\int (Re\widetilde{f})_{\lambda_1}d\mu \\
&= \left[(C)\int (Re\widetilde{f})^-_{\lambda_1}d\mu, (C)\int (Re\widetilde{f})^+_{\lambda_1}d\mu\right] \\
&\supset \left[(C)\int (Re\widetilde{f})^-_{\lambda_2}d\mu, (C)\int (Re\widetilde{f})^+_{\lambda_2}d\mu\right] \\
&= (C)\int (Re\widetilde{f})_{\lambda_2}d\mu = \left((C)\int Re\widetilde{f}d\mu\right)_{\lambda_2}.
\end{aligned}$$

Similarly, we obtain the followings.

$$\left((C)\int Im\widetilde{f}d\mu\right)_{\lambda_1} \supset \left((C)\int Im\widetilde{f}d\mu\right)_{\lambda_2}.$$

(ii) Let $\{\lambda_k\}$ be an increasing sequence in $(0,1]$ converging to λ. Then, by Definition 2.5 (4) and the monotone convergence theorem for Lebesgue integral, we can obtain the followings.

$$
\begin{aligned}
&(C)\int (Re\widetilde{f})^-_\lambda d\mu = \int_0^\infty \mu(\{x|(Re\widetilde{f})^-_\lambda(x) > \alpha\})d\alpha \\
&= \int_0^\infty \lim_{n\to\infty} \mu(\{x|(Re\widetilde{f})^-_{\lambda_n}(x) > \alpha\})d\alpha \\
&= \lim_{n\to\infty} \int_0^\infty \mu(\{x|(Re\widetilde{f})^-_{\lambda_n}(x) > \alpha\})d\alpha \\
&= \lim_{n\to\infty} (C)\int (Re\widetilde{f})_{\lambda_n} d\mu = \cap_{n=1}^\infty (C)\int (Re\widetilde{f})^-_{\lambda_n} d\mu.
\end{aligned}
$$

Similarly, we obtain the following three equalities.

$$(C)\int (Re\widetilde{f})^+_\lambda d\mu = \cap_{n=1}^\infty (C)\int (Re\widetilde{f})^+_{\lambda_n} d\mu,$$

$$(C)\int (Im\widetilde{f})^-_\lambda d\mu = \cap_{n=1}^\infty (C)\int (Im\widetilde{f})^-_{\lambda_n} d\mu,$$

and

$$(C)\int (Im\widetilde{f})^+_\lambda d\mu = \cap_{n=1}^\infty (C)\int (Im\widetilde{f})^+_{\lambda_n} d\mu.$$

Thus we have

$$
\begin{aligned}
\left((C)\int Re\widetilde{f} d\mu\right)_\lambda &= \left[(C)\int (Re\widetilde{f})^-_\lambda d\mu, (C)\int (Re\widetilde{f})^+_\lambda d\mu\right] \\
&= \left[\cap_{n=1}^\infty (C)\int (Re\widetilde{f})^-_{\lambda_n} d\mu, \cap_{n=1}^\infty (C)\int (Re\widetilde{f})^+_{\lambda_n} d\mu\right] \\
&= \cap_{n=1}^\infty \left[(C)\int (Re\widetilde{f})^-_{\lambda_n} d\mu, \int (Re\widetilde{f})^+_{\lambda_n} d\mu\right] \\
&= \cap_{n=1}^\infty (C)\int (Re\widetilde{f})_{\lambda_n} d\mu = \cap_{n=1}^\infty \left((C)\int Re\widetilde{f} d\mu\right)_{\lambda_n}.
\end{aligned}
$$

By the same method of the above equality's proof for $Re\widetilde{f}$, we can obtain

$$\left((C)\int Im\widetilde{f} d\mu\right)_\lambda = \cap_{n=1}^\infty \left((C)\int Re\widetilde{f} d\mu\right)_{\lambda_n}.$$

From Theorem 3.12, we can obtain the following Remark which is the existence of the Choquet integral with respect to a fuzzy measure of an integrably bounded fuzzy complex valued measurable function.

Remark 3.13. *By Theorem 3.12 and Lemma 3.11, there exists a fuzzy number $\widetilde{u}, \widetilde{v} \in FN(\mathbb{C}^+)$ such that*

$$(\widetilde{u})_\lambda = \left((C)\int Re\widetilde{f}d\widetilde{\mu}\right)_\lambda \quad \text{and} \quad (\widetilde{v})_\lambda = \left((C)\int Im\widetilde{f}d\widetilde{\mu}\right)_\lambda.$$

for all $\lambda \in (0,1]$. If we put $C = (\widetilde{u}, \widetilde{v})$, then $C \in FCN(\mathbb{C}^+)$ and

$$C_\lambda = (\widetilde{u}_\lambda, \widetilde{v}_\lambda) = \left(\left((C)\int Re\widetilde{f}d\widetilde{\mu}\right)_\lambda, \left((C)\int Im\widetilde{f}d\widetilde{\mu}\right)_\lambda\right) = \left((C)\int \widetilde{f}d\widetilde{\mu}\right)_\lambda.$$

That is, if a fuzzy complex valued function $\widetilde{f}$ is integrably bounded, then $\widetilde{f}$ is integrable.

Thus, we have the following basic properties of Choquet integrals of fuzzy complex valued measurable functions.

Theorem 3.14. *Let μ be a semi-continuous fuzzy measure. The Choquet of integrably bounded fuzzy complex valued measurable functions has the following properties: for any two fuzzy complex valued measurable functions widetildef and widetildeg, then*

(1) if $\widetilde{f} \leq \widetilde{g}$, then $(C)\int \widetilde{f}d\mu \leq (C)\int \widetilde{g}d\mu$,

(2) if we define $(\widetilde{f} \vee \widetilde{g})(z) = \widetilde{f}(z) \vee \widetilde{g}(z)$ and $(\widetilde{f} \wedge \widetilde{g})(z) = \widetilde{f}(z) \wedge \widetilde{g}(z)$ for all $z \in \mathbb{C}^+$, then

$$(C)\int \widetilde{f} \vee \widetilde{g}d\mu \geq (C)\int \widetilde{f}d\mu \vee (C)\int \widetilde{g}d\mu$$

and

$$(C)\int \widetilde{f} \wedge \widetilde{g}d\mu \leq (C)\int \widetilde{f}d\mu \wedge (C)\int \widetilde{g}d\mu$$

§4. Some convergence properties of the fuzzy complex valued Choquet integral

In this section, we introduce some convergence properties of the Choquet integral, for examples, Denneberg's convergence theorem and monotone convergence theorem for Choquet integrals with respect to a fuzzy measure of real-valued measurable functions(see [11,12]).

Definition 4.1 ([10]). *A sequence $\{f_n\}$ of measurable functions is said to converge to f in distribution, in symbols $G - \lim_{n\to\infty} f_n = f$, if*

$$\lim_{n\to\infty} G_{f_n}(r) = G_f(r), \quad e.c.,$$

where "e.c." stands for "except at most countably many values of r".

Theorem 4.2 ([10]). *If $\{f_n\}$ is a sequence of measurable functions that converges to f in distribution and if g and h are integrable functions such that*

$$G_h \leq G_{f_n} \leq G_g \quad e.c., n = 1, 2, \cdots,$$

then f is integrable and

$$\lim_{n\to\infty} (C) \int f_n d\mu = (C) \int f d\mu.$$

Theorem 4.3 ([9]). *(1) If a fuzzy measure μ is semi-continuous and $\{f_n\}$ is an increasing sequence of measurable functions which converges to f, $\mu - a.e.$, then we have*

$$\lim_{n\to\infty} (C) \int f_n d\mu = (C) \int f d\mu,$$

where "P is $\mu - a.e.$" means $\mu(\{x \in \mathbb{R}^+ | P(x) \text{ is not true }\}) = 0$.

(2) If a fuzzy measure μ is upper semi-continuous and $\{f_n\}$ is an decreasing sequence of measurable functions which converges to f, $\mu - a.e.$, and if there exists an integrable function g such that $f_1 \leq g$, then we have

$$\lim_{n\to\infty} (C) \int f_n d\mu = (C) \int f d\mu.$$

We discuss some convergence theorems for Choquet integrals with respect to a fuzzy measure of fuzzy complex valued measurable functions and define the new metric on $FCN(\mathbb{C}^+)$.

Definition 4.4. *A mapping $D : FCN(\mathbb{C}^+) \times FCN(\mathbb{C}^+) \longrightarrow \bar{\mathbb{R}}^+$ is defined by*

$$D(C_1, C_2) = \max\{\triangle(ReC_1, ReC_2), \triangle(ImC_1, ImC_2)\},$$

where $\triangle(\widetilde{u}, \widetilde{v}) = \sup_{\lambda\in(0,1]} d_H(\widetilde{u}_\lambda, \widetilde{v}_\lambda)$ for all $\widetilde{u}, \widetilde{v} \in FN(\mathbb{R}^+)$.

Note that $(FCN(\mathbb{C}^+, D)$ is a metric space. By using this metric D, we define the concept of convergence of a sequence in $(FCN(\mathbb{C}^+, D)$.

Definition 4.5. *A sequence $\{C_n\}$ of fuzzy complex numbers in $FCN(\mathbb{C}^+)$ is said to converge to a fuzzy complex number C in the metric D, in symbols $D - \lim_{n\to\infty} C_n = C$, if*

$$lim_{n\to\infty} D(C_n, C) = 0.$$

From the definition of metric D on $FCN(\mathbb{C}^+)$, we can define the following definitions.

Definition 4.6. *A sequence $\{\widetilde{f}_n\}$ of integrably bounded fuzzy complex valued measurable functions on $FCN(\mathbb{C}^+)$ is said to converges to $\widetilde{f}$ in distribution, in symbols $G-\lim_{n\to\infty}\widetilde{f}_n=\widetilde{f}$ if four sequences $\{(Re\widetilde{f}_n)^-_\lambda\}$, $\{(Re\widetilde{f}_n)^+_\lambda\}$, $\{(Im\widetilde{f}_n)^-_\lambda\}$, and $\{(Im\widetilde{f}_n)^+_\lambda\}$ converge to $\{(Re\widetilde{f})^-_\lambda\}$, $\{(Re\widetilde{f})^+_\lambda\}$, $\{(Im\widetilde{f})^-_\lambda\}$, and $\{(Im\widetilde{f})^+_\lambda\}$ in distribution, respectively.*

By using Definition 4.6 and Theorem 2.5 and the definition of the metric D, we can obtain the following theorem under some sufficient conditions which is Denneberg-type convergence theorem for Choquet integral with respect to a fuzzy measure of integrably bounded fuzzy complex valued functions.

Theorem 4.7. *Assume that a fuzzy complex valued function $\widetilde{f}$ is integrably bounded and μ is a semi-continuous fuzzy measure. If $\{\widetilde{f}_n\}$ is a sequence of fuzzy complex valued measurable functions that converges to $\widetilde{f}$ in distribution, and if g and h are integrable functions such that*

$$h\leq(Re\widetilde{f}_n)^-_\lambda\leq(Re\widetilde{f}_n)^+_\lambda\leq g \ \ \text{and}\ h\leq(Im\widetilde{f}_n)^-_\lambda\leq(Im\widetilde{f}_n)^+_\lambda\leq g$$

for all $\lambda\in(0,1]$ and a.c. for $n=1,2,\cdots$, then $\widetilde{f}$ is integrably bounded and

$$D-\lim_{n\to\infty}(C)\int\widetilde{f}_n d\mu=(C)\int\widetilde{f}d\mu.$$

Proof. Clearly, if we take $z=x+iy\in\mathbb{C}^+$, then we have

$$\|(Re\widetilde{f})_\lambda(x)\|\leq(Re\widetilde{f})^+_\lambda\leq g(x)\text{ and }\|(Im\widetilde{f})_\lambda(x)\|\leq(Im\widetilde{f})^+_\lambda\leq g(x),$$

for all $\lambda\in(0,1]$. Thus, $\widetilde{f}$ is integrably bounded. Since $h\leq(Re\widetilde{f}_n)^-_\lambda\leq(Re\widetilde{f}_n)^+_\lambda\leq g$ and $h\leq(Im\widetilde{f}_n)^-_\lambda\leq(Im\widetilde{f}_n)^+_\lambda\leq g$, $G_h\leq G_{(Re\widetilde{f}_n)^-_\lambda}\leq G_{(Re\widetilde{f}_n)^+_\lambda}\leq G_g$ and $G_h\leq G_{(Im\widetilde{f}_n)^-_\lambda}\leq G_{(Im\widetilde{f}_n)^+_\lambda}\leq G_g$. Then, by Definition 4.6 and Theorem 4.2, we obtain

$$\lim_{n\to\infty}(C)\int(Re\widetilde{f}_n)^-_\lambda d\mu=(C)\int(Re\widetilde{f})^-_\lambda d\mu,$$

$$\lim_{n\to\infty}(C)\int(Re\widetilde{f}_n)^+_\lambda d\mu=(C)\int(Re\widetilde{f})^+_\lambda d\mu,$$

$$\lim_{n\to\infty}(C)\int(Im\widetilde{f}_n)^-_\lambda d\mu=(C)\int(Im\widetilde{f})^-_\lambda d\mu,$$

and

$$\lim_{n\to\infty}(C)\int(Im\widetilde{f}_n)^+_\lambda d\mu=(C)\int(Im\widetilde{f})^+_\lambda d\mu,$$

for all $\lambda \in (0, 1]$. Thus, by the definition of the metric Δ, we have

$$
\begin{aligned}
&\Delta\left((C)\int Re\widetilde{f}_n d\mu, (C)\int Re\widetilde{f} d\mu\right) \\
&= \sup_{\lambda\in(0,1]} d_H\left((C)\int (Re\widetilde{f}_n)_\lambda d\mu, (C)\int (Re\widetilde{f})_\lambda d\mu\right) \\
&= \sup_{\lambda\in(0,1]} \max\left\{|(C)\int (Re\widetilde{f}_n)^-_\lambda d\mu - (C)\int (Re\widetilde{f})^-_\lambda d\mu|,\right. \\
&\qquad\qquad \left.|(C)\int (Re\widetilde{f}_n)^+_\lambda d\mu - (C)\int (Re\widetilde{f})^+_\lambda d\mu|\right\} \\
&\longrightarrow 0,
\end{aligned}
$$

for all $\lambda \in (0, 1]$ as $n \to \infty$ and

$$
\begin{aligned}
&\Delta\left((C)\int Im\widetilde{f}_n d\mu, (C)\int Im\widetilde{f} d\mu\right) \\
&= \sup_{\lambda\in(0,1]} d_H\left((C)\int (Im\widetilde{f}_n)_\lambda d\mu, (C)\int (Im\widetilde{f})_\lambda d\mu\right) \\
&= \sup_{\lambda\in(0,1]} \max\left\{|(C)\int (Im\widetilde{f}_n)^-_\lambda d\mu - (C)\int (Im\widetilde{f})^-_\lambda d\mu|,\right. \\
&\qquad\qquad \left.|(C)\int (Im\widetilde{f}_n)^+_\lambda d\mu - (C)\int (Im\widetilde{f})^+_\lambda d\mu|\right\} \\
&\longrightarrow 0.
\end{aligned}
$$

Therefore, by Definition 4.4, we obtain

$$
\begin{aligned}
&D - \lim_{n\to\infty} (C)\int \widetilde{f}_n d\mu = (C)\int \widetilde{f} d\mu \\
&= \lim_{n\to\infty} \max\left\{\Delta\left((C)\int Re\widetilde{f}_n d\mu, (C)\int Re\widetilde{f} d\mu\right),\right. \\
&\qquad\qquad \left.\Delta\left((C)\int Re\widetilde{f}_n d\mu, (C)\int Re\widetilde{f} d\mu\right)\right\} \\
&= 0.
\end{aligned}
$$

Finally, we can obtain monotone convergence theorems for Choquet integrals with respect to a fuzzy measure of integrably bounded fuzzy complex valued functions as follows.

Theorem 4.8. *Assume that $\widetilde{f}$ is integrably bounded and that a fuzzy measure μ is semi-continuous.*

(1) If $\{\widetilde{f}_n\}$ is an increasing sequence of integrably bounded fuzzy complex valued measurable functions that converges to $\widetilde{f}$ in the metric D,then we have

$$D-\lim_{n\to\infty}(C)\int \widetilde{f}_n d\mu = (C)\int \widetilde{f} d\mu.$$

(2) If $\{\widetilde{f}_n\}$ is a decreasing sequence of integrably bounded fuzzy complex valued measurable functions that converges to $\widetilde{f}$ in the metric D and if there exists an integrabe function g such that

$$(Re\widetilde{f}_n)^-_\lambda \leq (Re\widetilde{f}_n)^+_\lambda \leq g \ \text{ and } (Im\widetilde{f}_n)^-_\lambda \leq (Im\widetilde{f}_n)^+_\lambda \leq g, \quad \mu - a.e.,$$

for all $\lambda \in (0,1]$ and for all $n = 1, 2, \cdots$,, then we have

$$D-\lim_{n\to\infty}(C)\int \widetilde{f}_n d\mu = (C)\int \widetilde{f} d\mu.$$

Proof. Note that if $\{\widetilde{f}_n\}$ is an increasing sequence of fuzzy complex valued measurable functions that converges to $\widetilde{f}$ in the metric D, then four increasing sequences $\{(Re\widetilde{f}_n)^-_\lambda\}$, $\{(Re\widetilde{f}_n)^+_\lambda\}$, $\{(Im\widetilde{f}_n)^-_\lambda\}$, and $\{(Im\widetilde{f}_n)^+_\lambda\}$ converge to $\{(Re\widetilde{f})^-_\lambda\}$, $\{(Re\widetilde{f})^+_\lambda\}$, $\{(Im\widetilde{f})^-_\lambda\}$, and $\{(Im\widetilde{f})^+_\lambda\}$, $\mu - a.e.$, respectively for all $\lambda \in (0,1]$. By Theorem 4.3 (1), we have

$$\lim_{n\to\infty}(C)\int (Re\widetilde{f}_n)^-_\lambda d\mu = (C)\int (Re\widetilde{f})^-_\lambda d\mu,$$

$$\lim_{n\to\infty}(C)\int (Re\widetilde{f}_n)^+_\lambda d\mu = (C)\int (Re\widetilde{f})^+_\lambda d\mu,$$

$$\lim_{n\to\infty}(C)\int (Im\widetilde{f}_n)^-_\lambda d\mu = (C)\int (Im\widetilde{f})^-_\lambda d\mu,$$

and

$$\lim_{n\to\infty}(C)\int (Im\widetilde{f}_n)^+_\lambda d\mu = (C)\int (Im\widetilde{f})^+_\lambda d\mu,$$

for all $\lambda \in (0,1]$. Thus, by Definition 4.4 and the same method of the proof of Theorem 4.7, we have

$$\lim_{n\to\infty} D\left((C)\int \widetilde{f}_n d\mu, (C)\int \widetilde{f} d\mu\right) = 0.$$

(2) The proof is similar to the proof of (1).

§5. Conclusions

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In this paper, by using, we use the Choquet integral with respect to a fuzzy measure instead of the Lebesgue integral with respect to a classical measure, we define the new concept of the Choquet integral with respect to a fuzzy measure of fuzzy complex valued functions in Definition 3.8 and Theorems 3.10, 3.12. In Definitions 4.4, 4.5, 4.6, and Theorems 4.7, 4.8, we investigate the existence of the fuzzy complex valued Choquet integral and some convergence properties of the Choquet integrals of integrably bounded fuzzy complex valued functions.

In the future, we will study a probability measure approach to rank fuzzy complex numbers and the theoretical fundamentals of leaning theory based on fuzzy complex random samples, etc.

Acknowledgement: This paper was supported by Konkuk University in 2013.

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J. COMPUTATIONAL ANALYSIS AND APPLICATIONS, VOL. 15, NO.6, 1085-1095, 2013, COPYRIGHT 2013 EUDOXUS PRESS, LLC

INTUITIONISTIC FUZZY STABILITY OF EULER-LAGRANGE TYPE QUARTIC MAPPINGS

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ABSTRACT. We investigate some stability results and intuitionistic fuzzy continuities concerning the following Euler-Lagrange type quartic functional equation

$$\begin{aligned} &f(ax+y)+f(x+ay)+\frac{1}{2}a(a-1)^2 f(x-y) \\ &= \frac{1}{2}a(a+1)^2 f(x+y)+(a^2-1)^2(f(x)+f(y)) \end{aligned}$$

in intuitionistic fuzzy normed spaces.

1. INTRODUCTION

In 1965, Zadeh [19] introduced the theory of fuzzy sets. After the pioneering work of Zadeh, there has been a great effort to obtain fuzzy analogues of classical theories. It has useful applications in various fields such as population dynamics, chaos control, computer programming, nonlinear dynamical systems, nonlinear operators, etc. Also, many mathematicians considered the fuzzy metric spaces in different view. In particular, In 1984, Katsaras [8] defined a fuzzy norm on a linear space to construct a fuzzy vector topological structure on the space.

Stability problem of a functional equation was first originated by S.M. Ulam [18] concerning the stability of group homomorphisms. It was answered by Hyers [5] on the assumption that the spaces are Banach spaces and generalized by T. Aoki [1] for the stability of the additive mapping involving a sum of powers of p-norms and Th.M. Rassias [16] for the stability of the linear mapping by considering the Cauchy difference to be unbounded.

During the last three decades, several stability problems of a large variety of functional equations have been extensively studied and generalized by a number of authors [3], [4], [6], [16], and [2] and various fuzzy stability results have been studied in [9], [10], [11], and [12].

In particular, J. M. Rassias [15] introduced the Euler-Lagrange type quadratic functional equation

$$f(rx+sy)+f(sx-ry)=(r^2+s^2)[f(x)+f(y)]\,, \tag{1.1}$$

for fixed reals r,s with $r\neq 0\,,s\neq 0$. Also, K-W. Jun and H-M. Kim [7] proved the Hyers-Ulam-Rassias stability of a Euler-Lagrange type cubic mapping as follows:

$$\begin{aligned} &f(ax+y)+f(x+ay) \\ &=(a+1)(a-1)^2[f(x)+f(y)]+a(a+1)f(x+y)\,, \end{aligned} \tag{1.2}$$

2000 *Mathematics Subject Classification.* 39B52.

Key words and phrases. stability problem, Euler-Lagrange functional equation, quartic functional equation, intuitionistic fuzzy stability, intuitionistic fuzzy continuity.

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where $a \neq 0, \pm 1$, for all $x, y \in X$.

In this paper, we investigate the stability problem for the Euler-Lagrange type quartic functional equation as follows:

$$\begin{aligned} f(ax+y) + f(x+ay) + \frac{1}{2}a(a-1)^2 f(x-y) \\ = \frac{1}{2}a(a+1)^2 f(x+y) + (a^2-1)^2(f(x)+f(y)), \end{aligned} \tag{1.3}$$

for fixed integer a with $a \neq 0, \pm 1$.

In fact, $f(x) = x^4$ is a solution of (1.3) by virtue of the identity

$$\begin{aligned} (ax+y)^4 + (x+ay)^4 + \frac{1}{2}a(a-1)^2(x-y)^4 \\ = \frac{1}{2}a(a+1)^2(x+y)^4 + (a^2-1)^2(x^4+y^4). \end{aligned}$$

In this paper, we investigate some stability results and intuitionistic fuzzy continuities concerning the equation (1.3) in intuitionistic fuzzy normed spaces.

Definition 1.1. A binary operation $* : [0,1] \times [0,1] \to [0,1]$ is said to be a continuous t-norm if it satisfies the following conditions:

(1) $*$ is associative and commutative, (2) $*$ is continuous, (3) $a * 1 = a$ for all $a \in [0,1]$, (4) $a * b \leq c * d$ whenever $a \leq c$ and $b \leq d$, for each $a, b, c, d \in [0,1]$.

Definition 1.2. A binary operation $\diamondsuit : [0,1] \times [0,1] \to [0,1]$ is said to be a continuous t-conorm if it satisfies the following conditions:

(1) $\diamondsuit$ is associative and commutative, (2) $\diamondsuit$ is continuous, (3) $a \diamondsuit 0 = a$ for all $a \in [0,1]$, (4) $a \diamondsuit b \leq c \diamondsuit d$ whenever $a \leq c$ and $b \leq d$, for each $a, b, c, d \in [0,1]$.

Saadati and Park introduced the concept of intuitionistic fuzzy normed space; [17].

Definition 1.3. The five-tuple $(X, \mu, \nu, *, \diamondsuit)$ is called an intuitionistic fuzzy normed space(for short, IFNS) if X is a vector space, $*$ is a continuous t-norm, $\diamondsuit$ is continuous t-conorm, and μ and ν are fuzzy sets on $X \times (0,1)$ satisfying the following conditions. For all $x, y \in X$ and $s, t > 0$,

(1) $\mu(x,t) + \nu(x,y) \leq 1$,
(2) $\mu(x,t) > 0$,
(3) $\mu(x,t) = 1$ if and only if $x = 0$,
(4) $\mu(\alpha x, t) = \mu(x, \frac{t}{|\alpha|})$ for each $\alpha \neq 0$,
(5) $\mu(x,t) * \mu(y,s) \leq \mu(x+y, t+s)$,
(6) $\mu(x, \cdot) : (0,\infty) \to [0.1]$ is continuous,
(7) $\lim_{t\to\infty} \mu(x,t) = 1$ and $\lim_{t\to 0} \mu(x,t) = 0$,
(8) $\nu(x,t) < 1$,
(9) $\nu(x,t) = 0$ if and only if $x = 0$,
(10) $\nu(\alpha x, t) = \nu(x, \frac{t}{|\alpha|})$ for each $\alpha \neq 0$,
(11) $\nu(x,t) \diamondsuit \nu(y,s) \geq \nu(x+y, t+s)$,
(12) $\nu(x, \cdot) : (0,\infty) \to [0.1]$ is continuous,
(13) $\lim_{t\to\infty} \nu(x,t) = 0$ and $\lim_{t\to 0} \nu(x,t) = 1$.

In this case (μ, ν) is said to be an intuitionistic fuzzy norm.

Also, they investigated the concepts of convergence and Cauchy sequences in an intuittionistic fuzzy normed space as follows:

Let $(X, \mu, \nu, *, \diamondsuit)$ be an IFNS. A sequence (x_k) is said to be intuittionistic fuzzy convergent to $L \in X$ if $\lim_{k\to\infty} \mu(x_k - L, t) = 1$ and $\lim_{k\to\infty} \nu(x_k - L, t) = 0$, for all $t > 0$. A sequence (x_k) is said to be intuittionistic fuzzy Cauchy sequence if $\lim_{k\to\infty} \mu(x_{k+p} - x_k, t) = 1$ and $\lim_{k\to\infty} \nu(x_{k+p} - x_k, t) = 0$, for all $t > 0$ and $p = 1, 2, \cdots$. Also, $(X, \mu, \nu, *, \diamondsuit)$ is said to be complete if every intuitionistic fuzzy Cauchy sequence in $(X, \mu, \nu, *, \diamondsuit)$ is intuitionistic fuzzy convergent in $(X, \mu, \nu, *, \diamondsuit)$.

2. Intuitionistic Fuzzy Stability

Throughout this section, let X be a linear space and let Y be a intuitionistic fuzzy Banach space. Let a be a fixed integer with $a \neq 0, \pm 1$, For convenience, we use the following abbreviation:

$$D_a f(x,y) := f(ax+y) + f(x+ay) + \frac{1}{2}a(a-1)^2 f(x-y) - \frac{1}{2}a(a+1)^2 f(x+y) - (a^2-1)^2(f(x)+f(y)), \tag{2.1}$$

for all $x, y \in X$.

Theorem 2.1. *Let a be an integer with $a \neq 0, \pm 1$, and let X be a linear space and let (Z, μ', ν') be an intuitionistic fuzzy normed space(IFNS). Let $\phi : X \times X \to Z$ be a function such that for some $0 < \alpha < a^4$*

$$\mu'(\phi(ax,0),t) \geq \mu'(\alpha\phi(x,0),t) \text{ and } \nu'(\phi(ax,0),t) \leq \nu'(\alpha\phi(x,0),t), \tag{2.2}$$

and $\lim_{n\to\infty} \mu'(\phi(a^n x, a^n y), a^{4n}t) = 1$ and $\lim_{n\to\infty} \nu'(\phi(a^n x, a^n y), a^{4n}t) = 0$, for all $x, y \in X$ and $t > 0$. Suppose (Y, μ, ν) is an intuitionistic fuzzy Banach space and $f : X \to Y$ is a ϕ-approximately mapping such that $f(0) = 0$ and

$$\mu\Big(D_a f(x,y),\, t\Big) \geq \mu'(\phi(x,y),t) \tag{2.3}$$

and

$$\nu\Big(D_a f(x,y),\, t\Big) \leq \nu'(\phi(x,y),t) \tag{2.4}$$

for all $t > 0$ and all $x, y \in X$. Then there exists a unique Euler-Lagrange type quartic mapping $Q : X \to Y$ such that

$$\mu(Q(x) - f(x), t) \geq \mu'(\phi(x,0), \frac{1}{2}(a^4 - \alpha)t), \tag{2.5}$$

and

$$\nu(Q(x) - f(x), t) \leq \nu'(\phi(x,0), \frac{1}{2}(a^4 - \alpha)t), \tag{2.6}$$

for all $x \in X$ and all $t > 0$.

Proof. By letting $y = 0$ in inequalities (2.3) and (2.4), we have

$$\mu(f(ax) - a^4 f(x), t) \geq \mu'(\phi(x,0),t) \text{ and } \nu(f(ax) - a^4 f(x), t) \leq \nu'(\phi(x,0),t), \tag{2.7}$$

that is,

$$\mu(\frac{f(ax)}{a^4} - f(x), \frac{t}{a^4}) \geq \mu'(\phi(x,0),t), \tag{2.8}$$

and

$$\nu(\frac{f(ax)}{a^4} - f(x), \frac{t}{a^4}) \leq \nu'(\phi(x,0),t), \tag{2.9}$$

for all $x \in X$ and $t > 0$. For each $n \in \mathbb{N}$, letting $x = a^n x$ in inequalities (2.8) and (2.9), we get

$$\begin{aligned}\mu\Big(a^{4n}(\frac{f(a^{n+1}x)}{a^{4(n+1)}} - \frac{f(a^n x)}{a^{4n}}), \frac{t}{a^4}\Big) &\geq \mu'(\phi(a^n x, 0), t)\\ \nu\Big(a^{4n}(\frac{f(a^{n+1}x)}{a^{4(n+1)}} - \frac{f(a^n x)}{a^{4n}}), \frac{t}{a^4}\Big) &\leq \nu'(\phi(a^n x, 0), t)\,.\end{aligned}$$

By using the inequality (2.2), these previous inequalities imply that

$$\begin{aligned}\mu\Big(\frac{f(a^{n+1}x)}{a^{4(n+1)}} - \frac{f(a^n x)}{a^{4n}}, \frac{t}{a^{4(n+1)}}\Big) &\geq \mu'(\phi(a^n x, 0), t) = \mu'(\phi(x, 0), \frac{t}{\alpha^n})\\ \nu\Big(\frac{f(a^{n+1}x)}{a^{4(n+1)}} - \frac{f(a^n x)}{a^{4n}}, \frac{t}{a^{4(n+1)}}\Big) &\leq \nu'(\phi(x, 0), \frac{t}{\alpha^n})\,,\end{aligned}$$

for all $x \in X$, $t > 0$, and $n \geq 0$. Now, switching t by $\alpha^n t$ in the previous inequalities, we have

$$\begin{aligned}\mu\Big(\frac{f(a^{n+1}x)}{a^{4(n+1)}} - \frac{f(a^n x)}{a^{4n}}, \frac{1}{a^4}(\frac{\alpha}{a^4})^n t\Big) &\geq \mu'(\phi(x, 0), t)\,,\\ \nu\Big(\frac{f(a^{n+1}x)}{a^{4(n+1)}} - \frac{f(a^n x)}{a^{4n}}, \frac{1}{a^4}(\frac{\alpha}{a^4})^n t\Big) &\leq \nu'(\phi(x, 0), t)\,,\end{aligned}$$

for all $x \in X$, $t > 0$, and $n \geq 0$. Then

$$\begin{aligned}\mu\Big(\frac{f(a^n x)}{a^{4n}} - f(x), \sum_{k=0}^{n-1}\frac{1}{a^4}(\frac{\alpha}{a^4})^k t\Big) &= \mu\Big(\sum_{k=0}^{n-1}(\frac{f(a^{k+1}x)}{a^{4(k+1)}} - \frac{f(a^k x)}{a^{4k}}), \sum_{k=0}^{n-1}\frac{1}{a^4}(\frac{\alpha}{a^4})^k t\Big)\\ &\geq \prod_{k=0}^{n-1}\mu\Big(\frac{f(a^{k+1}x)}{a^{4(k+1)}} - \frac{f(a^k x)}{a^{4k}}, \frac{1}{a^4}(\frac{\alpha}{a^4})^k t\Big) \geq \mu'(\phi(x, 0), t)\,,\end{aligned}$$

and

$$\begin{aligned}\nu\Big(\frac{f(a^n x)}{a^{4n}} - f(x), \sum_{k=0}^{n-1}\frac{1}{a^4}(\frac{\alpha}{a^4})^k t\Big) &= \nu\Big(\sum_{k=0}^{n-1}(\frac{f(a^{k+1}x)}{a^{4(k+1)}} - \frac{f(a^k x)}{a^{4k}}), \sum_{k=0}^{n-1}\frac{1}{a^4}(\frac{\alpha}{a^4})^k t\Big)\\ &\leq \coprod_{k=0}^{n-1}\nu\Big(\frac{f(a^{k+1}x)}{a^{4(k+1)}} - \frac{f(a^k x)}{a^{4k}}, \frac{1}{a^4}(\frac{\alpha}{a^4})^k t\Big) \leq \nu'(\phi(x, 0), t)\,,\end{aligned}$$

for all $x \in X$, $t > 0$, and $n \geq 1$, where $\prod_{j=1}^n a_j = a_1 * \cdots * a_n$ and $\coprod_{j=1}^n a_j = a_1 \diamondsuit \cdots \diamondsuit a_n$. For any integer $s \geq 0$, replacing x with $a^s x$ in the previous inequalities, we have

$$\mu\Big(a^{4s}[\frac{f(a^{n+s}x)}{a^{4(n+s)}} - \frac{f(a^s x)}{a^{4s}}], \sum_{k=0}^{n-1}\frac{1}{a^4}(\frac{\alpha}{a^4})^k t\Big) \geq \mu'(\phi(a^s x, 0), t)\,,$$

and

$$\nu\Big(a^{4s}[\frac{f(a^{n+s}x)}{a^{4(n+s)}} - \frac{f(a^s x)}{a^{4s}}], \sum_{k=0}^{n-1}\frac{1}{a^4}(\frac{\alpha}{a^4})^k t\Big) \leq \nu'(\phi(a^s x, 0), t)\,,$$

that is,

$$\mu\Big(\frac{f(a^{n+s}x)}{a^{4(n+s)}} - \frac{f(a^s x)}{a^{4s}}, \frac{1}{a^{4s}}\sum_{k=0}^{n-1}\frac{1}{a^4}(\frac{\alpha}{a^4})^k t\Big) \geq \mu'(\phi(x, 0), \frac{t}{\alpha^s})\,,$$

and

$$\mu\Big(\frac{f(a^{n+s}x)}{a^{4(n+s)}} - \frac{f(a^s x)}{a^{4s}}, \frac{1}{a^{4s}}\sum_{k=0}^{n-1}\frac{1}{a^4}(\frac{\alpha}{a^4})^k t\Big) \geq \mu'(\phi(x,0), \frac{t}{\alpha^s}),$$

for all $x \in X$, $t > 0$, $n \geq 0$, and $s \geq 0$. Now, switching t by $\alpha^s t$, we get

$$\begin{aligned} &\mu\Big(\frac{f(a^{n+s}x)}{a^{4(n+s)}} - \frac{f(a^s x)}{a^{4s}}, \frac{1}{a^{4s}}\sum_{k=0}^{n-1}\frac{\alpha^s}{a^4}(\frac{\alpha}{a^4})^k t\Big) \\ = \;&\mu\Big(\frac{f(a^{n+s}x)}{a^{4(n+s)}} - \frac{f(a^s x)}{a^{4s}}, \sum_{k=s}^{n+s-1}\frac{1}{a^4}(\frac{\alpha}{a^4})^k t\Big) \geq \mu'(\phi(x,0), t), \end{aligned}$$

and

$$\begin{aligned} &\nu\Big(\frac{f(a^{n+s}x)}{a^{4(n+s)}} - \frac{f(a^s x)}{a^{4s}}, \frac{1}{a^{4s}}\sum_{k=0}^{n-1}\frac{\alpha^s}{a^4}(\frac{\alpha}{a^4})^k t\Big) \\ = \;&\nu\Big(\frac{f(a^{n+s}x)}{a^{4(n+s)}} - \frac{f(a^s x)}{a^{4s}}, \sum_{k=s}^{n+s-1}\frac{1}{a^4}(\frac{\alpha}{a^4})^k t\Big) \leq \nu'(\phi(x,0), t), \end{aligned}$$

for all $x \in X$, $t > 0$, $n \geq 0$, and $s \geq 0$. By putting t with $\frac{t}{\sum_{k=s}^{n+s-1}\frac{1}{a^4}(\frac{\alpha}{a^4})^k}$, we have

$$\mu\Big(\frac{f(a^{n+s}x)}{a^{4(n+s)}} - \frac{f(a^s x)}{a^{4s}}, t\Big) \geq \mu'(\phi(x,0), \frac{t}{\sum_{k=s}^{n+s-1}\frac{1}{a^4}(\frac{\alpha}{a^4})^k}), \tag{2.10}$$

and

$$\nu\Big(\frac{f(a^{n+s}x)}{a^{4(n+s)}} - \frac{f(a^s x)}{a^{4s}}, t\Big) \leq \nu'(\phi(x,0), \frac{t}{\sum_{k=s}^{n+s-1}\frac{1}{a^4}(\frac{\alpha}{a^4})^k}), \tag{2.11}$$

for all $x \in X$, $t > 0$, $n \geq 0$, and $s \geq 0$. Since $0 < \alpha < a^4$, $\sum_{k=0}^{\infty}\left(\frac{\alpha}{a^4}\right)^k < \infty$. Hence $\lim_{t\to\infty}\mu'(\phi(x,0), \frac{t}{\sum_{k=s}^{n+s-1}\frac{1}{a^4}(\frac{\alpha}{a^4})^k}) = 1$, and $\lim_{t\to\infty}\nu'(\phi(x,0), \frac{t}{\sum_{k=s}^{n+s-1}\frac{1}{a^4}(\frac{\alpha}{a^4})^k}) = 0$. Let $\varepsilon > 0$ and $\delta > 0$. Then there exists a $t_0 > 0$ such that $\mu'(\phi(x,0), \frac{t_0}{\sum_{k=s}^{n+s-1}\frac{1}{a^4}(\frac{\alpha}{a^4})^k}) \geq 1-\varepsilon$, and $\nu'(\phi(x,0), \frac{t_0}{\sum_{k=s}^{n+s-1}\frac{1}{a^4}(\frac{\alpha}{a^4})^k}) \leq \varepsilon$. Since $\sum_{k=0}^{\infty}\frac{t_0}{a^4}\left(\frac{\alpha}{a^4}\right)^k < \infty$, there exists a $n_0 \in \mathbb{N}$ such that $\sum_{k=s}^{n+s-1}\frac{t_0}{a^4}\left(\frac{\alpha}{a^4}\right)^k < \delta$, for all $n+s > s \geq n_0$. Hence the sequence $\left(\frac{f(a^n x)}{a^{4n}}\right)$ is a Cauchy sequence in (Y, μ, ν). Since (Y, μ, ν) is a Banach space, the sequence $\left(\frac{f(a^n x)}{a^{4n}}\right)$ converges. Hence we can define a function $Q : X \to Y$ by

$$Q(x) = \lim_{n\to\infty}\frac{f(a^n x)}{a^{4n}},$$

for all $x \in X$. Letting $s = 0$ in the inequalities (2.10) and (2.11), we have

$$\mu\Big(\frac{f(a^n x)}{a^{4n}} - f(x), t\Big) \geq \mu'(\phi(x,0), \frac{t}{\sum_{k=0}^{n-1}\frac{1}{a^4}(\frac{\alpha}{a^4})^k}),$$

and

$$\nu\Big(\frac{f(a^n x)}{a^{4n}} - f(x), t\Big) \leq \nu'(\phi(x,0), \frac{t}{\sum_{k=0}^{n-1}\frac{1}{a^4}(\frac{\alpha}{a^4})^k}),$$

for all $t > 0$ and $n > 0$. Hence we have

$$\begin{aligned}\mu(Q(x)-f(x),t) &= \mu(Q(x)-\frac{f(a^nx)}{a^{4n}}+\frac{f(a^nx)}{a^{4n}}-f(x),\frac{t}{2}+\frac{t}{2})\\ &\geq \mu\Big(Q(x)-\frac{f(a^nx)}{a^{4n}},\frac{t}{2}\Big)*\mu\Big(\frac{f(a^nx)}{a^{4n}}-f(x),\frac{t}{2}\Big)\\ &\geq \mu'\Big(\phi(x,0),\frac{1}{2}\frac{t}{\sum_{k=0}^{n-1}\frac{1}{a^4}\left(\frac{\alpha}{a^4}\right)^k}\Big),\end{aligned}$$

and

$$\begin{aligned}\nu(Q(x)-f(x),t) &= \nu(Q(x)-\frac{f(a^nx)}{a^{4n}}+\frac{f(a^nx)}{a^{4n}}-f(x),\frac{t}{2}+\frac{t}{2})\\ &\leq \nu\Big(Q(x)-\frac{f(a^nx)}{a^{4n}},\frac{t}{2}\Big)*\nu\Big(\frac{f(a^nx)}{a^{4n}}-f(x),\frac{t}{2}\Big)\\ &\leq \nu'\Big(\phi(x,0),\frac{1}{2}\frac{t}{\sum_{k=0}^{n-1}\frac{1}{a^4}\left(\frac{\alpha}{a^4}\right)^k}\Big),\end{aligned}$$

that is,

$$\mu(Q(x)-f(x),t)\geq \mu'(\phi(x,0),\frac{1}{2}(a^4-\alpha)t),$$

and

$$\nu(Q(x)-f(x),t)\leq \nu'(\phi(x,0),\frac{1}{2}(a^4-\alpha)t),$$

as $n\to\infty$. Respectively, replacing x , y , and t by a^nx , a^ny , and $a^{4n}t$ in inequalities (2.3) and (2.4), we have

$$\mu\Big(\frac{D_af(a^nx,a^ny)}{a^{4n}},t\Big)\geq \mu'(\phi(a^nx,a^ny),a^{4n}t),$$

and

$$\nu\Big(\frac{D_af(a^nx,a^ny)}{a^{4n}},t\Big)\leq \nu'(\phi(a^nx,a^ny),a^{4n}t),$$

for all $x\in X$, $t>0$, and $n\in\mathbb{N}$. Since $\lim_{n\to\infty}\mu'(\phi(a^nx,a^ny),a^{4n}t)=1$ and $\lim_{n\to\infty}\nu'(\phi(a^nx,a^ny),a^{4n}t)=0$, the mapping $Q:X\to Y$ satisfies the equation (1.3), that is, it is the Euler-Lagrange type quartic mapping. It only remains to show that the mapping $Q:X\to Y$ is unique. Assume $Q':X\to Y$ is another Euler-Lagrange type quartic mapping satisfying the inequalities (2.5) and (2.6). It is easy to show that $Q(a^nx)=a^{4n}Q(x)$ and $Q'(a^nx)=a^{4n}Q'(x)$, for all $n\in\mathbb{N}$.

$$\begin{aligned}&\mu\Big(Q(x)-Q'(x),t\Big)=\mu\Big(\frac{Q(a^nx)}{a^{4n}}-\frac{Q'(a^nx)}{a^{4n}},t\Big)\\ \geq\ &\mu\Big(\frac{Q(a^nx)}{a^{4n}}-\frac{f(a^nx)}{a^{4n}},\frac{t}{2}\Big)*\mu\Big(\frac{f(a^nx)}{a^{4n}}-\frac{Q'(a^nx)}{a^{4n}},\frac{t}{2}\Big)\\ \geq\ &\mu'\Big(\phi(a^nx,0),\frac{a^{4n}(a^4-\alpha)}{4}t\Big)\geq \mu'\Big(\phi(x,0),\frac{a^4-\alpha}{4}\Big(\frac{a^4}{\alpha}\Big)^nt\Big),\end{aligned}$$

and

$$\nu\Big(Q(x)-Q'(x),t\Big)\leq \nu'\Big(\phi(x,0),\frac{a^4-\alpha}{4}\Big(\frac{a^4}{\alpha}\Big)^nt\Big),$$

for all $x \in X$ and all $t > 0$. Since $\lim_{n\to\infty}\left(\frac{a^4}{\alpha}\right)^n = \infty$,

$$\lim_{n\to\infty}\mu'\Big(\phi(x,0),\frac{a^4-\alpha}{4}\Big(\frac{a^4}{\alpha}\Big)^n t\Big) = 1 \text{ and } \lim_{n\to\infty}\nu'\Big(\phi(x,0),\frac{a^4-\alpha}{4}\Big(\frac{a^4}{\alpha}\Big)^n t\Big) = 0\,.$$

Hence

$$\mu\Big(Q(x)-Q'(x),t\Big) = 1 \text{ and } \nu\Big(Q(x)-Q'(x),t\Big) = 0\,,$$

for all $x \in X$ and all $t > 0$. We may conclude that $Q(x) = Q'(x)$, for all $x \in X$, that is, the mapping $Q : X \to Y$ is unique, as desired. □

Theorem 2.2. *Let a be an integer with $a \neq 0, \pm 1$, and let X be a linear space and let (Z,μ',ν') be an intuitionistic fuzzy normed space(IFNS). Let $\phi : X \times X \to Z$ be a function such that for some $\alpha > a^4$*

$$\mu'(\phi(\frac{x}{a},0),t) \geq \mu'(\phi(x,0),\alpha t) \text{ and } \nu'(\phi(\frac{x}{a},0),t) \leq \nu'(\phi(x,0),\alpha t)\,, \tag{2.12}$$

and $\lim_{n\to\infty}\mu'(\phi(a^{-n}x,a^{-n}y),a^{-4n}t) = 1$ and $\lim_{n\to\infty}\nu'(\phi(a^{-n}x,a^{-n}y),a^{-4n}t) = 0$, for all $x,y \in X$ and $t > 0$. Suppose (Y,μ,ν) is an intuitionistic fuzzy Banach space and $f : X \to Y$ is a ϕ-approximately mapping with $f(0) = 0$ satisfying the inequalities (2.3) and (2.4). Then there exists a unique Euler-Lagrange type quartic mapping $Q : X \to Y$ such that

$$\mu(Q(x)-f(x),t) \geq \mu'(\phi(x,0),\frac{(\alpha-a^4)}{2}t)\,, \tag{2.13}$$

and

$$\nu(Q(x)-f(x),t) \leq \nu'(\phi(x,0),\frac{(\alpha-a^4)}{2}t)\,, \tag{2.14}$$

for all $x \in X$ and all $t > 0$.

Proof. Letting $x = \frac{x}{a}$ in inequalities (2.7) of proof of Theorem 2.1, we have

$$\mu(f(x)-a^4f(\frac{x}{a}),t) \geq \mu'(\phi(x,0),\alpha t) \text{ and } \nu(f(x)-a^4f(\frac{x}{a}),t) \leq \nu'(\phi(x,0),\alpha t)\,, \tag{2.15}$$

for all $x \in X$ and $t > 0$. Similar to the proof of Theorem 2.1, we can deduce

$$\mu\Big(a^{4(n+s)}f(a^{-(n+s)}x) - a^{4s}f(a^{-s}x),t\Big) \geq \mu'(\phi(x,0),\frac{t}{\sum_{k=s}^{n+s-1}\frac{a^{4k}}{\alpha^{k+1}}})\,, \tag{2.16}$$

and

$$\nu\Big(a^{4(n+s)}f(a^{-(n+s)}x) - a^{4s}f(a^{-s}x),t\Big) \leq \nu'(\phi(x,0),\frac{t}{\sum_{k=s}^{n+s-1}\frac{a^{4k}}{\alpha^{k+1}}})\,, \tag{2.17}$$

for all $x \in X$, $t > 0$, and $s \geq 0$ and $n \geq 0$. Since $\alpha > a^4$ and $\sum_{k=0}^{\infty}\left(\frac{a^4}{\alpha}\right)^k < \infty$, the Cauchy criterion for convergence in IFNS implies that $\left(a^{4n}f(\frac{x}{a^n})\right)$ is a Cauchy sequence in the Banach space (Y,μ,ν). A function $Q : X \to Y$ by

$$Q(x) = \lim_{n\to\infty} a^{4n}f(\frac{x}{a^n})\,,$$

for all $x \in X$. Also, letting $s = 0$ and taking $n \to \infty$ in the inequalities (2.16) and (2.17), we have the inequalities (2.13) and (2.14). The remains follows from the proof of Theorem 2.1. □

3. Intutionistic fuzzy continuity

Throughout this section, let $(X, ||\cdot||)$ be a normed space. In [13], they defined and studied the intuitionistic fuzzy continuity. In this section, we will investigate interesting results of continuous approximately Euler-Lagrange type quartic mappings. Before proceeding the proof, we will state the definition of intuitionistic fuzzy continuity as follows.

Definition 3.1. [[14, Definition 3.1]] Let $f : \mathbb{R} \to X$ be a function, where $\mathbb{R}$ is endowed with the Euclidean topology and X is an intuitionistic fuzzy normed space equipped with intuitionistic fuzzy norm $(\mu\,,\nu)$. Them f is called intuitionistic fuzzy continuous at a point $s_0 \in \mathbb{R}$ if for all $\varepsilon > 0$ and all $0 < \alpha < 1$ there exists $\delta > 0$ such that for each s with $0 < |s - s_0| < \delta$

$$\mu(f(sx) - f(s_0x), \varepsilon) \geq \alpha \text{ and } \nu(f(sx) - f(s_0x), \varepsilon) \leq 1 - \alpha\,.$$

Theorem 3.2. *Let a be an integer with $a \neq 0, \pm 1$,and let X be a normed space and (Z, μ', ν') be an IFNS. Let (Y, μ, ν) be an intuitionistic fuzzy Banach space and $f : X \to Y$ be a (p, q)-approximately mapping with $f(0) = 0$ in the sense that for some p, q and some $z_0 \in Z$*

$$\mu\Big(D_af(x, y),\, t\Big) \geq \mu'((||x||^p + ||y||^q)z_0, t) \tag{3.1}$$

and

$$\nu\Big(D_af(x, y),\, t\Big) \leq \nu'((||x||^p + ||y||^q)z_0, t) \tag{3.2}$$

for all $t > 0$ and all $x, y \in X$. If $p,\, q < 4$, then there exists a unique Euler-Lagrange type quartic mapping $Q : X \to Y$ such that

$$\mu(C(x) - f(x), t) \geq \mu'(||x||^p z_0, \frac{1}{2}(a^4 - |a|^p)t)\,, \tag{3.3}$$

and

$$\nu(C(x) - f(x), t) \leq \nu'(||x||^p z_0, \frac{m^2}{2}(a^4 - |a|^p)t)\,, \tag{3.4}$$

for all $x \in X$ and all $t > 0$. Furthermore, if for some $x \in X$ and all $n \in \mathbb{N}$, the mapping $g : \mathbb{R} \to Y$ defined by $g(s) = f(a^n sx)$ is intuitionistic fuzzy continuous, then the mappings $s \mapsto Q(sx)$ from $\mathbb{R}$ to Y is intuitionistic fuzzy continuous.

Proof. For $x, y \in X$ and for some $z_0 \in Z$, we define the function $\phi : X \times X \to Z$ by $\phi(x, y) = (||x||^p + ||y||^q)z_0$ in Theorem 2.1. Since $p < 4$, we have $\alpha = |a|^p < a^4$. Hence Theorem 2.1 implies the existence and uniqueness of the Euler-Lagrange type quartic mapping $Q : X \to Y$ satisfying inequalities (3.3) and (3.4). Now, we will show the intuitionistic fuzzy continuity. For each $x \in X$, $t \in \mathbb{R}$ and $n \in \mathbb{N}$, we have

$$\begin{aligned}\mu(Q(x) - \frac{f(a^n x)}{a^{4n}}, t) &= \mu(\frac{Q(a^n x)}{a^{4n}} - \frac{f(a^n x)}{a^{4n}}, t) = \mu(Q(a^n x) - f(a^n x), a^{4n} t)\\ &\geq \mu'(|a|^{np}||x||^p z_0, \frac{a^{4n}}{2}(a^4 - |a|^p)t) = \mu'(||x||^p z_0, \frac{a^{4n}(a^4 - |a|^p)}{2\cdot |a|^{np}} t)\,,\end{aligned}$$

and

$$\nu(Q(x) - \frac{f(a^n x)}{a^{4n}}, t) \leq \nu'(||x||^p z_0, \frac{a^{4n}(a^4 - |a|^p)}{2\cdot |a|^{np}} t)\,.$$

Let $x \in X$ and $s_0 \in \mathbb{R}$ be fixed and $\varepsilon > 0$ and $0 < \beta < 1$ be given. For all $s \in \mathbb{R}$ with $|s - s_0| < 1$, by replacing x with sx in the previous inequalities,

$$\begin{aligned}\mu(Q(sx) - \frac{f(a^n sx)}{a^{4n}}, t) &\geq \mu'(||sx||^p z_0, \frac{a^{4n}(a^4 - |a|^p)}{2 \cdot |a|^{np}} t) \\ &\geq \mu'(||x||^p z_0, \frac{a^{4n}(a^4 - |a|^p)}{2 \cdot |a|^{np}(1 + |s_0|)^p} t),\end{aligned}$$

and

$$\nu(Q(sx) - \frac{f(a^n sx)}{a^{4n}}, t) \leq \nu'(||x||^p z_0, \frac{a^{4n}(a^4 - |a|^p)}{2 \cdot |a|^{np}(1 + |s_0|)^p} t).$$

Since $a^p < a^4$, we have

$$\lim_{n\to\infty} \frac{a^{4n}(a^4 - |a|^p)}{2 \cdot |a|^{np}(1 + |s_0|)^p} = \infty.$$

Hence there exists $n_0 \in \mathbb{N}$ such that

$$\mu\Big(Q(sx) - \frac{f(a^{n_0} sx)}{a^{4n_0}}, \frac{\varepsilon}{3}\Big) \geq \beta \text{ and } \nu\Big(Q(sx) - \frac{f(a^{n_0} sx)}{a^{4n_0}}, \frac{\varepsilon}{3}\Big) \leq 1 - \beta,$$

for all $|s - s_0| < 1$ and $s \in \mathbb{R}$. The intuitionistic fuzzy continuity of the mapping $t \mapsto f(a^{n_0} tx)$ implies that there exists $\delta < 1$ such that for each s with $0 < |s - s_0| < \delta$, we get

$$\mu(\frac{f(a^{n_0} sx)}{a^{4n_0}} - \frac{f(a^{n_0} s_0 x)}{a^{4n_0}}, \frac{\varepsilon}{3}) \geq \beta \text{ and } \nu(\frac{f(a^{n_0} sx)}{a^{4n_0}} - \frac{f(a^{n_0} s_0 x)}{a^{4n_0}}, \frac{\varepsilon}{3}) \leq 1 - \beta.$$

Thus

$$\begin{aligned}\mu(Q(sx) - Q(s_0 x), \varepsilon) \geq \mu(Q(sx) - \frac{f(a^{n_0} sx)}{a^{4n_0}}, \frac{\varepsilon}{3}) * \\ \mu(\frac{f(a^{n_0} sx)}{a^{4n_0}} - \frac{f(a^{n_0} s_0 x)}{a^{4n_0}}, \frac{\varepsilon}{3}) * \mu(Q(s_0 x) - \frac{f(a^{n_0} s_0 x)}{a^{4n_0}}, \frac{\varepsilon}{3}) \geq \beta\end{aligned}$$

and

$$\nu(Q(sx) - Q(s_0 x), \varepsilon) \leq 1 - \beta,$$

for all $s \in \mathbb{R}$ with $0 < |s - s_0| < \delta$, that is, the mapping $s \mapsto Q(sx)$ is intuitionistic fuzzy continuous. $\square$

Theorem 3.3. *Let a be an integer with $a \neq 0, \pm 1$, and let X be a normed space and (Z, μ', ν') be an IFNS. Let (Y, μ, ν) be an intuitionistic fuzzy Banach space and $f : X \to Y$ be a (p, q)-approximately mapping with $f(0) = 0$ satisfying (3.1) and (3.2) for some p, q and some $z_0 \in Z$. If $p, q > 4$, then there exists a unique Euler-Lagrange type quartic mapping $Q : X \to Y$ such that*

$$\mu(Q(x) - f(x), t) \geq \mu'(||x||^p z_0, \frac{1}{2}(|a|^p - a^4)t), \tag{3.5}$$

and

$$\nu(Q(x) - f(x), t) \leq \nu'(||x||^p z_0, \frac{1}{2}(|a|^p - a^4)t), \tag{3.6}$$

for all $x \in X$ and all $t > 0$. Furthermore, if for some $x \in X$ and all $n \in \mathbb{N}$, the mapping $g : \mathbb{R} \to Y$ defined by $g(s) = f(a^n sx)$ is intuitionistic fuzzy continuous, then the mappings $s \mapsto Q(sx)$ from $\mathbb{R}$ to Y is intuitionistic fuzzy continuous.

Proof. Similar to the proof of Theorem 3.2, we may define the function $\phi : X \times X \to Z$ by $\phi(x, y) = (||x||^p + ||y||^q)z_0$. Then we have

$$\mu'(\phi(\frac{x}{2}, 0), t) = \mu'(||x||^p z_0, |a|^p t) \text{ and } \nu'(\phi(\frac{x}{2}, 0), t) = \nu'(||x||^p z_0, |a|^p t)\,,$$

for all $x \in X$ and all $t > 0$. Since $p > 4$, we have $\alpha = |a|^p > a^4$. Hence Theorem 2.2 implies the existence and uniqueness of the Euler-Lagrange type quartic mapping $Q : X \to Y$ satisfying inequalities (3.5) and (3.6). The remains follow from the proof of Theorem 3.2. □

Acknowledgement

This work was supported by the University of Incheon Research Grant in 2011.

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J. COMPUTATIONAL ANALYSIS AND APPLICATIONS, VOL. 15, NO.6, 1096-1103, 2013, COPYRIGHT 2013 EUDOXUS PRESS, LLC

STABILITY FOR AN n-DIMENSIONAL FUNCTIONAL EQUATION OF QUADRATIC-ADDITIVE TYPE WITH THE FIXED POINT APPROACH

ICK-SOON CHANG AND YANG-HI LEE

ABSTRACT. In this paper, we investigate the stability of a functional equation

$$\sum_{1\le i,j\le n, i\neq j}[f(x_i+x_j)+f(x_i-x_j)]-(n-1)\sum_{j=1}^{n}f(2x_j)=0$$

by using the fixed point methd in the sense of Cădariu and Radu.

1. Introduction and peliminaries

It is of interest to consider the concept of stability for a functional equation arising when we replace the functional equation by an inequality which acts as a perturbation of the equation. The study of stability problems had been formulated by Ulam [17] during a talk: *under what condition does there exists a homomorphism near an approximate homomorphism?* In the following year, Hyers [6] was answered affirmatively the question of Ulam for Banach spaces, which states that *if* $\varepsilon \ge 0$ *and* $f : \mathcal{X} \to \mathcal{Y}$ *is a mapping with* $\mathcal{X}$ *a normed space,* $\mathcal{Y}$ *a Banach space such that*

$$||f(x+y)-f(x)-f(y)|| \le \varepsilon \tag{1.1}$$

for all $x, y \in \mathcal{X}$, *then there exists a unique additive mapping* $T : \mathcal{X} \to \mathcal{Y}$ *such that*

$$||f(x)-T(x)|| \le \varepsilon$$

for all $x \in \mathcal{X}$. A generalized version of the theorem of Hyers for approximately additive mappings was given by Aoki [1] and for the theorem of Hyers for approximately linear mappings it was presented by Rassias [15] by considering the case when the inequality (1.1) is unbounded. Since then, more generalizations and applications of the stability to a number of functional equations and mappings have been investigated (for example, [5], [7], [8]-[14]).

In this very active area, almost all subsequent proofs have used the method of Hyers [6]. On the other hand, Cădariu and Radu [2] observed that the existence of the solution for a functional equation and the estimation of the difference with the given mapping can be obtained from the fixed point alternative. This method is called a *fixed point method*. In particular, they [3, 4] applied this method to prove the stability theorems of the *additive functional equation*

$$f(x+y)-f(x)-f(y)=0. \tag{1.2}$$

and the *quadratic functional equation*

$$f(x+y)+f(x-y)-2f(x)-2f(y)=0. \tag{1.3}$$

Note that the *additive mapping* $f_1(x)=ax$ and *quadratic mapping* $f_2(x)=ax^2$ are solution of the functional equations (1.2) and (1.3).

We now take account of the functional equation:

$$\sum_{1\le i,j\le n, i\neq j}[f(x_i+x_j)+f(x_i-x_j)]-(n-1)\sum_{j=1}^{n}f(2x_j)=0. \tag{1.4}$$

2000 *Mathematics Subject Classification*: 39B52.

Keywords and phrases: stability: fixed point method: n-dimensional quadratic-additive type functional equation. The first author was supported by Basic Science Research Program through the National Research Foundation of Korea (NRF) funded by the Ministry of Education, Science and Technology (No. 2012-0002410).

Hence, throughout this paper, we promise that the equation (1.4) is said to be an *quadratic-additive type functional equation* and every solution of the equation (1.4) is called a *quadratic-additive mapping.*

In this paper, we will deal with the stability of the functional equation (1.4) by using the fixed point method: The stability of (1.4) can be obtained by handling the odd part and the even part of the given mapping. But, in violation of this processing, we can take the desired solution at once instead of splitting the given mapping into two parts.

Here and now, we recall the following result of the fixed point theory by Margolis and Diaz:

Theorem 1.1. (*The alternative of fixed point*) ([14] or [16]) *Suppose that a complete generalized metric space (X, d), which means that the metric d may assume infinite values, and a strictly contractive mapping $J : X \to X$ with the Lipschitz constant $0 < L < 1$ are given. Then, for each given element $x \in X$, either*

$$d(J^n x, J^{n+1}x) = +\infty, \ \forall n \in \mathbb{N} \cup \{0\},$$

or there exists a nonnegative integer k such that:

(1) *$d(J^n x, J^{n+1}x) < +\infty$ for all $n \geq k$;*
(2) *the sequence $\{J^n x\}$ is convergent to a fixed point y^* of J;*
(3) *y^* is the unique fixed point of J in $Y := \{y \in X, d(J^k x, y) < +\infty\}$;*
(4) *$d(y, y^*) \leq (1/(1-L))d(y, Jy)$ for all $y \in Y$.*

2. A general fixed point method

Throughout this paper, let V be a real or complex linear space and Y a Banach space. For a given mapping $f : V \to Y$, we use the following abbreviation

$$f(x_1, x_2, \cdots, x_n) := \sum_{1 \leq i,j \leq n, i \neq j} [f(x_i + x_j) + f(x_i - x_j)] - (n-1)\sum_{j=1}^{n} f(2x_j)$$

for all $x_1, x_2, \cdots, x_n \in V$. Now we can prove some stability results of the functional equation (1.4).

Theorem 2.1. *Let $\varphi : V^n \to [0, \infty)$ be a given function with $\varphi(x, 0, \cdots, 0) = \varphi(-x, 0, \cdots, 0)$ for all $x \in V$. Suppose that the mapping $f : V \to Y$ satisfies*

$$\|Df(x_1, x_2, \cdots, x_n)\| \leq \varphi(x_1, x_2, \cdots, x_n) \tag{2.1}$$

for all $x_1, x_2, \cdots, x_n \in V$ with $f(0) = 0$. If there exists a constant $0 < L < 1$ such that φ has the property

$$\varphi(2x_1, 2x_2, \cdots, 2x_n) \leq 2L\varphi(x_1, x_2, \cdots, x_n) \tag{2.2}$$

for all $x_1, x_2, \cdots, x_n \in V$, then there exists a unique quadratic-additive mapping $F : V \to Y$ such that

$$\|f(x) - F(x)\| \leq \frac{\varphi(x, 0, \cdots, 0)}{2(n-1)(1-L)} \tag{2.3}$$

for all $x \in V$. In particular, F is given by

$$F(x) = \lim_{m\to\infty} \left(\frac{f(2^m x) + f(-2^m x)}{2 \cdot 2^{2m}} + \frac{f(2^m x) - f(-2^m x)}{2 \cdot 2^m} \right) \tag{2.4}$$

for all $x \in V$.

Proof. Consider the set

$$S := \{g : g : V \to Y, \ g(0) = 0\}$$

and introduce a generalized metric on S by

$$d(g, h) = \inf \left\{ K \in \mathbb{R} \big| \|g(x) - h(x)\| \leq K\varphi(x, 0, \cdots, 0) \ \text{ for all } \ x \in V \right\}.$$

It is easy to see that (S, d) is a generalized complete metric space.

Now we define a mapping $J : S \to S$ by

$$Jg(x) := \frac{g(2x) - g(-2x)}{4} + \frac{g(2x) + g(-2x)}{8}$$

for all $x \in V$. Note that

$$J^m g(x) = \frac{g(2^m x) - g(-2^n x)}{2^{m+1}} + \frac{g(2^m x) + g(-2^m x)}{2 \cdot 4^m}$$

for all $m \in \mathbb{N}$ and $x \in V$. Let $g, h \in S$ and let $K \in [0, \infty]$ be an arbitrary constant with $d(g, h) \le K$. From the definition of d, we have

$$\begin{aligned}\|Jg(x) - Jh(x)\| \le& \left\| \frac{3(g(2x) - h(2x))}{8} \right\| + \left\| \frac{g(-2x) - h(-2x)}{8} \right\| \\ \le& \frac{K\varphi(2x, 0, \cdots, 0)}{2} \\ \le& KL\varphi(x, 0, \cdots, 0)\end{aligned}$$

for all $x \in V$, which implies that $d(Jg, Jh) \le Ld(g, h)$ for any $g, h \in S$. That is, J is a strictly contractive self-mapping of S with the Lipschitz constant L. Moreover, by (2.1), we see that

$$\|f(x) - Jf(x)\| = \frac{1}{n-1} \left\| \frac{3}{8} Df(x, 0, \cdots, 0) - \frac{1}{8} Df(-x, 0, \cdots, 0) \right\| \le \frac{\varphi(x, 0, \cdots, 0)}{2(n-1)}$$

for all $x \in V$. It means that $d(f, Jf) \le \frac{1}{2(n-1)} < \infty$ by the definition of d. Therefore, according to Theorem 1.1, the sequence $\{J^m f\}$ converges to the unique fixed point $F : V \to Y$ of J in the set $T = \{g \in S : d(f, g) < \infty\}$, which is given by (2.4) for all $x \in V$.

Observe that

$$d(f, F) \le \frac{1}{1-L} d(f, Jf) \le \frac{1}{2(n-1)(1-L)},$$

which implies (2.3).

By the definition of F, together with (2.1) and (2.4) that

$$\begin{aligned}&\|DF(x_1, x_2, \cdots, x_n)\| \\ =& \lim_{m\to\infty} \left\| \frac{Df(2^m x_1, 2^m x_2, \cdots, 2^m x_n) - Df(-2^m x_1, -2^m x_2, \cdots, -2^m x_n)}{2^{m+1}} \right. \\ &+ \left. \frac{Df(2^m x_1, 2^m x_2, \cdots, 2^m x_n) + Df(-2^m x_1, -2^m x_2, \cdots, -2^m x_n)}{2 \cdot 4^m} \right\| \\ \le& \lim_{m\to\infty} \frac{2^m + 1}{2 \cdot 4^m} \left(\varphi(2^m x_1, \cdots, 2^m x_n) + \varphi(-2^m x_1, \cdots, -2^m x_n) \right) \\ =& 0\end{aligned}$$

for all $x_1, x_2, \cdots, x_n \in V$, which completes the proof. $\square$

We continue our investigation with the following result.

Theorem 2.2. *Let $\varphi : V^n \to [0, \infty)$ with $\varphi(x, 0, \cdots, 0) = \varphi(-x, 0, \cdots, 0)$ for all $x, y \in V$. Suppose that $f : V \to Y$ satisfies the inequality (2.1) for all $x_1, x_2, \cdots, x_n \in V$ with $f(0) = 0$. If there exists $0 < L < 1$ such that the mapping φ has the property*

$$\varphi(2x_1, 2x_2, \cdots, 2x_n) \ge 4\varphi(x_1, x_2, \cdots, x_n) \tag{2.5}$$

for all $x_1, x_2, \cdots, x_n \in V$, then there exists a unique quadratic-additive mapping $F : V \to Y$ such that

$$\|f(x) - F(x)\| \le \frac{L\varphi(x, 0, \cdots, 0)}{4(n-1)(1-L)} \tag{2.6}$$

for all $x \in V$. In particular, F is represented by

$$F(x) = \lim_{m\to\infty} \left(2^{m-1} \left(f\left(\frac{x}{2^m}\right) - f\left(-\frac{x}{2^m}\right) \right) + \frac{4^m}{2} \left(f\left(\frac{x}{2^m}\right) + f\left(-\frac{x}{2^m}\right) \right) \right) \tag{2.7}$$

for all $x \in V$.

Proof. Let the set (S, d) be as in the proof of Theorem 2.1. Now we consider the mapping $J : S \to S$ defined by

$$Jg(x) := g\left(\frac{x}{2}\right) - g\left(-\frac{x}{2}\right) + 2\left(g\left(\frac{x}{2}\right) + g\left(-\frac{x}{2}\right)\right)$$

for all $g \in S$ and $x \in V$. We remark that

$$J^m g(x) = 2^{m-1} \left(g\left(\frac{x}{2^m}\right) - g\left(-\frac{x}{2^m}\right) \right) + \frac{4^m}{2} \left(g\left(\frac{x}{2^m}\right) + g\left(-\frac{x}{2^m}\right) \right)$$

and $J^0g(x) = g(x)$ for all $x \in V$. Let $g, h \in S$ and let $K \in [0, \infty]$ be an arbitrary constant with $d(g,h) \le K$. From the definition of d, we have

$$\begin{aligned}\|Jg(x) - Jh(x)\| &\le 3\left\|g\left(\frac{x}{2}\right) - h\left(\frac{x}{2}\right)\right\| + \left\|g\left(-\frac{x}{2}\right) - h\left(-\frac{x}{2}\right)\right\| \\ &\le 4K\varphi\left(\frac{x}{2}, 0, \cdots, 0\right) \le LK\varphi(x, 0, \cdots, 0)\end{aligned}$$

for all $x \in V$. So we find that J is a strictly contractive self-mapping of S with the Lipschitz constant L. Also, we see that

$$\begin{aligned}\|f(x) - Jf(x)\| =& \frac{1}{n-1}\left\|-Df\left(\frac{x}{2}, 0, \cdots, 0\right)\right\| \\ \le& \frac{1}{n-1}\varphi\left(\frac{x}{2}, 0, \cdots, 0\right) \le \frac{L}{4(n-1)}\varphi(x, 0, \cdots, 0)\end{aligned}$$

for all $x \in V$, which implies that $d(f, Jf) \le \frac{L}{4(n-1)} < \infty$. Therefore, according to Theorem 1.1, the sequence $\{J^m f\}$ converges to the unique fixed point F of J in the set $T := \{g \in S : d(f,g) < \infty\}$, which is represented by (2.7).

Since

$$d(f, F) \le \frac{1}{1-L} d(f, Jf) \le \frac{L}{4(n-1)(1-L)}$$

the inequality (2.6) holds.

From the definition of F, (2.1), and (2.5), we have

$$\begin{aligned}&\|DF(x_1, x_2, \cdots, x_n)\| \\ &= \lim_{m\to\infty}\left\|2^{m-1}\left(Df\left(\frac{x_1}{2^m}, \frac{x_2}{2^m}, \cdots, \frac{x_n}{2^m}\right) - Df\left(-\frac{x_1}{2^m}, -\frac{x_2}{2^m}, \cdots, -\frac{x_n}{2^m}\right)\right)\right. \\ &\quad \left. + \frac{4^m}{2}\left(Df\left(\frac{x_1}{2^m}, \frac{x_2}{2^m}, \cdots, \frac{x_n}{2^m}\right) + Df\left(-\frac{x_1}{2^m}, -\frac{x_2}{2^m}, \cdots, -\frac{x_n}{2^m}\right)\right)\right\| \\ &\le \lim_{m\to\infty}\left(2^{m-1} + \frac{4^m}{2}\right)\left(\varphi\left(\frac{x_1}{2^m}, \frac{x_2}{2^m}, \cdots, \frac{x_n}{2^m}\right) + \varphi\left(-\frac{x_1}{2^m}, -\frac{x_2}{2^m}, \cdots, -\frac{x_n}{2^m}\right)\right) \\ &= 0\end{aligned}$$

for all $x_1, x_2, \cdots, x_n \in V$. This completes the proof. □

3. Applications

For the sake of convenience, given a mapping $f : V \to Y$, we set

$$Af(x,y) := f(x+y) - f(x) - f(y)$$

for all $x, y \in V$.

Corollary 3.1. *Let $f_k : V \to Y, k = 1, 2,$ be mappings for which there exist functions $\phi_k : V^2 \to [0, \infty), k = 1, 2,$ such that*

$$\|Af_k(x,y)\| \le \phi_k(x,y) \tag{3.1}$$

for all $x, y \in V$. If $f_k(0) = 0$, $\phi_k(0) = 0$, $\phi_k(x,y) = \phi_k(-x,-y), k = 1,2,$ for all $x, y \in V$ and there exists $0 < L < 1$ such that

$$\phi_1(2x, 2y) \le 2L\phi_1(x,y), \tag{3.2}$$

$$4\phi_2(x,y) \le L\phi_2(2x, 2y) \tag{3.3}$$

for all $x, y \in V$, then there exist unique additive mappings $F_k : V \to Y, k = 1, 2,$ such that

$$\|f_1(x) - F_1(x)\| \le \frac{\phi_1(x,x) + \phi_1(x,-x)}{2(1-L)}, \tag{3.4}$$

$$\|f_2(x) - F_2(x)\| \le \frac{L(\phi_k(x,x) + \phi_k(x,-x))}{4(1-L)} \tag{3.5}$$

for all $x \in V$. In particular, the mappings F_1, F_2 are represented by

$$F_1(x) = \lim_{m\to\infty} \frac{f_1(2^m x)}{2^m}, \tag{3.6}$$

$$F_2(x) = \lim_{m\to\infty} 2^m f_2\left(\frac{x}{2^m}\right) \tag{3.7}$$

for all $x \in V$.

Proof. Now we note that

$$Df_k(x_1, x_2, \cdots, x_n) = \sum_{1\le i,j\le n, i\ne j} A_k(x_i + x_j, x_i - x_j)$$

for all $x_1, x_2, \cdots, x_n \in V$ and $k = 1, 2$. Put

$$\varphi_k(x_1, x_2, \cdots, x_n) := \sum_{1\le i,j\le n, i\ne j} \phi_k(x_i + x_j, x_i - x_j)$$

for all $x_1, x_2, \cdots, x_n \in V$ and $k = 1, 2$, then

$$\|Df_k(x_1, x_2, \cdots, x_n)\| \le \varphi_k(x_1, x_2, \cdots, x_n)$$

and φ_1 and φ_2 satisfies (2.2) and (2.5), respectively. According to Theorem 2.1, there exists a unique mapping $F_1 : V \to Y$ satisfying (3.4), which is represented by (2.4).

Observe that, by (3.1) and (3.2),

$$\begin{aligned}
\lim_{m\to\infty} \left\| \frac{f_1(2^m x) + f_1(-2^m x)}{2^{m+1}} \right\| &= \lim_{m\to\infty} \frac{1}{2^{m+1}} \|Af_1(2^m x, -2^m x)\| \\
&\le \lim_{m\to\infty} \frac{1}{2^{m+1}} \phi_1(2^m x, -2^m x) \\
&\le \lim_{m\to\infty} \frac{L^m}{2} \phi_1(x, -x) = 0
\end{aligned}$$

as well as

$$\lim_{m\to\infty} \left\| \frac{f_1(2^m x) + f_1(-2^m x)}{2\cdot 4^m} \right\| \le \lim_{m\to\infty} \frac{2^m L^m}{2\cdot 4^m} \phi_1(x, -x) = 0$$

for all $x \in V$. From these and (2.4), we get (3.6).

Moreover, we have

$$\left\| \frac{Af_1(2^m x, 2^m y)}{2^m} \right\| \le \frac{\phi_1(2^m x, 2^m y)}{2^m} \le L^m \phi_1(x, y)$$

for all $x, y \in V$. Taking the limit as $m \to \infty$ in the above inequality, we get $AF_1(x, y) = 0$ for all $x, y \in V$. On the other hand, according to Theorem 2.4, there exists a unique mapping $F_2 : V \to Y$ satisfying (3.5), which is represented by (2.7).

Observe that, by (3.1) and (3.3),

$$\begin{aligned}
\lim_{m\to\infty} 2^{2m-1} \left\| f_2\left(\frac{x}{2^m}\right) + f_2\left(\frac{-x}{2^m}\right) \right\| &= \lim_{m\to\infty} 2^{2m-1} \left\| Af_2\left(\frac{x}{2^m}, -\frac{x}{2^m}\right) \right\| \\
&\le \lim_{m\to\infty} 2^{2m-1} \phi_2\left(\frac{x}{2^m}, -\frac{x}{2^m}\right) \\
&\le \lim_{m\to\infty} \frac{L^m}{2} \phi_2(x, -x) = 0
\end{aligned}$$

as well as

$$\lim_{m\to\infty} 2^{m-1} \left\| f_2\left(\frac{x}{2^m}\right) + f_2\left(\frac{-x}{2^m}\right) \right\| \le \lim_{m\to\infty} \frac{L^m}{2^{m+1}} \phi_2(x, -x) = 0$$

for all $x \in V$. From these and (2.5), we get (3.10). Moreover, we have

$$\left\| 2^m Af_2\left(\frac{x}{2^m}, \frac{y}{2^m}\right) \right\| \le 2^m \phi_2\left(\frac{x}{2^m}, \frac{y}{2^m}\right) \le \frac{L^m}{2^m} \phi_2(x, y)$$

for all $x, y \in V$. Taking the limit as $m \to \infty$ in the above inequality, we get

$$AF_2(x, y) = 0$$

for all $x, y \in V$. This completes the proof. □

Corollary 3.2. *Let $f_k : V \to Y, k = 1, 2,$ be mappings for which there exist functions $\phi_k : V^2 \to [0, \infty), k = 1, 2,$ such that*

$$\|Qf_k(x, y)\| \le \phi_k(x, y)$$

for all $x, y \in V$. If $f_k(0) = 0, \phi_k(0) = 0, \phi_k(x, y) = \phi_i(-x, -y), k = 1, 2,$ for all $x, y \in V$, and there exists $0 < L < 1$ such that the mapping ϕ_1 satisfies (3.2) and ϕ_2 satisfies (3.3) for all $x, y \in V$, then there exist unique quadratic mappings $F_k : V \to Y, k = 1, 2,$ such that

$$\|f_1(x) - F_1(x)\| \le \frac{\phi_1(x, x) + \phi_1(x, -x) + 3\phi_1(x, 0) + \phi_1(0, -x)}{4(1 - L)}, \tag{3.8}$$

$$\|f_2(x) - F_2(x)\| \le \frac{L(\phi_2(x, x) + \phi_2(x, -x) + 3\phi_2(x, 0) + \phi_2(0, -x))}{8(1 - L)} \tag{3.9}$$

for all $x \in V$. In particular, the mappings $F_k, k = 1, 2,$ are represented by

$$F_1(x) = \lim_{m\to\infty} \frac{f_1(2^m x)}{4^m}, \tag{3.10}$$

$$F_2(x) = \lim_{m\to\infty} 4^m f_2\left(\frac{x}{2^m}\right) \tag{3.11}$$

for all $x \in V$.

Proof. Notice that

$$\begin{aligned} Df_k(x_1, \cdots, x_n) =& \frac{1}{2} \sum_{1\le i,j\le n, i\ne j} (Q_k(x_i, x_j) + Q_k(x_i, -x_j)) \\ &- \frac{n-1}{2} \sum_{i=1}^{n} (Q_k(x_i, x_i) + Q_k(x_i, -x_i)) \end{aligned}$$

for all $x_1, x_2, \cdots, x_n \in V$ and $k = 1, 2$. Put

$$\begin{aligned} \varphi_k(x_1, \cdots, x_n) =& \frac{1}{2} \sum_{1\le i,j\le n, i\ne j} (\phi_k(x_i, x_j) + \phi_k(x_i, -x_j)) \\ &+ \frac{n-1}{2} \sum_{i=1}^{n} (\phi_k(x_i, x_i) + \phi_k(x_i, -x_i)) \end{aligned}$$

for all $x_1, x_2, \cdots, x_n \in V$ and $k = 1, 2$, then φ_1 satisfies (2.2) and φ_2 satisfies (2.5). Moreover,

$$\|Df_k(x_1, x_2, \cdots, x_n)\| \le \varphi_k(x_1, x_2, \cdots, x_n)$$

for all $x_1, x_2, \cdots, x_n \in V$ and $k = 1, 2$. According to Theorem 2.1, there exists a unique mapping $F_1 : V \to Y$ satisfying (3.8) which is represented by (2.4).

Observe that

$$\begin{aligned} \lim_{m\to\infty} \left\| \frac{f_1(2^m x) - f_1(-2^m x)}{2^{m+1}} \right\| &= \lim_{m\to\infty} \frac{1}{2^{m+1}} \left\| Qf_1(0, -2^{m-1}x) \right\| \\ &\le \lim_{m\to\infty} \frac{1}{2^{m+1}} \phi_1(0, -2^{m-1}x) \\ &\le \lim_{m\to\infty} \frac{L^m}{2} \left(\phi_1 0, -\frac{x}{2}\right) \\ &=0 \end{aligned}$$

as well as

$$\lim_{m\to\infty} \left\| \frac{f_1(2^m x) - f_1(-2^m x)}{2 \cdot 4^m} \right\| \le \lim_{m\to\infty} \frac{L^m}{2^{m+1}} \phi_1\left(0, -\frac{x}{2}\right) = 0$$

for all $x \in V$. From these and (2.4), we get (3.10) for all $x \in V$.

Moreover, we have

$$\left\| \frac{Qf_1(2^m x, 2^m y)}{4^m} \right\| \le \frac{\phi_1(2^m x, 2^m y)}{4^m} \le \frac{L^m}{2^m} \phi_1(x, y)$$

for all $x, y \in V$. Taking the limit as $m \to \infty$ in the above inequality, we get $QF_1(x, y) = 0$ for all $x, y \in V$.

On the other hand, according to Theorem 2.2, there exists a unique mapping $F_2 : V \to Y$ satisfying (3.9) which is represented by (2.7).

Observe that

$$\begin{aligned} 4^m \left\| f_2\left(\frac{x}{2^m}\right) - f_2\left(-\frac{x}{2^m}\right) \right\| &= 4^m \left\| Qf_2\left(0, -\frac{x}{2^{m+1}}\right) \right\| \\ &\le 4^m \phi_2\left(0, -\frac{x}{2^{m+1}}\right) \\ &\le L^m \phi_2\left(0, -\frac{x}{2}\right) \end{aligned}$$

for all $x \in V$. It leads us to get

$$\lim_{m\to\infty} 4^m \left(f_2\left(\frac{x}{2^m}\right) - f_2\left(-\frac{x}{2^m}\right)\right) = 0, \ \lim_{m\to\infty} 2^m \left(f_2\left(\frac{x}{2^m}\right) - f_2\left(-\frac{x}{2^m}\right)\right) = 0$$

for all $x \in V$. From these and (2.7), we obtain (3.11).

Moreover, we have

$$\left\| 4^m Qf_2\left(\frac{x}{2^m}, \frac{y}{2^m}\right) \right\| \le 4^m \phi_2\left(\frac{x}{2^m}, \frac{y}{2^m}\right) \le L^m \phi_2(x, y)$$

for all $x, y \in V$. Taking the limit as $m \to \infty$ in the above inequality, we get $QF_2(x, y) = 0$ for all $x, y \in V$, which completes the proof. □

Corollary 3.3. *Let X be a normed space and Y a Banach space. Suppose that the mapping $f : X \to Y$ satisfies the inequality*

$$\|Df(x_1, x_2, \cdots, x_n)\| \le \|x_1\|^p + \|x_2\|^p + \cdots + \|x_n\|^p$$

for all $x_1, x_2, \cdots, x_n \in X$, where $p \in (0, 1) \cup (2, \infty)$. Then there exists a unique quadratic-additive mapping $F : X \to Y$ such that

$$\|f(x) - F(x)\| \le \begin{cases} \frac{\|x\|^p}{(n-1)(2^p-4)} & \text{if } p > 2, \\ \frac{\|x\|^p}{(n-1)(2-2^p)} & \text{if } p < 1 \end{cases}$$

for all $x \in X$.

Proof. This follows from Theorem 2.1 and Theorem 2.2, by putting

$$\varphi(x_1, x_2, \cdots, x_n) := \|x_1\|^p + \|x_2\|^p + \cdots + \|x_n\|^p$$

for all $x_1, x_2, \cdots, x_n \in X$ with $L = 2^{p-1} < 1$ if $0 < p < 1$ and $L = 2^{2-p} < 1$ if $p > 2$. □

Corollary 3.4. *Let X be a normd space and Y a Banach space. Suppose that the mapping $f : X \to Y$ satisfies the inequality*

$$\|Df(x_1, x_2, \cdots, x_n)\| \le \theta \|x_1\|^{p_1} \|x_2\|^{p_2} \cdots \|x_n\|^{p_n}$$

for all $x_1, x_2, \cdots, x_n \in X$, where $\theta \ge 0$ and $p_1, p_2, \cdots, p_n$, $p_1 + p_2 + \cdots + p_n \in (0, 1) \cup (2, \infty)$. Then f is itself a quadratic additive mapping.

Proof. This follows from Theorem 2.1 and Theorem 2.2, by letting

$$\varphi(x_1, x_2, \cdots, x_n) := \|x_1\|^{p_1} \|x_2\|^{p_2} \cdots \|x_n\|^{p_n}$$

for all $x_1, x_2, \cdots, x_n \in X$ with $L = 2^{p-1} < 1$ if $0 < p < 1$ and $L = 2^{2-p} < 1$ if $p > 2$. □

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J. COMPUTATIONAL ANALYSIS AND APPLICATIONS, VOL. 15, NO.6, 1104-1109, 2013, COPYRIGHT 2013 EUDOXUS PRESS, LLC

An identity of the q-Euler polynomials associated with the p-adic q-integrals on $\mathbb{Z}_p$

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Abstract : We introduce the q-Euler numbers and polynomials. By using these numbers and polynomials, we investigate the alternating sums of powers of consecutive integers. By applying the symmetry of the fermionic p-adic q-integral on $\mathbb{Z}_p$, we give recurrence identities the q-Euler polynomials and q-analogue of alternating sums of powers of consecutive integers.

2000 Mathematics Subject Classification - 11B68, 11S40, 11S80.

Key words : Euler numbers and polynomials, q-Euler numbers and polynomials, alternating sums.

1. Introduction

Throughout this paper, we always make use of the following notations: $\mathbb{C}$ denotes the set of complex numbers, $\mathbb{Z}_p$ denotes the ring of p-adic rational integers, $\mathbb{Q}_p$ denotes the field of p-adic rational numbers, and $\mathbb{C}_p$ denotes the completion of algebraic closure of $\mathbb{Q}_p$.

Let ν_p be the normalized exponential valuation of $\mathbb{C}_p$ with $|p|_p = p^{-\nu_p(p)} = p^{-1}$. When one talks of q-extension, q is considered in many ways such as an indeterminate, a complex number $q \in \mathbb{C}$, or p-adic number $q \in \mathbb{C}_p$. If $q \in \mathbb{C}$ one normally assume that $|q| < 1$. If $q \in \mathbb{C}_p$, we normally assume that $|q-1|_p < p^{-\frac{1}{p-1}}$ so that $q^x = \exp(x \log q)$ for $|x|_p \leq 1$. For

$$g \in UD(\mathbb{Z}_p) = \{g | g : \mathbb{Z}_p \to \mathbb{C}_p \text{ is uniformly differentiable function}\},$$

the fermionic p-adic q-integral on $\mathbb{Z}_p$ is defined by Kim as follows:

$$I_{-q}(g) = \int_{\mathbb{Z}_p} g(x) d\mu_{-q}(x) = \lim_{N\to\infty} \frac{[2]_q}{1+q^{p^N}} \sum_{x=0}^{p^N-1} g(x)(-q)^x, \text{ see [1-10] }. \tag{1.1}$$

If we take $g_1(x) = g(x+1)$ in (1.1), then we easily see that

$$qI_{-q}(g_1) + I_{-q}(g) = [2]_q g(0). \tag{1.2}$$

For $q \in \mathbb{C}_p$ with $|1-q|_p \leq 1$, the q-Euler polynomials $\widetilde{E}_{n,q}(x)$ are defined by

$$\widetilde{F}_q(x,t) = \sum_{n=0}^{\infty} \widetilde{E}_{n,q}(x) \frac{t^n}{n!} = \frac{[2]_q}{qe^t+1} e^{xt}. \tag{1.3}$$

The q-Euler numbers $\widetilde{E}_{n,q}$ are defined by the generating function:

$$\widetilde{F}_q(t) = \sum_{n=0}^{\infty} \widetilde{E}_{n,q} \frac{t^n}{n!} = \frac{[2]_q}{qe^t+1}. \tag{1.4}$$

The following elementary properties of the q-Euler numbers $\widetilde{E}_{n,q}$ and polynomials $\widetilde{E}_{n,q}(x)$ are readily derived form (1.1), (1.2), (1.3) and (1.4). We, therefore, choose to omit details involved.

Theorem 1(Witt formula). For $q \in \mathbb{C}_p$ with $|1-q|_p < 1$, we have

$$\widetilde{E}_{n,q} = \int_{\mathbb{Z}_p} x^n d\mu_{-q}(x),$$

$$\widetilde{E}_{n,q}(x) = \int_{\mathbb{Z}_p} (x+y)^n d\mu_{-q}(y).$$

Theorem 2. For any positive integer n, we have

$$\widetilde{E}_{n,q}(x) = \sum_{k=0}^{n} \binom{n}{k} \widetilde{E}_{k,q} x^{n-k}.$$

2. The alternating sums of powers of consecutive q-integers

Let q be a complex number with $|q| < 1$. By using (1.3), we give the alternating sums of powers of consecutive q-integers as follows:

$$\sum_{n=0}^{\infty} \widetilde{E}_{n,q} \frac{t^n}{n!} = \frac{[2]_q}{qe^t + 1} = [2]_q \sum_{n=0}^{\infty} (-1)^n q^n e^{nt}.$$

From the above, we obtain

$$-\sum_{n=0}^{\infty} (-1)^n q^n e^{(n+k)t} + \sum_{n=0}^{\infty} (-1)^{n-k} q^{n-k} e^{nt} = \sum_{n=0}^{k-1} (-1)^{n-k} q^{n-k} e^{nt}.$$

Thus, we have

$$\begin{aligned} &-[2]_q \sum_{n=0}^{\infty} (-1)^n q^n e^{(n+k)t} + [2]_q (-1)^{-k} q^{-k} \sum_{n=0}^{\infty} (-1)^n q^n e^{nt} \\ &= [2]_q (-1)^{-k} q^{-k} \sum_{n=0}^{k-1} (-1)^n q^n e^{nt}. \end{aligned} \tag{2.1}$$

By using (1.3)and (1.4), and (2.1), we obtain

$$-\sum_{j=0}^{\infty} \widetilde{E}_{j,q}(k) \frac{t^j}{j!} + (-1)^{-k} q^{-k} \sum_{j=0}^{\infty} \widetilde{E}_{j,q} \frac{t^j}{j!} = [2]_q \sum_{j=0}^{\infty} \left((-1)^{-k} q^{-k} \sum_{n=0}^{k-1} (-1)^n q^n n^j \right) \frac{t^j}{j!}.$$

By comparing coefficients of $\frac{t^j}{j!}$ in the above equation, we obtain

$$\sum_{n=0}^{k-1} (-1)^n q^n n^j = \frac{(-1)^{k+1} q^k \widetilde{E}_{j,q}(k) + \widetilde{E}_{j,q}}{[2]_q}.$$

By using the above equation we arrive at the following theorem:

Theorem 3. Let k be a positive integer and $q \in \mathbb{C}$ with $|q| < 1$. Then we obtain

$$\widetilde{T}_{j,q}(k-1) = \sum_{n=0}^{k-1} (-1)^n q^n n^j = \frac{(-1)^{k+1} q^k \widetilde{E}_{j,q}(k) + \widetilde{E}_{j,q}}{[2]_q}.$$

Remark 4. Let k be a positive integer and $q \in \mathbb{C}$ with $|q| < 1$. Then we have

$$\lim_{q \to 1} \widetilde{T}_{j,q}(k-1) = \sum_{n=0}^{k-1} (-1)^n n^j = \frac{(-1)^{k+1} E_j(k) + E_j}{2},$$

where $E_j(x)$ and E_j denote the Euler polynomials and Euler numbers, respectively.

Next, we assume that $q \in \mathbb{C}_p$. We obtain recurrence identities the q-Euler polynomials and the q-analogue of alternating sums of powers of consecutive integers.

By using (1.1), we have

$$q^n I_{-q}(g_n) + (-1)^{n-1} I_{-q}(g) = [2]_q \sum_{l=0}^{n-1} (-1)^{n-1-l} q^l g(l),$$

where $g_n(x) = g(x+n)$. If n is odd from the above, we obtain

$$q^n I_{-q}(g_n) + I_{-q}(g) = [2]_q \sum_{l=0}^{n-1} (-1)^{n-1-l} q^l g(l) \text{ (cf. [1-5]).} \tag{2.2}$$

It will be more convenient to write (2.2) as the equivalent integral form

$$q^n \int_{\mathbb{Z}_p} g(x+n) d\mu_{-q}(x) + \int_{\mathbb{Z}_p} g(x) d\mu_{-q}(x) = [2]_q \sum_{k=0}^{n-1} (-1)^k q^k g(k). \tag{2.3}$$

Substituting $g(x) = e^{xt}$ into the above, we obtain

$$q^n \int_{\mathbb{Z}_p} e^{(x+n)t} d\mu_{-q}(x) + \int_{\mathbb{Z}_p} e^{xt} d\mu_{-q}(x) = [2]_q \sum_{j=0}^{n-1} (-1)^j q^j e^{jt}. \tag{2.4}$$

After some elementary calculations, we have

$$\begin{aligned} \int_{\mathbb{Z}_p} e^{xt} d\mu_{-q}(x) &= \frac{[2]_q}{qe^t+1}, \\ \int_{\mathbb{Z}_p} e^{(x+n)t} d\mu_{-q}(x) &= e^{nt} \frac{[2]_q}{qe^t+1}. \end{aligned} \tag{2.5}$$

By using (2.4) and (2.5), we have

$$q^n \int_{\mathbb{Z}_p} e^{(x+n)t} d\mu_{-q}(x) + \int_{\mathbb{Z}_p} e^{xt} d\mu_{-q}(x) = \frac{[2]_q(1+q^n e^{nt})}{qe^t+1}.$$

From the above, we get

$$\frac{[2]_q(1+q^n e^{nt})}{qe^t+1} = \frac{[2]_q \int_{\mathbb{Z}_p} e^{xt} d\mu_{-q}(x)}{\int_{\mathbb{Z}_p} q^{(n-1)x} e^{ntx} d\mu_{-q}(x)}. \tag{2.6}$$

By substituting Taylor series of e^{xt} into (2.4), we obtain

$$\begin{aligned} &\sum_{m=0}^{\infty} \left(q^n \int_{\mathbb{Z}_p} (x+n)^m d\mu_{-q}(x) + \int_{\mathbb{Z}_p} x^m d\mu_{-q}(x) \right) \frac{t^m}{m!} \\ &= \sum_{m=0}^{\infty} \left([2]_q \sum_{j=0}^{n-1} (-1)^j q^j j^m \right) \frac{t^m}{m!}. \end{aligned}$$

By comparing coefficients $\frac{t^m}{m!}$ in the above equation, we obtain

$$q^n \sum_{k=0}^{m} \binom{m}{k} n^{m-k} \int_{\mathbb{Z}_p} x^k d\mu_{-q}(x) + \int_{\mathbb{Z}_p} x^m d\mu_{-q}(x) = [2]_q \sum_{j=0}^{n-1} (-1)^j q^j j^m.$$

By using Theorem 3, we have

$$q^n \sum_{k=0}^{m} \binom{m}{k} n^{m-k} \int_{\mathbb{Z}_p} x^k d\mu_{-q}(x) + \int_{\mathbb{Z}_p} x^m d\mu_{-q}(x) = [2]_q \widetilde{T}_{m,q}(n-1). \tag{2.7}$$

By using (2.6) and (2.7), we arrive at the following theorem:

Theorem 5. Let n be odd positive integer. Then we have

$$\frac{\int_{\mathbb{Z}_p} e^{xt} d\mu_{-q}(x)}{\int_{\mathbb{Z}_p} q^{(n-1)x} e^{ntx} d\mu_{-q}(x)} = \sum_{m=0}^{\infty} \left(\widetilde{T}_{m,q}(n-1) \right) \frac{t^m}{m!}.$$

Let w_1 and w_2 be odd positive integers. By (2.5), Theorem 5, and after some elementary calculations, we obtain the following theorem.

Theorem 6. Let w_1 and w_2 be odd positive integers. Then we have

$$\frac{\int_{\mathbb{Z}_p} e^{w_2 xt} d\mu_{-q^{w_2}}(x)}{\int_{\mathbb{Z}_p} q^{(w_1 w_2 -1)x} e^{w_1 w_2 tx} d\mu_{-q}(x)} = \frac{[2]_{q^{w_2}}}{[2]_q} \sum_{m=0}^{\infty} \left(\widetilde{T}_{m,q^{w_2}}(w-1) w_2^m \right) \frac{t^m}{m!}. \tag{2.8}$$

By (1.1), we obtain

$$\begin{aligned} &\frac{\int_{\mathbb{Z}_p}\int_{\mathbb{Z}_p} e^{(w_1x_1+w_2x_2+w_1w_2x)t} d\mu_{-q^{w_1}}(x_1) d\mu_{-q^{w_2}}(x_2)}{\int_{\mathbb{Z}_p} q^{(w_1w_2-1)x} e^{w_1w_2xt} d\mu_{-q}(x)} \\ &= \frac{e^{w_1w_2xt}\int_{\mathbb{Z}_p} e^{w_1x_1t} d\mu_{-q^{w_1}}(x_1) \int_{\mathbb{Z}_p} e^{w_2x_2t} d\mu_{-q^{w_2}}(x_2)}{\int_{\mathbb{Z}_p} q^{(w_1w_2-1)x} e^{w_1w_2xt} d\mu_{-q}(x)}. \end{aligned} \tag{2.9}$$

By using (2.8) and (2.9), after elementary calculations, we obtain

$$\begin{aligned} a &= \left(\int_{\mathbb{Z}_p} e^{(w_1x_1+w_1w_2x)t} d\mu_{-q^{w_1}}(x_1) \right) \left(\frac{\int_{\mathbb{Z}_p} e^{x_2w_2t} d\mu_{-q^{w_2}}(x_2)}{\int_{\mathbb{Z}_p} q^{(w_1w_2-1)x} e^{w_1w_2xt} d\mu_{-q}(x)} \right) \\ &= \left(\sum_{m=0}^{\infty} \widetilde{E}_{m,q^{w_1}}(w_2x) w_1^m \frac{t^m}{m!} \right) \left(\frac{[2]_{q^{w_2}}}{[2]_q} \sum_{m=0}^{\infty} \widetilde{T}_{m,q^{w_2}}(w_1-1) w_2^m \frac{t^m}{m!} \right). \end{aligned} \tag{2.10}$$

By using Cauchy product in the above, we have

$$a = \sum_{m=0}^{\infty} \left(\frac{[2]_{q^{w_2}}}{[2]_q} \sum_{j=0}^{m} \binom{m}{j} \widetilde{E}_{j,q^{w_1}}(w_2x) w_1^j \widetilde{T}_{m-j,q^{w_2}}(w_1-1) w_2^{m-j} \right) \frac{t^m}{m!}. \tag{2.11}$$

By using the symmetry in (2.10), we obtain

$$\begin{aligned} a &= \left(\int_{\mathbb{Z}_p} e^{(w_2x_2+w_1w_2x)t} d\mu_{-q^{w_2}}(x_2) \right) \left(\frac{\int_{\mathbb{Z}_p} e^{x_1w_1t} d\mu_{-q^{w_1}}(x_1)}{\int_{\mathbb{Z}_p} q^{(w_1w_2-1)x} e^{w_1w_2xt} d\mu_{-q}(x)} \right) \\ &= \left(\sum_{m=0}^{\infty} \widetilde{E}_{m,q^{w_1}}(w_1x) w_2^m \frac{t^m}{m!} \right) \left(\frac{[2]_{q^{w_1}}}{[2]_q} \sum_{m=0}^{\infty} \widetilde{T}_{m,q^{w_1}}(w_2-1) w_1^m \frac{t^m}{m!} \right). \end{aligned}$$

Thus we obtain

$$a = \sum_{m=0}^{\infty} \left(\frac{[2]_{q^{w_1}}}{[2]_q} \sum_{j=0}^{m} \binom{m}{j} \widetilde{E}_{j,q^{w_2},}(w_1x) w_2^j \widetilde{T}_{m-j,q^{w_1}}(w_2-1) w_1^{m-j} \right) \frac{t^m}{m!}. \tag{2.12}$$

By comparing coefficients $\frac{t^m}{m!}$ in the both sides of (2.11) and (2.12), we arrive at the following theorem.

Theorem 7. Let w_1 and w_2 be odd positive integers. Then we obtain

$$
\begin{aligned}
&[2]_{q^{w_2}} \sum_{j=0}^{m} \binom{m}{j} \widetilde{E}_{j,q^{w_1}}(w_2 x) w_1^j \widetilde{T}_{m-j,q^{w_2}}(w_1-1) w_2^{m-j} \\
&= [2]_{q^{w_1}} \sum_{j=0}^{m} \binom{m}{j} \widetilde{E}_{j,q^{w_2}}(w_1 x) w_2^j \widetilde{T}_{m-j,q^{w_1}}(w_2-1) w_1^{m-j},
\end{aligned}
$$

where $\widetilde{E}_{k,q}(x)$ and $\widetilde{T}_{m,q}(k)$ denote the q-Euler polynomials and the q-analogue of alternating sums of powers of consecutive integers, respectively.

By using Theorem 2, we have the following corollary.

Corollary 8. Let w_1 and w_2 be odd positive integers. Then we obtain

$$
\begin{aligned}
&[2]_{q^{w_1}} \sum_{j=0}^{m} \sum_{k=0}^{j} \binom{m}{j} \binom{j}{k} w_1^{m-k} w_2^j x^{j-k} \widetilde{E}_{k,q^{w_2}} \widetilde{T}_{m-j,q^{w_1}}(w_2-1) \\
&= [2]_{q^{w_2}} \sum_{j=0}^{m} \sum_{k=0}^{j} \binom{m}{j} \binom{j}{k} w_1^j w_2^{m-k} x^{j-k} \widetilde{E}_{k,q^{w_1}} \widetilde{T}_{m-j,q^{w_2}}(w_1-1).
\end{aligned}
$$

By using (2.9), we have

$$
\begin{aligned}
a &= \left(e^{w_1 w_2 x t} \int_{\mathbb{Z}_p} e^{x_1 w_1 t} d\mu_{-q^{w_1}}(x_1) \right) \left(\frac{\int_{\mathbb{Z}_p} e^{x_2 w_2 t} d\mu_{-q^{w_2}}(x_2)}{\int_{\mathbb{Z}_p} q^{(w_1 w_2 - 1)x} e^{w_1 w_2 x t} d\mu_{-q}(x)} \right) \\
&= \frac{[2]_{q^{w_2}}}{[2]_q} \sum_{j=0}^{w_1-1} (-1)^j q^{w_2 j} \int_{\mathbb{Z}_p} e^{\left(x_1 + w_2 x + j \frac{w_2}{w_1}\right)(w_1 t)} d\mu_{-q^{w_1}}(x_1) \\
&= \sum_{n=0}^{\infty} \left(\frac{[2]_{q^{w_2}}}{[2]_q} \sum_{j=0}^{w_1-1} (-1)^j q^{w_2 j} \widetilde{E}_{n,q^{w_1}} \left(w_2 x + j \frac{w_2}{w_1} \right) w_1^n \right) \frac{t^n}{n!}.
\end{aligned}
\tag{2.13}
$$

By using the symmetry property in (2.13), we also have

$$
\begin{aligned}
a &= \left(e^{w_1 w_2 x t} \int_{\mathbb{Z}_p} e^{x_2 w_2 t} d\mu_{-q^{w_2}}(x_2) \right) \left(\frac{\int_{\mathbb{Z}_p} e^{x_1 w_1 t} d\mu_{-q^{w_1}}(x_1)}{\int_{\mathbb{Z}_p} q^{(w_1 w_2 - 1)x} e^{w_1 w_2 x t} d\mu_{-q}(x)} \right) \\
&= \frac{[2]_{q^{w_1}}}{[2]_q} \sum_{j=0}^{w_2-1} (-1)^j q^{w_1 j} \int_{\mathbb{Z}_p} e^{\left(x_2 + w_1 x + j \frac{w_1}{w_2}\right)(w_2 t)} d\mu_{-q^{w_2}}(x_2) \\
&= \sum_{n=0}^{\infty} \left(\frac{[2]_{q^{w_1}}}{[2]_q} \sum_{j=0}^{w_2-1} (-1)^j q^{w_1 j} \widetilde{E}_{n,q^{w_2}} \left(w_1 x + j \frac{w_1}{w_2} \right) w_2^n \right) \frac{t^n}{n!}.
\end{aligned}
\tag{2.14}
$$

By comparing coefficients $\frac{t^n}{n!}$ in the both sides of (2.13) and (2.14), we have the following theorem.

Theorem 9. Let w_1 and w_2 be odd positive integers. Then we have

$$
\begin{aligned}
&[2]_{q^{w_2}} \sum_{j=0}^{w_1-1} (-1)^j q^{w_2 j} \widetilde{E}_{n,q^{w_1}} \left(w_2 x + j \frac{w_2}{w_1} \right) w_1^n \\
=&[2]_{q^{w_1}} \sum_{j=0}^{w_2-1} (-1)^j q^{w_1 j} \widetilde{E}_{n,q^{w_2}} \left(w_1 x + j \frac{w_1}{w_2} \right) w_2^n.
\end{aligned}
\tag{2.15}
$$

Remark 10. Let w_1 and w_2 be odd positive integers. If $q \to 1$, we have

$$\sum_{j=0}^{w_1-1} (-1)^j E_n \left(w_2 x + j \frac{w_2}{w_1} \right) w_1^n = \sum_{j=0}^{w_2-1} (-1)^j E_n \left(w_1 x + j \frac{w_1}{w_2} \right) w_2^n.$$

Substituting $w_1 = 1$ into (2.15), we arrive at the following corollary.

Corollary 11. Let w_2 be odd positive integer. Then we obtain

$$\widetilde{E}_{n,q}(x) = \frac{[2]_q}{[2]_{q^{w_2}}} \sum_{j=0}^{w_2-1} (-1)^j q^j \widetilde{E}_{n,q^{w_2}} \left(\frac{x+j}{w_2} \right) w_2^n.$$

ACKNOWLEDGEMENT

This paper has been supported by the 2013 Hannam University Research Fund.

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J. COMPUTATIONAL ANALYSIS AND APPLICATIONS, VOL. 15, NO.6, 1110-1119, 2013, COPYRIGHT 2013 EUDOXUS PRESS, LLC

Approximate septic and octic mappings in quasi-β-normed spaces

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Abstract In this paper, we achieve the general solution of the septic and octic functional equations. Moreover, we prove the stability of the septic and octic functional equations in quasi-β-normed spaces.

Keywords Quasi-β-normed spaces; Septic mapping; Octic mapping; (β, p)-Banach spaces; Hyers–Ulam stability.

MR(2000) Subject Classification: 39B52, 39B82.

1. Introduction and preliminaries

The concept of stability for a functional equation arises when one replaces a functional equation by an inequality which acts as a perturbation of the equation. The first stability problem concerning group homomorphisms was raised by Ulam [1] in 1940 and affirmatively solved by Hyers [2]. The result of Hyers was generalized by Rassias [3] for approximate linear mappings by allowing the Cauchy difference operator $CDf(x,y) = f(x+y) - [f(x)+f(y)]$ to be controlled by $\epsilon(\|x\|^p + \|y\|^p)$. In 1994, a generalization of Rassias' theorem was obtained by Găvruţa [4], who replaced $\epsilon(\|x\|^p + \|y\|^p)$ by a general control function $\varphi(x,y)$ in the spirit of Rassias' approach. The reader is referred to [5–20] and references therein for more information on stability of functional equations.

In this paper, we achieve the general solutions of the septic functional equation

$$f(x+4y)-7f(x+3y)+21f(x+2y)-35f(x+y)+35f(x)-21f(x-y)+7f(x-2y)-f(x-3y) = 5040f(y) \quad (1.1)$$

and the octic functional equation

$$\begin{aligned} f(x+4y) - 8f(x+3y) + 28f(x+2y) - 56f(x+y) + 70f(x) - 56f(x-y) + 28f(x-2y) \\ -8f(x-3y) + f(x-4y) = 40320f(y). \end{aligned} \quad (1.2)$$

Moreover, we prove the stability of the septic and octic functional equations in quasi-β-normed spaces. Since $f(x) = x^7$ is a solutions of (1.1), we say it quintic functional equation. Similarly, $f(x) = x^8$ is a solutions of (1.2), we say it septic functional equation. Every solution of the septic or octic functional equation is said to be a septic or an octic mapping, respectively.

Let us recall some basic concepts concerning quasi-β-normed spaces (see [9, 16]). Let β be a fix real number with $0 < \beta \leq 1$ and let $\mathbb{K}$ denote either $\mathbb{R}$ or $\mathbb{C}$. Let X be a linear space over $\mathbb{K}$. A quasi-β-norm $\|\cdot\|$ is a real-valued function on X satisfying the following:

(1) $\|x\| \geq 0$ for all $x \in X$ and $\|x\| = 0$ if and only if $x = 0$.

(2) $\|\lambda x\| = |\lambda|^\beta \|x\|$ for all $\lambda \in \mathbb{K}$ and all $x \in X$.

(3) There is a constant $K \geq 1$ such that $\|x+y\| \leq K(\|x\| + \|y\|)$ for all $x, y \in X$.

A quasi-β-normed space is a pair $(X, \|\cdot\|)$, where $\|\cdot\|$ is a quasi-β-norm on X. The smallest possible K is called the modulus of concavity of $\|\cdot\|$. A quasi-β-Banach space is a complete quasi-β-normed space.

A quasi-β-norm $\|\cdot\|$ is called a (β, p)-norm $(0 < p \leq 1)$ if $\|x+y\|^p \leq \|x\|^p + \|y\|^p$ for all $x, y \in X$. In this case, a quasi-β-Banach space is called a (β, p)-Banach space. We can refer to [13] for the concept of quasi-normed spaces and p-Banach spaces.

Given a p-norm, the formula $d(x,y) := \|x-y\|^p$ gives us a translation invariant metric on X. By the Aoki-Rolewicz theorem, each quasi-norm is equivalent to some p-norm. Since it is much easier to work with p-norms than quasi-norms, henceforth we restrict our attention mainly to p-norms.

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The first author was supported by the National Natural Science Foundation of China (NNSFC)(Grant No. 11171022).

2. General solutions to the septic and octic functional equations

In this section, let X and Y be vector spaces. Some basic facts on n-additive symmetric mappings can be found in [11, 17, 20].

Theorem 2.1. *A function $f : X \to Y$ is a solution of the functional equation (1.1) if and only if f is of the form $f(x) = A^7(x)$ for all $x \in X$, where $A^7(x)$ is the diagonal of the 7-additive symmetric map $A_7 : X^7 \to Y$.*

Proof. Assume that f satisfies the functional equation (1.1). Replacing $x = y = 0$ in equation (1.1), one finds $f(0) = 0$. Replacing (x, y) with $(0, x)$ and $(x, -x)$ in (1.1), respectively, and adding the two resulting equations, we obtain $f(-x) = -f(x)$. Replacing (x, y) with $(4x, x)$ and $(0, 2x)$ in (1.1), respectively, and subtracting the two resulting equations, we get

$$7f(7x) - 27f(6x) + 35f(5x) - 21f(4x) + 21f(3x) - 5061f(2x) + 5041f(x) = 0 \tag{2.1}$$

Replacing (x, y) with $(3x, x)$ in (1.1), and multiplying the resulting equation by 7, one obtains

$$7f(7x) - 49f(6x) + 147f(5x) - 245f(4x) + 245f(3x) - 147f(2x) - 35231f(x) = 0 \tag{2.2}$$

for all $x \in X$. Subtracting equations (2.1) and (2.1), we get

$$22f(6x) - 112f(5x) + 224f(4x) - 224f(3x) - 4914f(2x) + 40272f(x) = 0 \tag{2.3}$$

Replacing (x, y) with $(2x, x)$ in (1.1), and multiplying the resulting equation by 22, one finds

$$22f(6x) - 154f(5x) + 462f(4x) - 770f(3x) + 770f(2x) - 111320f(x) = 0 \tag{2.4}$$

for all $x \in X$. Subtracting equations (2.3) and (2.4), we arrive at

$$42f(5x) - 238f(4x) + 546f(3x) - 5684f(2x) + 151592f(x) = 0 \tag{2.5}$$

for all $x \in X$. Replacing (x, y) with (x, x) in (1.1), and multiplying the resulting equation by 42, one finds

$$42f(5x) - 294f(4x) + 882f(3x) - 1428f(2x) - 210504f(x) = 0 \tag{2.6}$$

for all $x \in X$. Subtracting equations (2.5) and (2.6), one gets

$$56f(4x) - 336f(3x) - 4256f(2x) + 362096f(x) = 0 \tag{2.7}$$

for all $x \in X$. Replacing (x, y) with $(0, x)$ in (1.1), and multiplying the resulting equation by 56, one finds

$$56f(4x) - 336f(3x) + 784f(2x) - 283024f(x) = 0 \tag{2.8}$$

for all $x \in X$. Subtracting equations (2.7) and (2.8), we arrive at

$$f(2x) = 2^7 f(x) \tag{2.9}$$

for all $x \in X$.

On the other hand, one can rewrite the functional equation (1.1) in the form

$$\begin{aligned} &f(x) + \tfrac{1}{35} f(x + 4y) - \tfrac{1}{5} f(x + 3y) + \tfrac{3}{5} f(x + 2y) - f(x + y) - \tfrac{3}{5} f(x - y) + \tfrac{1}{5} f(x - 2y) \\ &\quad = \tfrac{1}{35} f(x - 3y) + 144f(y) \end{aligned} \tag{2.10}$$

for all $x \in X$. By Theorems 3.5 and 3.6 in [11], f is a generalized polynomial function of degree at most 6, that is, f is of the form

$$f(x) = A^7(x) + A^6(x) + A^5(x) + A^4(x) + A^3(x) + A^2(x) + A^1(x) + A^0(x), \quad \forall\, x \in X, \tag{2.11}$$

where $A^0(x) = A^0$ is an arbitrary element of Y, and $A^i(x)$ is the diagonal of the i-additive symmetric map $A_i : X^i \to Y$ for $i = 1, 2, 3, 4, 5$. By $f(0) = 0$ and $f(-x) = -f(x)$ for all $x \in X$, we get $A^0(x) = A^0 = 0$ and the function f is odd. Thus we have $A^6(x) = A^4(x) = A^2(x) = 0$. It follows that $f(x) = A^7(x)+A^5(x)+A^3(x)+A^1(x)$. By (2.9) and $A^n(rx) = r^n A^n(x)$ whenever $x \in X$ and $r \in \mathbb{Q}$, we obtain $2^7(A^7(x) + A^5(x) + A^3(x) + A^1(x)) = 2^7A^7(x) + 2^5A^5(x) + 2^3A^3(x) + 2A^1(x)$. It follows that $A^5(x) = A^3(x) = A^1(x) = 0$ for all $x \in X$. Hence $f(x) = A^7(x)$.

Conversely, assume that $f(x) = A^7(x)$ for all $x \in X$, where $A^7(x)$ is the diagonal of the 7-additive symmetric map $A_7 : X^7 \to Y$. From $A^7(x+y) = A^7(x) + A^7(y) + 7A^{6,1}(x,y) + 21A^{5,2}(x,y) + 35A^{4,3}(x,y) + 35A^{3,4}(x,y) + 21A^{2,5}(x,y) + 7A^{1,6}(x,y)$, $A^7(rx) = r^7A^5(x), A^{6,1}(x,ry) = rA^{6,1}(x,y)$, $A^{5,2}(x,ry) = r^2A^{5,2}(x,y)$, $A^{4,3}(x,ry) = r^3A^{4,3}(x,y)$, $A^{3,4}(x,ry) = r^4A^{3,4}(x,y)$, $A^{2,5}(x,ry) = r^5A^{2,5}(x,y)$, and $A^{1,6}(x,ry) = r^6A^{1,6}(x,y)$ $(x,y \in X, r \in \mathbb{Q})$, we see that f satisfies (1.1), which completes the proof of Theorem 2.1. □

Theorem 2.2. *A function $f : X \to Y$ is a solution of the functional equation* (1.2) *if and only if f is of the form $f(x) = A^8(x)$ for all $x \in X$, where $A^8(x)$ is the diagonal of the* 8*-additive symmetric map $A_8 : X^8 \to Y$.*

Proof. Assume that f satisfies the functional equation (1.2). Replacing $x = y = 0$ in equation (1.2), one gets $f(0) = 0$. Substituting y by $-y$ in (1.2) and subtracting the resulting equation from equation (1.2) and then y by x, we obtain $f(-x) = f(x)$. Replacing (x,y) with $(0,2x)$ and $(4x,x)$ in (1.2), respectively, we get

$$f(8x) - 8f(6x) + 28f(4x) - 20216f(x) = 0 \tag{2.12}$$

and

$$f(8x) - 8f(7x) + 28f(6x) - 56f(5x) + 70f(4x) - 56f(3x) + 28f(2x) - 40328f(x) = 0 \tag{2.13}$$

for all $x \in X$. Subtracting equations (2.12) and (2.13), we find

$$8f(7x) - 36f(6x) + 56f(5x) - 42f(4x) + 56f(3x) - 20244f(2x) + 40328f(x) = 0 \tag{2.14}$$

for all $x \in X$. Replacing (x,y) with $(3x,x)$ in (1.2), and multiplying the resulting equation by 8, one obtains

$$8f(7x) - 64f(6x) + 224f(5x) - 448f(4x) + 560f(3x) - 448f(2x) - 322328f(x) = 0 \tag{2.15}$$

for all $x \in X$. Subtracting equations (2.14) and (2.15), one gets

$$28f(6x) - 168f(5x) + 406f(4x) - 504f(3x) - 19796f(2x) + 362656f(x) = 0 \tag{2.16}$$

for all $x \in X$. Replacing (x,y) with $(2x,x)$ in (1.2), and multiplying the resulting equation by 28, one finds

$$28f(6x) - 224f(5x) + 784f(4x) - 1568f(3x) + 1988f(2x) - 1130752f(x) = 0 \tag{2.17}$$

for all $x \in X$. Subtracting equations (2.16) and (2.17), one gets

$$56f(5x) - 378f(4x) + 1064f(3x) - 21784f(2x) + 1493408f(x) = 0 \tag{2.18}$$

for all $x \in X$. Replacing (x,y) with (x,x), and multiplying the resulting equation by 56, one finds

$$56f(5x) - 448f(4x) + 1624f(3x) - 3584f(2x) - 2252432f(x) = 0 \tag{2.19}$$

for all $x \in X$. Subtracting equations (2.18) and (2.19), we arrive at

$$70f(4x) - 560f(3x) - 18200f(2x) + 3745840f(x) = 0 \tag{2.20}$$

for all $x \in X$. Replacing (x,y) with $(0,x)$, and multiplying the resulting equation by 70, one finds

$$70f(4x) - 560f(3x) + 1960f(2x) - 1415120f(x) = 0 \tag{2.21}$$

for all $x \in X$. Subtracting equations (2.20) and (2.21), we arrive at

$$f(2x) = 2^8 f(x) \tag{2.22}$$

for all $x \in X$.

On the other hand, one can rewrite the functional equation (1.2) in the form

$$\begin{aligned} &f(x) + \tfrac{1}{70}f(x+4y) - \tfrac{4}{35}f(x+3y) + \tfrac{2}{5}f(x+2y) - \tfrac{4}{5}f(x+y) - \tfrac{4}{5}f(x-y) + \tfrac{2}{5}f(x-2y) \\ &\quad = \tfrac{4}{35}f(x-3y) - \tfrac{1}{70}f(x-4y) + \tfrac{1}{576}f(y) \end{aligned} \tag{2.23}$$

for all $x \in X$. By Theorems 3.5 and 3.6 in [11], f is a generalized polynomial function of degree at most 6, that is f is of the form

$$f(x) = A^8(x) + A^7(x) + \cdots + A^1(x) + A^0(x), \quad \forall\, x \in X, \tag{2.24}$$

where $A^0(x) = A^0$ is an arbitrary element of Y, and $A^i(x)$ is the diagonal of the i-additive symmetric map $A_i : X^i \to Y$ for $i = 1, 2, \ldots, 8$. By $f(0) = 0$ and $f(-x) = f(x)$ for all $x \in X$, we get $A^0(x) = A^0 = 0$

and the function f is even. Thus we have $A^7(x) = A^5(x) = A^3(x) = A^1(x) = 0$. It follows that $f(x) = A^8(x) + A^6(x) + A^4(x) + A^2(x)$. By (2.22) and $A^n(rx) = r^n A^n(x)$ whenever $x \in X$ and $r \in \mathbb{Q}$, we obtain $2^8(A^8(x) + A^6(x) + A^4(x) + A^2(x)) = 2^8A^8(x) + 2^6A^6(x) + 2^4A^4(x) + 2^2A^2(x)$. It follows that $A^6(x) = A^4(x) = A^2(x) = 0$, $x \in X$. Therefore, $f(x) = A^8(x)$. The rest of the proof is similar to the proof of Theorem 2.1. $\square$

3. Stability of the septic and octic functional equations

Throughout this section, we assume that X is a linear space and Y is a (β, p)-Banach space with (β, p)-norm $\|\cdot\|_Y$. For a given mapping $f : X \to Y$, we define the difference operators

$$D_s f(x, y) := f(x+4y)-7f(x+3y)+21f(x+2y)-35f(x+y)+35f(x)-21f(x-y)+7f(x-2y)-f(x-3y)-5040f(y)$$

and

$$\begin{aligned} D_o f(x, y) \quad := \quad & f(x + 4y) - 8f(x + 3y) + 28f(x + 2y) - 56f(x + y) + 70f(x) - 56f(x - y) \\ & +28f(x - 2y) - 8f(x - 3y) + f(x - 4y) - 40320f(y) \end{aligned}$$

for all $x, y \in X$.

Lemma 3.1(see [16]). *Let $j \in \{-1, 1\}$ be fixed, $s, a \in \mathbb{N}$ with $a \geq 2$, and $\psi : X \to [0, \infty)$ be a function such that there exists an $L < 1$ with $\psi(a^j x) \leq a^{js\beta} L\psi(x)$ for all $x \in X$. Let $f : X \to Y$ be a mapping satisfying*

$$\|f(ax) - a^s f(x)\|_Y \leq \psi(x) \tag{3.1}$$

for all $x \in X$, then there exists a uniquely determined mapping $F : X \to Y$ such that $F(ax) = a^s F(x)$ and

$$\|f(x) - F(x)\|_Y \leq \frac{1}{a^{s\beta}|1 - L^j|}\psi(x) \tag{3.2}$$

for all $x \in X$.

Theorem 3.2. *Let $j \in \{-1, 1\}$ be fixed, $\varphi : X \times X \to [0, \infty)$ be a function such that there exists an $L < 1$ with $\varphi(2^j x, 2^j y) \leq 128^{j\beta} L\varphi(x, y)$ for all $x, y \in X$. Let $f : X \to Y$ be a mapping satisfying*

$$\|D_s f(x, y)\|_Y \leq \varphi(x, y) \tag{3.3}$$

for all $x, y \in X$. Then there exists a unique septic mapping $S : X \to Y$ such that

$$\|f(x) - S(x)\|_Y \leq \frac{1}{128^\beta|1 - L^j|}\varphi_s(x) \tag{3.4}$$

for all $x \in X$, where

$$\begin{aligned} \varphi_s(x) \quad = \; & \tfrac{1}{5040^\beta}[K^5\varphi(4x, x) + K^6\varphi(0, 2x) + 7^\beta K^5\varphi(3x, x) + 22^\beta K^4\varphi(2x, x) + 42^\beta K^3\varphi(x, x) \\ & +(\tfrac{K^7}{144^\beta} + \tfrac{11^\beta K^5}{360^\beta} + \tfrac{K^5}{720^\beta} + \tfrac{7^\beta K^4}{40^\beta} + \tfrac{7^\beta K^3}{36^\beta})\varphi(0, 0) + \tfrac{K^{10}}{5040^\beta}(\varphi(0, 6x) + \varphi(6x, -6x)) \\ & +\tfrac{K^{10}}{720^\beta}(\varphi(0, 4x) + \varphi(4x, -4x)) + (\tfrac{K^9}{240^\beta} + \tfrac{K^6}{120^\beta} + \tfrac{7^\beta K^6}{90^\beta})(\varphi(0, 2x) + \varphi(2x, -2x)) \\ & +56^\beta K^2\varphi(0, x) + (\tfrac{11^\beta K^6}{2520^\beta} + \tfrac{7^\beta K^6}{120^\beta} + \tfrac{7^\beta K^5}{30^\beta})(\varphi(0, x) + \varphi(x, -x)) + \tfrac{K^6}{5040^\beta}(\varphi(0, 3x) + \varphi(3x, -3x))]. \end{aligned}$$

Proof. Replacing $x = y = 0$ in (3.3), we get

$$\|f(0)\|_Y \leq \frac{1}{5040^\beta}\varphi(0, 0). \tag{3.5}$$

Replacing x and y by 0 and x in (3.3), respectively, we get

$$\|f(4x) - 7f(3x) + 21f(2x) - 5075f(x) + 35f(0) - 21f(-x) + 7f(-2x) - f(-3x)\|_Y \leq \varphi(0, x) \tag{3.6}$$

for all $x \in X$. Replacing x and y by x and $-x$ in (3.3), respectively, we have

$$\|f(-3x) - 7f(-2x) - 35f(0) + 35f(x) - 21f(2x) + 7f(3x) - f(4x) - 5019f(-x)\|_Y \leq \varphi(x, -x) \tag{3.7}$$

for all $x \in X$. By (3.6) and (3.7), we obtain

$$\|f(x) + f(-x)\|_Y \leq \frac{K}{5040^\beta}(\varphi(0, x) + \varphi(x, -x)) \tag{3.8}$$

for all $x \in X$. Replacing x and y by 0 and $2x$ in (3.3), respectively, we find

$$\|f(8x) - 7f(6x) + 21f(4x) - 5075f(2x) + 35f(0) - 21f(-2x) + 7f(-4x) - f(-6x)\|_Y \leq \varphi(0, 2x) \tag{3.9}$$

for all $x \in X$. By (3.5), (3.8) and (3.9), one obtains

$$\begin{aligned} &\|f(8x) - 6f(6x) + 14f(4x) - 5054f(2x)\|_Y \\ &\leq K\varphi(0, 2x) + \tfrac{K^2}{144^\beta}\varphi(0, 0) + \tfrac{K^4}{240^\beta}(\varphi(0, 2x) + \varphi(2x, -2x)) \\ &\quad + \tfrac{K^5}{720^\beta}(\varphi(0, 4x) + \varphi(4x, -4x)) + \tfrac{K^5}{5040^\beta}(\varphi(0, 6x) + \varphi(6x, -6x)) \end{aligned} \tag{3.10}$$

for all $x \in X$. Replacing x and y by $4x$ and x in (3.3), respectively, we get

$$\|f(8x) - 7f(7x) + 21f(6x) - 35f(5x) + 35f(4x) - 21f(3x) + 7f(2x) - 5041f(x)\|_Y \leq \varphi(4x, x) \tag{3.11}$$

for all $x \in X$. By (3.10) and (3.11), we obtain

$$\begin{aligned} &\|7f(7x) - 27f(6x) + 35f(5x) - 21f(4x) + 21f(3x) - 5061f(2x) + 5041f(x)\|_Y \\ &\leq K\varphi(4x, x) + K^2\varphi(0, 2x) + \tfrac{K^3}{144^\beta}\varphi(0, 0) + \tfrac{K^5}{240^\beta}(\varphi(0, 2x) + \varphi(2x, -2x)) \\ &\quad + \tfrac{K^6}{720^\beta}(\varphi(0, 4x) + \varphi(4x, -4x)) + \tfrac{K^6}{5040^\beta}(\varphi(0, 6x) + \varphi(6x, -6x)) \end{aligned} \tag{3.12}$$

for all $x \in X$. Replacing x and y by $3x$ and x in (3.3), respectively, we get

$$\|f(7x) - 7f(6x) + 21f(5x) - 35f(4x) + 35f(3x) - 21f(2x) - f(0) - 5033f(x)\|_Y \leq \varphi(3x, x) \tag{3.13}$$

for all $x \in X$. Using (3.5), we have

$$\begin{aligned} &\|7f(7x) - 49f(6x) + 147f(5x) - 245f(4x) + 245f(3x) + 147f(2x) - 35231f(x)\|_Y \\ &\leq 7^\beta K\varphi(3x, x) + \tfrac{K}{720^\beta}\varphi(0, 0) \end{aligned} \tag{3.14}$$

for all $x \in X$. By (3.12) and (3.14), one obtains

$$\begin{aligned} &\|22f(6x) - 112f(5x) + 224f(4x) - 224f(3x) - 4914f(2x) + 40272f(x)\|_Y \\ &\leq K^2\varphi(4x, x) + K^3\varphi(0, 2x) + \tfrac{K^4}{144^\beta}\varphi(0, 0) + \tfrac{K^6}{240^\beta}(\varphi(0, 2x) + \varphi(2x, -2x)) \\ &\quad + \tfrac{K^7}{720^\beta}(\varphi(0, 4x) + \varphi(4x, -4x)) + \tfrac{K^7}{5040^\beta}(\varphi(0, 6x) + \varphi(6x, -6x)) + 7^\beta K^2\varphi(3x, x) + \tfrac{K^2}{720^\beta}\varphi(0, 0) \end{aligned} \tag{3.15}$$

for all $x \in X$. Replacing x and y by $2x$ and x in (3.3), respectively, we get

$$\|f(6x) - 7f(5x) + 21f(4x) - 35f(3x) + 35f(2x) - 5061f(x) + 7f(0) - f(-x)\|_Y \leq \varphi(2x, x) \tag{3.16}$$

for all $x \in X$. Using (3.5), (3.8) and (3.16), we have

$$\begin{aligned} &\|f(6x) - 7f(5x) + 21f(4x) - 35f(3x) + 35f(2x) - 5060f(x)\|_Y \\ &\leq K\varphi(2x, x) + \tfrac{K^2}{720^\beta}\varphi(0, 0) + \tfrac{K^3}{5040^\beta}(\varphi(0, x) + \varphi(x, -x)) \end{aligned} \tag{3.17}$$

for all $x \in X$. Hence

$$\begin{aligned} &\|22f(6x) - 154f(5x) + 462f(4x) - 770f(3x) + 770f(2x) - 111320f(x)\|_Y \\ &\leq 22^\beta K\varphi(2x, x) + \tfrac{11^\beta K^2}{360^\beta}\varphi(0, 0) + \tfrac{11^\beta K^3}{2520^\beta}(\varphi(0, x) + \varphi(x, -x)) \end{aligned} \tag{3.18}$$

for all $x \in X$. By (3.15) and (3.18), one obtains

$$\begin{aligned} &\|42f(5x) - 238f(4x) + 546f(3x) - 5684f(2x) + 151592f(x)\|_Y \\ &\leq K^3\varphi(4x, x) + K^4\varphi(0, 2x) + \tfrac{K^5}{144^\beta}\varphi(0, 0) + \tfrac{K^7}{240^\beta}(\varphi(0, 2x) + \varphi(2x, -2x)) \\ &\quad + \tfrac{K^8}{720^\beta}(\varphi(0, 4x) + \varphi(4x, -4x)) + \tfrac{K^8}{5040^\beta}(\varphi(0, 6x) + \varphi(6x, -6x)) + 7^\beta K^3\varphi(3x, x) + \tfrac{K^3}{720^\beta}\varphi(0, 0) \\ &\quad + 22^\beta K^2\varphi(2x, x) + \tfrac{11^\beta K^3}{360^\beta}\varphi(0, 0) + \tfrac{11^\beta K^4}{2520^\beta}(\varphi(0, x) + \varphi(x, -x)) \end{aligned} \tag{3.19}$$

for all $x \in X$. Replacing x and y by x and x in (3.3), respectively, we have

$$\|f(5x) - 7f(4x) + 21f(3x) - 35f(2x) - 5005f(x) - 21f(0) + 7f(-x) - f(-2x)\|_Y \leq \varphi(x, x) \tag{3.20}$$

for all $x \in X$. By (3.5), (3.8), and (3.20), we have

$$\begin{aligned} &\|f(5x) - 7f(4x) + 21f(3x) - 34f(2x) - 5012f(x)\|_Y \\ &\quad \leq K\varphi(x,x) + \tfrac{K^2}{240^\beta}\varphi(0,0) + \tfrac{K^4}{720^\beta}(\varphi(0,x) + \varphi(x,-x)) + \tfrac{K^4}{5040^\beta}(\varphi(0,2x) + \varphi(2x,-2x)) \end{aligned} \tag{3.21}$$

for all $x \in X$. Hence

$$\begin{aligned} &\|42f(5x) - 294f(4x) + 882f(3x) - 1428f(2x) - 210504f(x)\|_Y \\ &\quad \leq 42^\beta K\varphi(x,x) + \tfrac{7^\beta K^2}{40^\beta}\varphi(0,0) + \tfrac{7^\beta K^4}{120^\beta}(\varphi(0,x) + \varphi(x,-x)) + \tfrac{K^4}{120^\beta}(\varphi(0,2x) + \varphi(2x,-2x)) \end{aligned} \tag{3.22}$$

for all $x \in X$. By (3.19) and (3.22), we obtain

$$\begin{aligned} &\|56f(4x) - 336f(3x) - 4256f(2x) + 362096f(x)\|_Y \\ &\quad \leq K^4\varphi(4x,x) + K^5\varphi(0,2x) + \tfrac{K^6}{144^\beta}\varphi(0,0) + \tfrac{K^8}{240^\beta}(\varphi(0,2x) + \varphi(2x,-2x)) \\ &\qquad + \tfrac{K^9}{720^\beta}(\varphi(0,4x) + \varphi(4x,-4x)) + \tfrac{K^9}{5040^\beta}(\varphi(0,6x) + \varphi(6x,-6x)) + 7^\beta K^4\varphi(3x,x) \\ &\qquad + \tfrac{K^4}{720^\beta}\varphi(0,0) + 22^\beta K^3\varphi(2x,x) + \tfrac{11^\beta K^4}{360^\beta}\varphi(0,0) + \tfrac{11^\beta K^5}{2520^\beta}(\varphi(0,x) + \varphi(x,-x)) \\ &\qquad + 42^\beta K^2\varphi(x,x) + \tfrac{7^\beta K^3}{40^\beta}\varphi(0,0) + \tfrac{7^\beta K^5}{120^\beta}(\varphi(0,x) + \varphi(x,-x)) + \tfrac{K^5}{120^\beta}(\varphi(0,2x) + \varphi(2x,-2x)) \end{aligned} \tag{3.23}$$

for all $x \in X$. Replacing x and y by 0 and x in (3.3), respectively, one gets

$$\|f(4x) - 7f(3x) + 21f(2x) - 5075f(x) + 35f(0) - 21f(-x) + 7f(-2x) - f(-3x)\|_Y \leq \varphi(0,x) \tag{3.24}$$

for all $x \in X$. By (3.5), (3.8) and (3.24), we obtain

$$\begin{aligned} &\|f(4x) - 6f(3x) + 14f(2x) - 5054f(x)\|_Y \\ &\quad \leq K\varphi(0,x) + \tfrac{K^2}{144^\beta}\varphi(0,0) + \tfrac{K^4}{240^\beta}(\varphi(0,x) + \varphi(x,-x)) + \tfrac{K^5}{720^\beta}(\varphi(0,2x) + \varphi(2x,-2x)) \\ &\qquad + \tfrac{K^5}{5040^\beta}(\varphi(0,3x) + \varphi(3x,-3x)) \end{aligned} \tag{3.25}$$

for all $x \in X$. Thus

$$\begin{aligned} &\|56f(4x) - 336f(3x) + 784f(2x) - 283024f(x)\|_Y \\ &\quad \leq 56^\beta K\varphi(0,x) + \tfrac{7^\beta K^2}{36^\beta}\varphi(0,0) + \tfrac{7^\beta K^4}{30^\beta}(\varphi(0,x) + \varphi(x,-x)) + \tfrac{7^\beta K^5}{90^\beta}(\varphi(0,2x) + \varphi(2x,-2x)) \\ &\qquad + \tfrac{K^5}{5040^\beta}(\varphi(0,3x) + \varphi(3x,-3x)) \end{aligned} \tag{3.26}$$

for all $x \in X$. By (3.23) and (3.26), we obtain $\|f(2x) - 2^7f(x)\|_Y \leq \varphi_s(x)$ for all $x \in X$. By Lemma 3.1, there exists a unique mapping $S : X \to Y$ such that $S(2x) = 2^7S(x)$ and

$$\|f(x) - S(x)\|_Y \leq \frac{1}{128^\beta|1 - L^j|}\varphi_s(x)$$

for all $x \in X$. It remains to show that S is a septic map. By (3.3), we have

$$\|D_sf(2^{jn}x, 2^{jn}y)/128^{jn}\|_Y \leq 128^{-jn\beta}\varphi(2^{jn}x, 2^{jn}y) \leq 128^{-jn\beta}(128^{j\beta}L)^n\varphi(x,y) = L^n\varphi(x,y)$$

for all $x, y \in X$ and $n \in \mathbb{N}$. So $\|D_sS(x,y)\|_Y = 0$ for all $x, y \in X$. Thus the mapping $S : X \to Y$ is septic. $\square$

Corollary 3.3. *Let X be a quasi-α-normed space with quasi-α-norm $\|\cdot\|_X$, Y be a (β,p)-Banach space with (β,p)-norm $\|\cdot\|_Y$. Let δ, λ be positive numbers with $\lambda \neq \frac{7\beta}{\alpha}$, and $f : X \to Y$ be a mapping satisfying*

$$\|D_sf(x,y)\|_Y \leq \delta(\|x\|_X^\lambda + \|y\|_X^\lambda)$$

for all $x, y \in X$. Then there exists a unique septic mapping $S : X \to Y$ such that

$$\|f(x) - S(x)\|_Y \leq \begin{cases} \frac{\delta\varepsilon_\lambda}{128^\beta - 2^{\alpha\lambda}}\|x\|_X^\lambda, & \lambda \in (0, \frac{7\beta}{\alpha}); \\ \frac{2^{\lambda\alpha}\delta\varepsilon_\lambda}{128^\beta(2^{\lambda\alpha} - 128^\beta)}\|x\|_X^\lambda, & \lambda \in (\frac{7\beta}{\alpha}, \infty); \end{cases}$$

for all $x \in X$, where

$$\begin{aligned} \varepsilon_\lambda &= \tfrac{1}{5040^\beta}[K^5(4^{\alpha\lambda} + 1) + K^62^{\alpha\lambda} + 7^\beta K^5(3^{\alpha\lambda} + 1) + 22^\beta K^4(2^{\alpha\lambda} + 1) + 2\cdot 42^\beta K^3 + \tfrac{3\cdot K^{10}6^{\alpha\lambda}}{5040^\beta} \\ &\quad + \tfrac{3\cdot K^{10}4^{\alpha\lambda}}{720^\beta} + 3\cdot 2^{\alpha\lambda}(\tfrac{K^9}{240^\beta} + \tfrac{K^6}{120^\beta} + \tfrac{7^\beta K^6}{90^\beta}) + 56^\beta K^2 + 3(\tfrac{11^\beta K^6}{2520^\beta} + \tfrac{7^\beta K^6}{120^\beta} + \tfrac{7^\beta K^5}{30^\beta}) + \tfrac{3K^63^{\alpha\lambda}}{5040^\beta}]. \end{aligned}$$

The following example shows that the assumption $\lambda \neq \frac{7\beta}{\alpha}$ cannot be omitted in Corollary 3.3. This example is a modification of the example of Gajda [21] for the additive functional inequality (see also [12] and [16]).

Example 3.4. Let $\phi : \mathbb{R} \to \mathbb{R}$ be defined by

$$\phi(x) = \begin{cases} x^7, & \text{for } |x| < 1, \\ 1, & \text{for } |x| \geq 1. \end{cases}$$

Consider the function $f : \mathbb{R} \to \mathbb{R}$ be defined by

$$f(x) = \sum_{n=0}^{\infty} 4^{-7n}\phi(4^n x)$$

for all $x \in \mathbb{R}$. Then f satisfies the functional inequality

$$|D_s f(x,y)| \leq \frac{5168 \cdot 16384^3}{16383}(|x|^7 + |y|^7) \tag{3.27}$$

for all $x, y \in \mathbb{R}$, but there do not exist a septic mapping $S : \mathbb{R} \to \mathbb{R}$ and a constant $d > 0$ such that $|f(x) - S(x)| \leq d\,|x|^7$ for all $x \in \mathbb{R}$.

Proof. It is clear that f is bounded by $16384/16383$ on $\mathbb{R}$. If $|x|^7 + |y|^7 = 0$ or $|x|^7 + |y|^7 \geq 1/16384$, then

$$|D_s f(x,y)| \leq \frac{5168 \cdot 16384}{16383} \leq \frac{5168 \cdot 16384^2}{16383}(|x|^7 + |y|^7).$$

Now suppose that $0 < |x|^5 + |y|^5 < 1/1024$. Then there exists a non-negative integer k such that

$$\frac{1}{16384^{k+2}} \leq |x|^7 + |y|^7 < \frac{1}{16384^{k+1}}. \tag{3.28}$$

Hence $16384^k|x|^7 < 1/16384, 16384^k|y|^7 < 1/16384$, and $4^n(x+3y), 4^n(x+2y), 4^n(x-2y), 4^n(x+y), 4^n(x-y), 4^n x, 4^n y \in (-1,1)$ for all $n = 0, 1, \ldots, k-1$. Hence, for $n = 0, 1, \ldots, k-1$, $D_s\phi(4^n x, 4^n y) = 0$. From the definition of f and the inequality (3.28), we obtain that

$$|D_s f(x,y)| \leq \sum_{n=k}^{\infty} 4^{-7n} \cdot 5168 = \frac{5168 \cdot 4^{7(1-k)}}{16383} \leq \frac{5168 \cdot 16384^3}{16383}(|x|^7 + |y|^7).$$

Therefore, f satisfies (3.27) for all $x, y \in \mathbb{R}$. Now, we claim that the functional equation (1.1) is not stable for $\lambda = 7$ in Corollary 3.3 ($\alpha = \beta = p = 1$). Suppose on the contrary that there exists a septic mapping $S : \mathbb{R} \to \mathbb{R}$ and constant $d > 0$ such that $|f(x) - S(x)| \leq d\,|x|^7$ for all $x \in \mathbb{R}$. Then there exists a constant $c \in \mathbb{R}$ such that $S(x) = cx^7$ for all rational numbers x. So we obtain that

$$|f(x)| \leq (d + |c|)|x|^5 \tag{3.29}$$

for all $x \in \mathbb{Q}$. Let $m \in \mathbb{N}$ with $m + 1 > d + |c|$. If x is a rational number in $(0, 4^{-m})$, then $4^n x \in (0,1)$ for all $n = 0, 1, \ldots, m$, and for this x we get

$$f(x) = \sum_{n=0}^{\infty} \frac{\phi(4^n x)}{4^{7n}} \geq \sum_{n=0}^{m} \frac{(4^n x)^7}{4^{7n}} = (m+1)x^7 > (d + |c|)x^7,$$

which contradicts (3.29).

Theorem 3.5. *Let $j \in \{-1, 1\}$ be fixed, $\varphi : X \times X \to [0, \infty)$ be a function such that there exists an $L < 1$ with $\varphi(2^j x, 2^j y) \leq 256^{j\beta} L\varphi(x,y)$ for all $x, y \in X$. Let $f : X \to Y$ be a mapping satisfying*

$$\|D_o f(x,y)\|_Y \leq \varphi(x,y) \tag{3.30}$$

for all $x, y \in X$. Then there exists a unique octic mapping $O : X \to Y$ such that

$$\|f(x) - O(x)\|_Y \leq \frac{1}{256^\beta |1 - L^j|}\varphi_o(x) \tag{3.31}$$

for all $x \in X$, where

$$\begin{array}{rcl}\varphi_o(x) & = & \frac{1}{20160^\beta}[\frac{K^6}{2^\beta}\varphi(0,2x) + (\frac{K^7}{1152^\beta} + \frac{K^6}{40320^\beta} + \frac{7^\beta K^5}{360^\beta} + \frac{7^\beta K^4}{90^\beta} + \frac{35^\beta K^3}{576^\beta} + \frac{K^6}{630^\beta})\varphi(0,0) \\ & & +35^\beta K^2\varphi(0,x) + 56^\beta K^3\varphi(x,x) + K^6\varphi(4x,x) + 8^\beta K^4\varphi(3x,x) + 28^\beta K^4\varphi(2x,x) \\ & & +(\frac{K^9}{1440^\beta} + \frac{K^7}{90^\beta} + \frac{7^\beta K^6}{288^\beta} + \frac{K^7}{1440^\beta})(\varphi(2x,2x) + \varphi(2x,-2x)) + \frac{K^{11}}{10080^\beta}(\varphi(6x,6x) + \varphi(6x,-6x)) \\ & & +\frac{K^{11}}{80640^\beta}(\varphi(8x,8x) + \varphi(8x,-8x)) + (\frac{K^7}{180^\beta} + \frac{K^7}{5040^\beta} + \frac{7^\beta K^6}{180^\beta} + \frac{7^\beta K^5}{144^\beta})(\varphi(x,x) + \varphi(x,-x)) \\ & & +(\frac{K^7}{720^\beta} + \frac{K^7}{144^\beta})(\varphi(3x,3x) + \varphi(3x,-3x)) + (\frac{K^7}{1152^\beta} + \frac{K^{10}}{2880^\beta})(\varphi(4x,4x) + \varphi(4x,-4x))].\end{array}$$

Proof. Replacing $x = y = 0$ in (3.30), we have

$$\|f(0)\|_Y \leq \frac{1}{40320^\beta}\varphi(0,0). \tag{3.32}$$

Replacing y by $-y$ in (3.30), we get

$$\begin{array}{c}\|f(x-4y) - 8f(x-3y) + 28f(x-2y) - 56f(x-y) + 70f(x) - 56f(x+y) \\ +28f(x+2y) - 8f(x+3y) + f(x+4y) - 40320f(-y)\|_Y \leq \varphi(x,-y)\end{array} \tag{3.33}$$

for all $x, y \in X$. By (3.30) and (3.33), one gets

$$\|f(x) - f(-x)\|_Y \leq \frac{K}{40320^\beta}(\varphi(x,x) + \varphi(x,-x)) \tag{3.34}$$

for all $x \in X$. Replacing x and y by 0 and $2x$ in (3.30), respectively, one obtains

$$\begin{array}{l}\|f(8x) - 8f(6x) + 28f(4x) - 56f(2x) + 70f(0) - 56f(-2x) + 28f(-4x) - 8f(-6x) \\ +f(-8x) - 40320f(2x)\|_Y \leq \varphi(0,2x)\end{array} \tag{3.35}$$

for all $x \in X$. By (3.32), (3.34), and (3.35), we have

$$\begin{array}{l}\|f(8x) - 8f(6x) + 28f(4x) - 20216f(2x)\|_Y \\ \leq \frac{K}{2^\beta}\varphi(0,2x) + \frac{K^2}{1152^\beta}\varphi(0,0) + \frac{K^4}{1440^\beta}(\varphi(2x,2x) + \varphi(2x,-2x)) + \frac{K^5}{2880^\beta}(\varphi(4x,4x) + \varphi(4x,-4x)) \\ +\frac{K^6}{10080^\beta}(\varphi(6x,6x) + \varphi(6x,-6x)) + \frac{K^6}{80640^\beta}(\varphi(8x,8x) + \varphi(8x,-8x))\end{array} \tag{3.36}$$

for all $x \in X$. Replacing x and y by $4x$ and x in (3.30), respectively, we get

$$\|f(8x) - 8f(7x) + 28f(6x) - 56f(5x) + 70f(4x) - 56f(3x) + 28f(2x) + f(0) - 40328f(x)\|_Y \leq \varphi(4x,x) \tag{3.37}$$

for all $x \in X$. Using (3.32), one gets

$$\begin{array}{l}\|f(8x) - 8f(7x) + 28f(6x) - 56f(5x) + 70f(4x) - 56f(3x) + 28f(2x) - 40328f(x)\|_Y \\ \leq K\varphi(4x,x) + \frac{K}{40320^\beta}\varphi(0,0)\end{array} \tag{3.38}$$

for all $x \in X$. By (3.36) and (3.38), we have

$$\begin{array}{l}\|8f(7x) - 36f(6x) + 56f(5x) - 70f(4x) + 56f(3x) - 28f(2x) + 40328f(x)\|_Y \\ \leq \frac{K^2}{2^\beta}\varphi(0,2x) + \frac{K^3}{1152^\beta}\varphi(0,0) + \frac{K^5}{1440^\beta}(\varphi(2x,2x) + \varphi(2x,-2x)) + \frac{K^6}{2880^\beta}(\varphi(4x,4x) + \varphi(4x,-4x)) \\ +\frac{K^7}{10080^\beta}(\varphi(6x,6x) + \varphi(6x,-6x)) + \frac{K^7}{80640^\beta}(\varphi(8x,8x) + \varphi(8x,-8x)) + K^2\varphi(4x,x) + \frac{K^2}{40320^\beta}\varphi(0,0)\end{array} \tag{3.39}$$

for all $x \in X$. Replacing x and y by $3x$ and x in (3.30), respectively, and then using (3.32) and (3.34), one obtains

$$\begin{array}{l}\|8f(7x) - 64f(6x) + 224f(5x) - 448f(4x) + 560f(3x) - 448f(2x) - 322328f(x)\|_Y \\ \leq 8^\beta\varphi(3x,x) + \frac{K^2}{630^\beta}\varphi(0,0) + \frac{K^3}{5040^\beta}(\varphi(x,x) + \varphi(x,-x))\end{array} \tag{3.40}$$

for all $x \in X$. Subtracting (3.39) $-$ (3.40), we obtain

$$\begin{array}{l}\|28f(6x) - 168f(5x) + 406f(4x) - 504f(3x) - 19796f(2x) + 362656f(x)\|_Y \\ \leq \frac{K^3}{2^\beta}\varphi(0,2x) + \frac{K^4}{1152^\beta}\varphi(0,0) + \frac{K^6}{1440^\beta}(\varphi(2x,2x) + \varphi(2x,-2x)) + \frac{K^7}{2880^\beta}(\varphi(4x,4x) + \varphi(4x,-4x)) \\ +\frac{K^8}{10080^\beta}(\varphi(6x,6x) + \varphi(6x,-6x)) + \frac{K^8}{80640^\beta}(\varphi(8x,8x) + \varphi(8x,-8x)) + K^3\varphi(4x,x) + \frac{K^3}{40320^\beta}\varphi(0,0) \\ +8^\beta K\varphi(3x,x) + \frac{K^3}{630^\beta}\varphi(0,0) + \frac{K^4}{5040^\beta}(\varphi(x,x) + \varphi(x,-x))\end{array} \tag{3.41}$$

for all $x \in X$. Replacing x and y by $2x$ and x in (3.30), respectively, and then using (3.32) and (3.34), we have

$$\begin{aligned} &\|28f(6x) - 224f(5x) + 784f(4x) - 1568f(3x) + 1988f(2x) - 1130752f(x)\|_Y \\ &\quad \le 28^\beta K\varphi(2x,x) + \tfrac{7^\beta K^2}{360^\beta}\varphi(0,0) + \tfrac{K^4}{180^\beta}(\varphi(x,x)+\varphi(x,-x)) + \tfrac{K^4}{1440^\beta}(\varphi(2x,2x)+\varphi(2x,-2x)) \end{aligned} \tag{3.42}$$

for all $x \in X$. Subtracting (3.41) − (3.42), one gets

$$\begin{aligned} &\|56f(5x) - 378f(4x) + 1064f(3x) - 21784f(2x) + 1493408f(x)\|_Y \\ &\le \tfrac{K^4}{2^\beta}\varphi(0,2x) + \tfrac{K^5}{1152^\beta}\varphi(0,0) + \tfrac{K^7}{1440^\beta}(\varphi(2x,2x)+\varphi(2x,-2x)) + \tfrac{K^8}{2880^\beta}(\varphi(4x,4x)+\varphi(4x,-4x)) \\ &\quad + \tfrac{K^9}{10080^\beta}(\varphi(6x,6x)+\varphi(6x,-6x)) + \tfrac{K^9}{80640^\beta}(\varphi(8x,8x)+\varphi(8x,-8x)) + K^4\varphi(4x,x) + \tfrac{K^4}{40320^\beta}\varphi(0,0) \\ &\quad + 8^\beta K^2\varphi(3x,x) + \tfrac{K^4}{630^\beta}\varphi(0,0) + \tfrac{K^5}{5040^\beta}(\varphi(x,x)+\varphi(x,-x)) + 28^\beta K^2\varphi(2x,x) + \tfrac{7^\beta K^3}{360^\beta}\varphi(0,0) \\ &\quad + \tfrac{K^5}{180^\beta}(\varphi(x,x)+\varphi(x,-x)) + \tfrac{K^5}{1440^\beta}(\varphi(2x,2x)+\varphi(2x,-2x)) \end{aligned} \tag{3.43}$$

for all $x \in X$. Replacing x and y by x and x in (3.30), respectively, and then using (3.32) and (3.34), we have

$$\begin{aligned} &\|f(5x) - 8f(4x) + 29f(3x) - 64f(2x) - 40222f(x)\|_Y \\ &\quad \le K\varphi(x,x) + \tfrac{K^2}{720^\beta}\varphi(0,0) + \tfrac{K^4}{1440^\beta}(\varphi(x,x)+\varphi(x,-x)) \\ &\qquad + \tfrac{K^5}{5040^\beta}(\varphi(2x,2x)+\varphi(2x,-2x)) + \tfrac{K^5}{40320^\beta}(\varphi(3x,3x)+\varphi(3x,-3x)) \end{aligned} \tag{3.44}$$

for all $x \in X$. Multiply each side of (3.44) by 56^β, one gets

$$\begin{aligned} &\|56f(5x) - 448f(4x) + 1624f(3x) - 3584f(2x) - 2252432f(x)\|_Y \\ &\quad \le 56^\beta K\varphi(x,x) + \tfrac{7^\beta K^2}{90^\beta}\varphi(0,0) + \tfrac{7^\beta K^4}{180^\beta}(\varphi(x,x)+\varphi(x,-x)) \\ &\qquad + \tfrac{K^5}{90^\beta}(\varphi(2x,2x)+\varphi(2x,-2x)) + \tfrac{K^5}{720^\beta}(\varphi(3x,3x)+\varphi(3x,-3x)) \end{aligned} \tag{3.45}$$

for all $x \in X$. By (3.43) and (3.45), we have

$$\begin{aligned} &\|70f(4x) - 560f(3x) - 18200f(2x) + 3745840f(x)\|_Y \\ &\le \tfrac{K^5}{2^\beta}\varphi(0,2x) + \tfrac{K^6}{1152^\beta}\varphi(0,0) + \tfrac{K^8}{1440^\beta}(\varphi(2x,2x)+\varphi(2x,-2x)) + \tfrac{K^9}{2880^\beta}(\varphi(4x,4x)+\varphi(4x,-4x)) \\ &\quad + \tfrac{K^{10}}{10080^\beta}(\varphi(6x,6x)+\varphi(6x,-6x)) + \tfrac{K^{10}}{80640^\beta}(\varphi(8x,8x)+\varphi(8x,-8x)) + K^5\varphi(4x,x) + \tfrac{K^5}{40320^\beta}\varphi(0,0) \\ &\quad + 8^\beta K^3\varphi(3x,x) + \tfrac{K^5}{630^\beta}\varphi(0,0) + \tfrac{K^6}{5040^\beta}(\varphi(x,x)+\varphi(x,-x)) + 28^\beta K^3\varphi(2x,x) + \tfrac{7^\beta K^4}{360^\beta}\varphi(0,0) \\ &\quad + \tfrac{K^6}{180^\beta}(\varphi(x,x)+\varphi(x,-x)) + \tfrac{K^6}{1440^\beta}(\varphi(2x,2x)+\varphi(2x,-2x)) + 56^\beta K^2\varphi(x,x) + \tfrac{7^\beta K^3}{90^\beta}\varphi(0,0) \\ &\quad + \tfrac{7^\beta K^5}{180^\beta}(\varphi(x,x)+\varphi(x,-x)) + \tfrac{K^6}{90^\beta}(\varphi(2x,2x)+\varphi(2x,-2x)) + \tfrac{K^6}{720^\beta}(\varphi(3x,3x)+\varphi(3x,-3x)) \end{aligned} \tag{3.46}$$

for all $x \in X$. Replacing x and y by 0 and x in (3.30), respectively, and then using (3.32) and (3.34), we have

$$\begin{aligned} &\|2f(4x) - 16f(3x) + 56f(2x) - 40432f(x)\|_Y \\ &\quad \le K\varphi(0,x) + \tfrac{K^2}{576^\beta}\varphi(0,0) + \tfrac{K^4}{720^\beta}(\varphi(x,x)+\varphi(x,-x)) + \tfrac{K^5}{1440^\beta}(\varphi(2x,2x)+\varphi(2x,-2x)) \\ &\qquad + \tfrac{K^6}{5040^\beta}(\varphi(3x,3x)+\varphi(3x,-3x)) + \tfrac{K^6}{40320^\beta}(\varphi(4x,4x)+\varphi(4x,-4x)) \end{aligned} \tag{3.47}$$

for all $x \in X$. Multiply each side of (3.47) by 35^β, one gets

$$\begin{aligned} &\|70f(4x) - 560f(3x) + 1960f(2x) - 1415120f(x)\|_Y \\ &\quad \le 35^\beta K\varphi(0,x) + \tfrac{35^\beta K^2}{576^\beta}\varphi(0,0) + \tfrac{7^\beta K^4}{144^\beta}(\varphi(x,x)+\varphi(x,-x)) + \tfrac{7^\beta K^5}{288^\beta}(\varphi(2x,2x)+\varphi(2x,-2x)) \\ &\qquad + \tfrac{K^6}{144^\beta}(\varphi(3x,3x)+\varphi(3x,-3x)) + \tfrac{K^6}{1152^\beta}(\varphi(4x,4x)+\varphi(4x,-4x)) \end{aligned} \tag{3.48}$$

for all $x \in X$. By (3.46) and (3.48), we obtain $\|f(2x) - 2^8 f(x)\|_Y \le \varphi_o(x)$ for all $x \in X$. By Lemma 3.1, there exists a unique mapping $O : X \to Y$ such that $O(2x) = 2^8 O(x)$ and

$$\|f(x) - O(x)\|_Y \le \frac{1}{256^\beta|1 - L^j|}\tilde{\varphi}(x)$$

for all $x \in X$. It remains to show that O is an octic mapping. By (3.30), we have

$$\|D_o f(2^{jn}x, 2^{jn}y)/256^{jn}\|_Y \le 256^{-jn\beta}\varphi(2^{jn}x, 2^{jn}y) \le 256^{-jn\beta}(256^{j\beta}L)^n\varphi(x,y) = L^n\varphi(x,y)$$

for all $x, y \in X$ and $n \in \mathbb{N}$. So $\|D_o O(x,y)\|_Y = 0$ for all $x, y \in X$. Thus the mapping $O : X \to Y$ is octic. □

Corollary 3.6. *Let X be a quasi-α-normed space with quasi-α-norm $\|\cdot\|_X$, Y be a (β,p)-Banach space with (β,p)-norm $\|\cdot\|_Y$. Let δ,λ be positive numbers with $\lambda \neq \frac{8\beta}{\alpha}$, and $f : X \to Y$ be a mapping satisfying $\|D_o f(x,y)\|_Y \leq \delta(\|x\|_X^\lambda + \|y\|_X^\lambda)$ for all $x,y \in X$. Then there exists a unique octic mapping $O : X \to Y$ such that*

$$\|f(x) - O(x)\|_Y \leq \begin{cases} \frac{\delta\varepsilon_\lambda}{256^\beta - 2^{\alpha\lambda}}\|x\|_X^\lambda, & \lambda \in (0, \frac{8\beta}{\alpha}); \\ \frac{2^{\lambda\alpha}\delta\varepsilon_\lambda}{256^\beta(2^{\lambda\alpha} - 256^\beta)}\|x\|_X^\lambda, & \lambda \in (\frac{8\beta}{\alpha}, \infty); \end{cases}$$

for all $x \in X$, where

$$\begin{aligned} \varepsilon_\lambda \;=\; & \tfrac{1}{20160^\beta}\Big[\tfrac{K^6}{2^\beta}2^{\alpha\lambda} + 35^\beta K^2 + 2\cdot 56^\beta K^3 + K^6(4^{\alpha\lambda}+1) + 8^\beta K^4(3^{\alpha\lambda}+1) + 28^\beta K^4(2^{\alpha\lambda}+1) \\ & +4\cdot 2^{\alpha\lambda}\big(\tfrac{K^9}{1440^\beta} + \tfrac{K^7}{90^\beta} + \tfrac{7^\beta K^6}{288^\beta} + \tfrac{K^7}{1440^\beta}\big) + 4\big(\tfrac{K^7}{180^\beta} + \tfrac{K^7}{5040^\beta} + \tfrac{7^\beta K^6}{180^\beta} + \tfrac{7^\beta K^5}{144^\beta}\big) \\ & + \tfrac{4\cdot K^{11}6^{\alpha\lambda}}{10080^\beta} + \tfrac{4\cdot K^{11}8^{\alpha\lambda}}{80640^\beta} + 4\cdot 3^{\alpha\lambda}\big(\tfrac{K^7}{720^\beta} + \tfrac{K^7}{144^\beta}\big) + 4\cdot 4^{\alpha\lambda}\big(\tfrac{K^7}{1152^\beta} + \tfrac{K^{10}}{2880^\beta}\big)\Big]. \end{aligned}$$

Remark 3.7. The Hyers–Ulam stability for the case of $\lambda = \frac{8\beta}{\alpha}$ was excluded in Corollary 3.6 (see Example 3.4).

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J. COMPUTATIONAL ANALYSIS AND APPLICATIONS, VOL. 15, NO.6, 1120-1137, 2013, COPYRIGHT 2013 EUDOXUS PRESS, LLC

Power harmonic operators and their applications in group decision making

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Abstract

The power average (PA) operator, power geometric (PG) operator, power ordered weighted average (POWA) operator and power ordered weighted geometric (POWG) operator are the nonlinear weighted aggregation tools whose weighting vectors depend on input arguments. In this paper, we develop a power harmonic (PH) operator and a power ordered weighted harmonic (POWH) operator, and study some properties of these operators. Then we extends the PH and POWH operators to uncertain environments, i.e, develop an uncertain PH (UPH) operator and its weighted form, and uncertain POWH (UPOWH) operator to aggregate the input arguments taking the form of interval numbers. Moreover, we utilize the weighted PH and POWH operators, respectively, to develop an approach to group decision making based on preference relations and utilize the weighted UPH and UPOWH operators, respectively, to develop an approach to group decision making based on uncertain preference relations. Finally, an example is used to illustrate the applicability of both the developed approaches.

Keywords: Group decision making, power harmonic (PH) operator, power ordered weighted harmonic (POWH) operator, uncertain PH (UPH) operator, uncertain POWH (UPOWH) operator.

2000 AMS Subject Classifications: 90B50, 91B06, 90C29

1 Introduction

Information aggregation is an essential process of gathering relevant information from multiple sources by using a proper aggregation technique. Many techniques, such as the weighted average operator [1], the weighted geometric mean operator [2], harmonic mean operator [3], weighted harmonic mean

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This study was supported by research funds from Dong-A University.

(WHM) operator [3], ordered weighted average (OWA) operator [4], ordered weighted geometric operator [5, 6], weighted OWA operator [7], induced OWA operator [8], induced ordered weighted geometric operator [9], uncertain OWA operator [10], hybrid aggregation operator [11], linguistic aggregation operators [12, 14, 15, 16, 17, 18] and so on have been developed to aggregate data information. However, yet most of existing aggregation operators do not take into account the information about the relationship between the values being fused. Yager [19] introduced a tool to provide more versatility in the information aggregation process, i.e., developed a power average (PA) operator and a power OWA (POWA) operator. In some situations, however, these two operators are unsuitable to deal with the arguments taking the forms of multiplicative variables because of lack of knowledge, or data, and decision makers' limited expertise related to the problem domain. So, based on this tool, Xu and Yager [20] developed additional new geometric aggregation operators, including the power geometric (PG) operator, weighted PG operator and power ordered weighted geometric (POWG) operator, whose weighting vectors depend upon the input arguments and allow values being aggregated to support and reinforce each other. In this paper, we will develop some new harmonic aggregation operators, including the power-harmonic (PH) operator, weighted PH operator, and power-ordered weighted harmonic (POWH) operator, and apply them to group decision making. In order to do this, the remainder of this paper is arranged in following sections. In Section 2, we first review some aggregation operators, including the PA, PG, POWA and POWG operators. Then, we develop a PH operator and its weighted form based on the PA (or PG) operator and the harmonic mean, and a POWH operator based on the POWA (POWG) operator and the harmonic mean, and investigate some of their properties, such as commutativity, idempotency and boundedness. The relationship among the PA, PG and PH operators and the relationship the POWA, POWG and POWH operators are also discussed. In Section 3, we utilize the weighted PH and POWH operators, respectively, to develop an approach to group decision making. In Section 4, we develop an uncertain PH (UPH) operator and its weighted form and uncertain POWH (UPOWH) operator to aggregate the input arguments, which are expressed in interval numbers, and also study the properties of these operators. In Section 5, we utilize the weighted UPH and UPOWH operators, respectively, to develop an approach to group decision making based on uncertain preference relations. Section 6 illustrates the presented approach with a practical example. Section 7 ends the paper with some concluding remarks.

2 Power harmonic operators

Yager [19] introduced a nonlinear weighted average aggregation operation tool, which is called PA operator, and can be defined as follows:

$$\mathrm{PA}(a_1, a_2, \ldots, a_n) = \frac{\sum_{i=1}^{n}(1+T(a_i))a_i}{\sum_{i=1}^{n}(1+T(a_i))} \tag{1}$$

where

$$T(a_i) = \sum_{j=1, j\neq i}^{n} \mathrm{Sup}(a_i, a_j) \tag{2}$$

and $\mathrm{Sup}(a, b)$ is the support for a from b, which satisfies the following three properties: 1) $\mathrm{Sup}(a, b) \in [0, 1]$, 2) $\mathrm{Sup}(a, b) = \mathrm{Sup}(b, a)$, 3) $\mathrm{Sup}(a, b) \geq \mathrm{Sup}(x, y)$ if

$|a-b| < |x-y|$.

Yager [19], based on the OWA operator [4] and PA operator, also defined a POWA operator as follows:

$$\mathrm{POWA}(a_1, a_2, \ldots, a_n) = \sum_{i=1}^{n} u_i a_{\mathrm{index}(i)} \tag{3}$$

where index is an indexing function such that index(i) is the index of the ith largest of the arguments a_j $(j = 1, 2, \ldots, n)$, and thus $a_{\mathrm{index}(i)}$ is the ith largest argument of a_j $(j = 1, 2, \ldots, n)$, and u_i $(i = 1, 2, \ldots, n)$ are a collection of weights such that

$$u_i = g\left(\frac{R_i}{TV}\right) - g\left(\frac{R_{i-1}}{TV}\right),\ R_i = \sum_{j=1}^{i} V_{\mathrm{index}(j)},\ TV = \sum_{i=1}^{n} V_{\mathrm{index}(i)},$$
$$V_{\mathrm{index}(i)} = 1 + T(a_{\mathrm{index}(i)}) \tag{4}$$

where $g : [0, 1] \to [0, 1]$ is a basic unit-interval monotone (BUM) function having the following properties: 1) $g(0) = 0$, 2) $g(1) = 1$, 3) $g(x) \geq g(y)$ if $x > y$, and $T(a_{\mathrm{index}(i)})$ denotes the support of the ith largest argument by all the other arguments, i.e.,

$$T(a_{\mathrm{index}(i)}) = \sum_{j=1, j\neq i}^{n} \mathrm{Sup}(a_{\mathrm{index}(i)}, a_{\mathrm{index}(j)}) \tag{5}$$

where $\mathrm{Sup}(a_{\mathrm{index}(i)}, a_{\mathrm{index}(j)})$ indicates the support of the jth largest argument for the ith largest argument.

Based on the PA operator and the geometric mean, in the following, Xu and Yager [20] defined the PG operator:

$$\mathrm{PG}(a_1, a_2, \ldots, a_n) = \prod_{i=1}^{n} a_i^{\frac{1+T(a_i)}{\sum_{i=1}^{n}(1+T(a_i))}} \tag{6}$$

where a_j $(j = 1, 2, \ldots, n)$ are a collection of arguments, and $T(a_i)$ satisfies the condition (2). Based on the POWA operator and the geometric mean, Xu and Yager [20] also defined the power ordered weighted geometric (POWG) operator as follows:

$$\mathrm{POWG}(a_1, a_2, \ldots, a_n) = \prod_{i=1}^{n} a_{\mathrm{index}(i)}^{u_i} \tag{7}$$

which satisfies the conditions (4) and (5), and $a_{\mathrm{index}(i)}$ is the ith largest argument of a_j $(j = 1, 2, \ldots, n)$.

Based on PA operator and the harmonic mean, in the following, we define a PH operator:

$$\mathrm{PH}(a_1, a_2, \ldots, a_n) = \frac{1}{\sum_{i=1}^{n} \frac{1+T(a_i)}{\sum_{i=1}^{n}(1+T(a_i))a_i}} \tag{8}$$

where a_j $(j = 1, 2, \ldots, n)$ are a collection of arguments, and $T(a_i)$ satisfies the condition (2). Clearly, the PH operator is a nonlinear weighted harmonic aggregation operator, and the weight $\frac{1+T(a_i)}{\sum_{i=1}^{n}(1+T(a_i))}$ of the argument a_i depends on all the input arguments a_j $(j = 1, 2, \ldots, n)$ and allows the argument values to support each other in the harmonic aggregation process.

Lemma 2.1 [22, 23, 24] *Letting $x_i > 0$, $\alpha_i > 0$, $i = 1, 2, \ldots, n$, and $\sum_{i=1}^{n} \alpha_i = 1$, then*

$$\frac{1}{\sum_{i=1}^{n} \frac{\alpha_i}{x_i}} \leq \prod_{i=1}^{n} (x_i)^{\alpha_i} \leq \sum_{i=1}^{n} \alpha_i x_i \tag{9}$$

with equality if and only if $x_1 = x_2 = \cdots = x_n$.

By Lemma 2.1, we have the following theorem.

Theorem 2.2 *Assuming that a_j $(j = 1, 2, \ldots, n)$ are a collection of arguments, then we have*

$$\mathrm{PH}(a_1, a_2, \ldots, a_n) \leq \mathrm{PG}(a_1, a_2, \ldots, a_n) \leq \mathrm{PA}(a_1, a_2, \ldots, a_n). \tag{10}$$

Now, we discuss some properties of the PH operator.

Theorem 2.3 *Letting $\mathrm{Sup}(a_i, a_j) = k$, for all $i \neq j$, then*

$$\mathrm{PH}(a_1, a_2, \ldots, a_n) = \frac{n}{\sum_{i=1}^{n} \frac{1}{a_i}} \tag{11}$$

which indicates that when all supports are the same, the PG operator is simply the harmonic mean.

Especially, if $\mathrm{Sup}(a_i, a_j) = 0$ for all $i \neq j$, i.e., all the supports are zero, then there is no support in the harmonic aggregation process, and in this case, by the condition (2), we have $T(a_i) = 0$, $i = 1, 2, \ldots, n$, then

$$\frac{1 + T(a_i)}{\sum_{i=1}^{n}(1 + T(a_i))} = \frac{1}{n},\ i = 1, 2, \ldots, n \tag{12}$$

and thus, by (8) and (12), it is clear that the PH operator reduces to the harmonic mean.

Theorem 2.4 *Let a_j $(j = 1, 2, \ldots, n)$ be a collection of arguments, then we have the following properties.*

1) (Commutativity): If $(a'_1, a'_2, \ldots, a'_n)$ is any permutation of $(a_1, a_2, \ldots, a_n)$, then

$$\mathrm{PH}(a_1, a_2, \ldots, a_n) = \mathrm{PH}(a'_1, a'_2, \ldots, a'_n). \tag{13}$$

2) (Idempotency): If $a_j = a$ for all j, then

$$\mathrm{PH}(a_1, a_2, \ldots, a_n) = a. \tag{14}$$

3) (Boundedness):

$$\min_i a_i \leq \mathrm{PH}(a_1, a_2, \ldots, a_n) \leq \max_i a_i. \tag{15}$$

In (8), all the objects that are being aggregated are of equal importance. In many situations, the weights of the objects should be taken into account, for example, in group decision making, the decision makers usually have different importance and thus, need to be assigned different weights. Suppose that each object that is being aggregated has a weight indicating its importance, then we define the weighted form of (8) as follows:

$$\mathrm{PH}_w(a_1, a_2, \ldots, a_n) = \frac{1}{\sum_{i=1}^{n} \frac{w_i(1+T'(a_i))}{\sum_{i=1}^{n} w_i(1+T'(a_i))a_i}} \tag{16}$$

where

$$T'(a_i) = \sum_{j=1, j\neq i}^{n} w_j \mathrm{Sup}(a_i, a_j) \tag{17}$$

with the condition

$$w_i \in [0,1],\ i = 1, 2, \ldots, n,\ \sum_{i=1}^{n} w_i = 1. \tag{18}$$

Obviously, the weighted PH operator has the properties, as described in Theorem 2.2, as well as 2) and 3) of Theorem 2.4. However, Theorem 2.3 and 1) of Theorem 2.4 do not hold for the weighted PH operator.

Based on the POWA operator and the harmonic mean, we define a power ordered weighted harmonic (POWH) operator as follows:

$$\mathrm{POWH}(a_1, a_2, \ldots, a_n) = \frac{1}{\sum_{i=1}^{n} \frac{u_i}{a_{\mathrm{index}(i)}}} \tag{19}$$

which satisfies the conditions (4) and (5), and $a_{\mathrm{index}(i)}$ is the ith largest argument of a_j $(j = 1, 2, \ldots, n)$.

Especially, if $g(x) = x$, then the POWH operator reduces to the PH operator, In fact, by (4), we have

$$\begin{aligned} \mathrm{POWH}(a_1, a_2, \ldots, a_n) &= \frac{1}{\sum_{i=1}^{n} \frac{u_i}{a_{\mathrm{index}(i)}}} = \frac{1}{\sum_{i=1}^{n} \frac{g\left(\frac{R_i}{TV}\right) - g\left(\frac{R_{i-1}}{TV}\right)}{a_{\mathrm{index}(i)}}} \\ &= \frac{1}{\sum_{i=1}^{n} \frac{\frac{R_i}{TV} - \frac{R_{i-1}}{TV}}{a_{\mathrm{index}(i)}}} = \frac{1}{\sum_{i=1}^{n} \frac{\frac{V_{\mathrm{index}(i)}}{TV}}{a_{\mathrm{index}(i)}}} \\ &= \frac{1}{\sum_{i=1}^{n} \frac{1+T(a_i)}{\sum_{i=1}^{n}(1+T(a_i))a_i}} \\ &= \mathrm{PH}(a_1, a_2, \ldots, a_n). \end{aligned} \tag{20}$$

By Lemma 2.1, we the following theorem.

Theorem 2.5 *Assuming that a_j $(j = 1, 2, \ldots, n)$ are a collection of arguments, then we have*

$$\mathrm{POWH}(a_1, a_2, \ldots, a_n) \leq \mathrm{POWG}(a_1, a_2, \ldots, a_n) \leq \mathrm{POWA}(a_1, a_2, \ldots, a_n). \tag{21}$$

From Theorem 2.3 and (20), we have the following corollary.

Corollary 2.6 *Letting* $\mathrm{Sup}(a_i, a_j) = k$ *for all* $i \neq j$*, and* $g(x) = x$*, then we have*

$$\mathrm{POWH}(a_1, a_2, \ldots, a_n) = \frac{n}{\sum_{i=1}^{n} \frac{1}{a_i}} \tag{22}$$

which indicates that when all supports are the same, the POWH operator is simply the harmonic mean.

Similar to Theorem 2.4, we have the following theorem.

Theorem 2.7 *Let* a_j $(j = 1, 2, \ldots, n)$ *be a collection of arguments, then we have the following properties.*

1) (Commutativity): If $(a_1', a_2', \ldots, a_n')$ *is any permutation of* $(a_1, a_2, \ldots, a_n)$*, then*

$$\mathrm{POWH}(a_1, a_2, \ldots, a_n) = \mathrm{POWH}(a_1', a_2', \ldots, a_n'). \tag{23}$$

2) (Idempotency): If $a_j = a$ *for all* j*, then*

$$\mathrm{POWH}(a_1, a_2, \ldots, a_n) = a. \tag{24}$$

3) (Boundedness):

$$\min_i a_i \leq \mathrm{POWH}(a_1, a_2, \ldots, a_n) \leq \max_i a_i. \tag{25}$$

From the above-mentioned theoretical analysis, the difference between the weighted PH and POWH operators is that the weighted PH operator emphasizes the importance of each argument, while the POWH operator weights the importance of the ordered position of each argument.

3 Approach to group decision making

Let us consider a group decision making problem. Let $X = \{x_1, x_2, \ldots, x_n\}$ be a finite set of alternatives and let $D = \{d_1, d_2, \ldots, d_m\}$ be a set of decision makers, whose weight vector is $w = (w_1, w_2, \ldots, w_m)^T$, with $w_k \geq 0$, $k = 1, 2, \ldots, m$, and $\sum_{k=1}^{m} w_k = 1$. The decision maker d_k compare each pair of alternatives (x_i, x_j) and provides his/her preference value $a_{ij}^{(k)}$ over them and constructs the preference relation A_k on the set X, which is defined as a matrix $A_k = (a_{ij}^{(k)})_{n\times n}$ under the following condition:

$$a_{ij}^{(k)} \geq 0, \quad a_{ij}^{(k)} + a_{ji}^{(k)} = 1, \quad a_{ii}^{(k)} = \frac{1}{2}, \quad \text{for all } i, j = 1, 2, \ldots, n. \tag{26}$$

Then, we utilize the weighted PH operator to develop an approach to group decision making based on preference relations, which involves the following steps.

Approach I.
Step 1: Calculate the supports

$$\mathrm{Sup}(a_{ij}^{(k)}, a_{ij}^{(l)}) = 1 - \frac{|a_{ij}^{(k)} - a_{ij}^{(l)}|}{\sum_{l=1, l\neq k}^{m} |a_{ij}^{(k)} - a_{ij}^{(l)}|}, \ l = 1, 2, \ldots, m \tag{27}$$

which satisfy the support condition 1)-3) in Section 2.

Especially, if $\sum_{l=1,l\neq k}^{m} |a_{ij}^{(k)} - a_{ij}^{(l)}| = 0$, then we stipulate $\mathrm{Sup}(a_{ij}^{(k)}, a_{ij}^{(l)}) = 1$.

Step 2: Utilize the weights w_k $(k = 1, 2, \ldots, m)$ of the decision makers d_k $(k = 1, 2, \ldots, m)$ to calculate the weighted support $T'(a_{ij}^{(k)})$ of the preference value $a_{ij}^{(k)}$ by the other preference values $a_{ij}^{(l)}$ $(l = 1, 2, \ldots, m$, and $l \neq k)$

$$T'(a_{ij}^{(k)}) = \sum_{l=1, l\neq k}^{m} w_l \mathrm{Sup}(a_{ij}^{(k)}, a_{ij}^{(l)}) \tag{28}$$

and calculate the weights $v_{ij}^{(k)}$ $(k = 1, 2, \ldots, m)$ associated with the preference values $a_{ij}^{(k)}$ $(k = 1, 2, \ldots, m)$

$$v_{ij}^{(k)} = \frac{w_k \left(1 + T'(a_{ij}^{(k)})\right)}{\sum_{k=1}^{m} w_k \left(1 + T'(a_{ij}^{(k)})\right)}, \ k = 1, 2, \ldots, m \tag{29}$$

where $v_{ij}^{(k)} \geq 0$, $k = 1, 2, \ldots, m$, and $\sum_{k=1}^{m} v_{ij}^{(k)} = 1$.

Step 3: Utilize the weighted PH operator to aggregate all the individual preference relations $A_k = (a_{ij}^{(k)})_{n\times n}$ $(k = 1, 2, \ldots, m)$ into the collective preference relation $A = (a_{ij})_{n\times n}$, where

$$a_{ij} = \mathrm{PH}_w(a_{ij}^{(1)}, a_{ij}^{(2)}, \ldots, a_{ij}^{(m)}) = \frac{1}{\sum_{k=1}^{m} \frac{v_{ij}^{(k)}}{a_{ij}^{(k)}}}, \quad i, j = 1, 2, \ldots, n. \tag{30}$$

Step 4: Utilize the normalizing rank aggregation method (NRAM) [25] given by

$$v_i = \frac{\sum_{j=1}^{n} a_{ij}}{\sum_{i=1}^{n}\sum_{j=1}^{n} a_{ij}}, \ i = 1, 2, \ldots, n \tag{31}$$

to derive the priority vector $v = (v_1, v_2, \ldots, v_n)^T$ of $A = (a_{ij})_{n\times n}$, where $v_i > 0$, $i = 1, 2, \ldots, n$, and $\sum_{i=1}^{n} v_i = 1$.

Step 5: Rank all alternatives x_i $(i = 1, 2, \ldots, n)$ in accordance with the priority weights v_i $(i = 1, 2, \ldots, n)$. The more the wight v_i, the better the alternative x_i will be.

In the case where the information about the weights of decision makers is unknown, then we utilize the POWH operator to develop an approach to group decision making based on preference relations, which can be described as follows.

Approach II.

Step 1: Calculate the supports

$$\mathrm{Sup}\big(a_{ij}^{\mathrm{index}(k)}, a_{ij}^{\mathrm{index}(l)}\big) = 1 - \frac{\big|a_{ij}^{\mathrm{index}(k)} - a_{ij}^{\mathrm{index}(l)}\big|}{\sum_{l=1,l\neq k}^{m} \big|a_{ij}^{\mathrm{index}(k)} - a_{ij}^{\mathrm{index}(l)}\big|},\ l = 1, 2, \ldots, m \quad (32)$$

which indicates the support of the lth largest preference value $a_{ij}^{\mathrm{index}(l)}$ for the kth largest preference value $a_{ij}^{\mathrm{index}(k)}$ of $a_{ij}^{(s)}$ $(s = 1, 2, \ldots, m)$. Especially, if $\sum_{l=1,l\neq k}^{m} |a_{ij}^{\mathrm{index}(k)} - a_{ij}^{\mathrm{index}(l)}| = 0$, then we stipulate $\mathrm{Sup}(a_{ij}^{\mathrm{index}(k)}, a_{ij}^{\mathrm{index}(l)}) = 1$. It is necessary to point out that the support measure is a similarity measure, which can be used to measure the degree that a preference value provided by a decision maker is close to another one provided by other decision maker in a group decision making problem. Thus, $\mathrm{Sup}\big(a_{ij}^{\mathrm{index}(k)}, a_{ij}^{\mathrm{index}(l)}\big)$ denotes the similarity degree between the kth largest preference value $a_{ij}^{\mathrm{index}(k)}$ and the lth largest preference value $a_{ij}^{\mathrm{index}(l)}$.

Step 2: Calculate the support $T(a_{ij}^{\mathrm{index}(k)})$ of the kth largest preference value $a_{ij}^{\mathrm{index}(k)}$ by the other preference values $a_{ij}^{(l)}$ $(l = 1, 2, \ldots, m$, and $l \neq k)$

$$T(a_{ij}^{\mathrm{index}(k)}) = \sum_{l=1,l\neq k}^{m} \mathrm{Sup}(a_{ij}^{\mathrm{index}(k)}, a_{ij}^{\mathrm{index}(l)}) \quad (33)$$

and by (4), calculate the weight $u_{ij}^{(k)}$ associated with the kth largest preference value $a_{ij}^{\mathrm{index}(k)}$, where

$$u_{ij}^{(k)} = g\left(\frac{R_{ij}^{(k)}}{TV_{ij}}\right) - g\left(\frac{R_{ij}^{(k-1)}}{TV_{ij}}\right),\ R_{ij}^{(k)} = \sum_{l=1}^{k} V_{ij}^{\mathrm{index}(l)},$$

$$TV_{ij} = \sum_{l=1}^{m} V_{ij}^{\mathrm{index}(l)}, V_{ij}^{\mathrm{index}(l)} = 1 + T(a_{ij}^{\mathrm{index}(l)}) \quad (34)$$

where $u_{ij}^{(k)} \geq 0$, $k = 1, 2, \ldots, m$, and $\sum_{k=1}^{m} u_{ij}^{(k)} = 1$, and g is the BUM function described in Section 2.

Step 3: Utilize the POWH operator to aggregate all the individual preference relations $A_k = (a_{ij}^{(k)})_{n\times n}$ $(k = 1, 2, \ldots, m)$ into the collective preference relation $A = (a_{ij})_{n\times n}$, where

$$a_{ij} = \mathrm{POWH}(a_{ij}^{(1)}, a_{ij}^{(2)}, \ldots, a_{ij}^{(m)}) = \frac{1}{\sum_{k=1}^{m} \frac{u_{ij}^{(k)}}{a_{ij}^{\mathrm{index}(k)}}},\quad i, j = 1, 2, \ldots, n. \quad (35)$$

Step 4: For this step, see Approach I.

Step 5: For this step, see Approach I.

In the above-mentioned two approaches, Approach I considers the situations where the weighted PH operator to aggregate all the individual preference relations into the collective preference relation and then the NRAM method to derive its priority vector, and using this, we can rank and select the given alternatives. While Approach II considers the situations where the information about the weights of decision makers is unknown and utilizes the POWH operator to aggregate all the individual preference relations into collective preference relation, then it also uses the NRAM method to find the final decision result.

4 Uncertain power harmonic operators

In this section, we consider the situations where the input arguments cannot be expressed in exact numerical values, but value range (i.e., interval numbers) can be obtained. We first review some operational laws, which are related to interval numbers [26, 27].

Let $\tilde{a} = [a^L, a^U]$ and $\tilde{b} = [b^L, b^U]$ be two interval numbers, where $a^U \geq a^L > 0$ and $b^U \geq b^L > 0$, then we have the following operational laws.

1) $\tilde{a} + \tilde{b} = [a^L, a^U] + [b^L, b^U] = [a^L + b^L, a^U + b^U]$.

2) $\tilde{a}\tilde{b} = [a^L, a^U] \cdot [b^L, b^U] = [a^l b^L, a^U, b^U]$.

3) $\lambda\tilde{a} = \lambda[a^L, a^U] = [\lambda a^L, \lambda a^U]$, where $\lambda > 0$.

4) $\frac{1}{\tilde{a}} = \frac{1}{[a^L, a^U]} = [\frac{1}{a^U}, \frac{1}{a^L}]$.

5) $\frac{\tilde{a}}{\tilde{b}} = \frac{[a^L, a^U]}{[b^L, b^U]} = [\frac{a^L}{b^U}, \frac{a^U}{b^L}]$.

In order to rank interval numbers, we now introduce a possibility degree formula [28] for the comparison between the interval numbers $\tilde{a} = [a^L, a^U]$ and $\tilde{b} = [b^L, b^U]$

$$p(\tilde{a} \geq \tilde{b}) = \min\left\{\max\left(\frac{a^U - b^L}{a^U - a^L + b^U - b^L}, 0\right), 1\right\} \tag{36}$$

where $p(\tilde{a} \geq \tilde{b})$ is called the possibility degree of $\tilde{a} \geq \tilde{b}$, which satisfies

$$0 \leq p(\tilde{a} \geq \tilde{b}) \leq 1,\ p(\tilde{a} \geq \tilde{b}) + p(\tilde{b} \geq \tilde{a}) = 1,\ p(\tilde{a} \geq \tilde{a}) = 0.5. \tag{37}$$

Let $\tilde{a}_j = [a_j^L, a_j^U]$ $(j = 1, 2, \ldots, n)$ be a collection of interval numbers, then based on the previous operational laws of interval numbers, we extend the PH operator to uncertain environments and define an UPH operator as follows:

$$\mathrm{UPH}(\tilde{a}_1, \tilde{a}_2, \ldots, \tilde{a}_n) = \frac{1}{\sum_{i=1}^{n} \frac{1+T(\tilde{a}_i)}{\sum_{i=1}^{n}(1+T(\tilde{a}_i))\tilde{a}_i}} \tag{38}$$

where

$$T(\tilde{a}_i) = \sum_{j=1, j\neq i}^{n} \mathrm{Sup}(\tilde{a}_i, \tilde{a}_j) \tag{39}$$

and $\mathrm{Sup}(\tilde{a}, \tilde{b})$ is the support for $\tilde{a}$ from $\tilde{b}$, which satisfies the following three properties: 1) $\mathrm{Sup}(\tilde{a}, \tilde{b}) \in [0, 1]$, 2) $\mathrm{Sup}(\tilde{a}, \tilde{b}) = \mathrm{Sup}(\tilde{b}, \tilde{a})$, 3) $\mathrm{Sup}(\tilde{a}, \tilde{b}) \geq \mathrm{Sup}(\tilde{x}, \tilde{y})$ if $d(\tilde{a}, \tilde{b}) < d(\tilde{x}, \tilde{y})$, where d is a distance measure.

Similar to the PH operator, the UPH operator has the following properties.

Theorem 4.1 *Letting* $\mathrm{Sup}(\tilde{a}_i, \tilde{a}_j) = k$ *for all* $i \neq j$, *then*

$$\mathrm{UPH}(\tilde{a}_1, \tilde{a}_2, \ldots, \tilde{a}_n) = \frac{n}{\sum_{i=1}^{n} \frac{1}{\tilde{a}_i}} \tag{40}$$

which indicates that when all the supports are the same, the UPH operator is simply the uncertain harmonic mean.

Theorem 4.2 *Let* $\tilde{a}_j$ $(j = 1, 2, \ldots, n)$ *be a collection of interval numbers, then we have the following properties.*

1) (Commutativity): If $(\tilde{a}'_1, \tilde{a}'_2, \ldots, \tilde{a}'_n)$ *is any permutation of* $(\tilde{a}_1, \tilde{a}_2, \ldots, \tilde{a}_n)$, *then*

$$\mathrm{UPH}(\tilde{a}_1, \tilde{a}_2, \ldots, \tilde{a}_n) = \mathrm{UPH}(\tilde{a}'_1, \tilde{a}'_2, \ldots, \tilde{a}'_n). \tag{41}$$

2) (Idempotency): If $\tilde{a}_j = \tilde{a}$ *for all* j, *then*

$$\mathrm{UPH}(\tilde{a}_1, \tilde{a}_2, \ldots, \tilde{a}_n) = \tilde{a}. \tag{42}$$

3) (Boundedness):

$$\min_i \tilde{a}_i \leq \mathrm{UPH}(\tilde{a}_1, \tilde{a}_2, \ldots, \tilde{a}_n) \leq \max_i \tilde{a}_i. \tag{43}$$

If the weights of the objects are taken into account, then we define the weighted form of (38) as follows:

$$\mathrm{UPH}_w(\tilde{a}_1, \tilde{a}_2, \ldots, \tilde{a}_n) = \frac{1}{\sum_{i=1}^{n} \frac{w_i(1+T'(\tilde{a}_i))}{\sum_{i=1}^{n} w_i(1+T'(\tilde{a}_i))\tilde{a}_i}} \tag{44}$$

where

$$T'(\tilde{a}_i) = \sum_{j=1, j\neq i}^{n} w_j \mathrm{Sup}(\tilde{a}_i, \tilde{a}_j) \tag{45}$$

with the condition

$$w_i \in [0, 1],\ i = 1, 2, \ldots, n,\ \sum_{i=1}^{n} w_i = 1. \tag{46}$$

Obviously, the weighted UPH operator has the properties of 2) and 3) in Theorem 4.2. However, Theorem 4.1 and 1) of Theorem 4.2 do not hold for the weighted UPH operator.

Based on the POWH operator and the possibility degree formula, we define a UPOWH operator as follows:

$$\mathrm{UPOWH}(\tilde{a}_1, \tilde{a}_2, \ldots, \tilde{a}_n) = \frac{1}{\sum_{i=1}^{n} \frac{u_i}{\tilde{a}_{\mathrm{index}(i)}}} \tag{47}$$

where $\tilde{a}_{\mathrm{index}(i)}$ is the ith largest interval number of $\tilde{a}_j$ $(j = 1, 2, \ldots, n)$, and

$$u_i = g\left(\frac{R_i}{TV}\right) - g\left(\frac{R_{i-1}}{TV}\right),\ R_i = \sum_{j=1}^{i} V_{\mathrm{index}(j)},$$

$$TV = \sum_{i=1}^{n} V_{\mathrm{index}(i)},\ V_{\mathrm{index}(j)} = 1 + T(\tilde{a}_{\mathrm{index}(i)}) \tag{48}$$

and $T(\tilde{a}_{\text{index}(i)})$ denotes the support of the ith largest interval number by all the other interval numbers, i.e.,

$$T(\tilde{a}_{\text{index}(i)}) = \sum_{j=1}^{n} \text{Sup}(\tilde{a}_{\text{index}(i)}, \tilde{a}_{\text{index}(j)}) \tag{49}$$

where $\text{Sup}(\tilde{a}_{\text{index}(i)}, \tilde{a}_{\text{index}(j)})$ indicates the support of the jth largest interval number for the ith largest interval number (here, we can use the possibility degree formula (36) to rank interval numbers).

Especially, if $g(x) = x$, then the UPOWH operator reduces to the UPH operator.

From Theorem 4.1, we have the following corollary.

Corollary 4.3 *Letting* $\text{Sup}(\tilde{a}_{\text{index}(i)}, \tilde{a}_{\text{index}(j)}) = k$ *for all* $i \neq j$*, and* $g(x) = x$*, then*

$$\text{UPOWH}(\tilde{a}_1, \tilde{a}_2, \ldots, \tilde{a}_n) = \frac{n}{\sum_{i=1}^{n} \frac{1}{\tilde{a}_i}} \tag{50}$$

which indicates that when the supports are the same, the UPOWH operator is simply the uncertain harmonic mean.

Similar to Theorem 4.2, we have the following theorem.

Theorem 4.4 *Let* $\tilde{a}_j$ $(j = 1, 2, \ldots, n)$ *be a collection of interval numbers, then we have the following properties.*

1) (Commutativity): If $(\tilde{a}'_1, \tilde{a}'_2, \ldots, \tilde{a}'_n)$ *is any permutation of* $(\tilde{a}_1, \tilde{a}_2, \ldots, \tilde{a}_n)$*, then*

$$\text{UPOWH}(\tilde{a}_1, \tilde{a}_2, \ldots, \tilde{a}_n) = \text{UPOWH}(\tilde{a}'_1, \tilde{a}'_2, \ldots, \tilde{a}'_n). \tag{51}$$

2) (Idempotency): If $\tilde{a}_j = \tilde{a}$ *for all* j*, then*

$$\text{UPOWH}(\tilde{a}_1, \tilde{a}_2, \ldots, \tilde{a}_n) = \tilde{a}. \tag{52}$$

3) (Boundedness):

$$\min_i \tilde{a}_i \leq \text{UPOWH}(\tilde{a}_1, \tilde{a}_2, \ldots, \tilde{a}_n) \leq \max_i \tilde{a}_i. \tag{53}$$

5 Approach to group decision making based on uncertain preference relations

As mentioned in Section 3, in this section, we will apply the weighted UPH and UPOWH operators to group decision making based on uncertain preference relations. Let $X = \{x_1, x_2, \ldots, x_n\}$ be a finite set of alternatives and let $D = \{d_1, d_2, \ldots, d_m\}$ be a set of decision makers, whose weight vector is $w = (w_1, w_2, \ldots, w_m)^T$, with $w_k \geq 0$, $k = 1, 2, \ldots, m$, and $\sum_{k=1}^{m} w_k = 1$. The decision maker d_k compare each pair of alternatives (x_i, x_j) and provides his/her preference value range $\tilde{a}_{ij}^{(k)} = [a_{ij}^{L(k)}, a_{ij}^{U(k)}]$ over them and constructs the uncertain preference relation $\tilde{A}_k$ on the set X, which is defined as a matrix $\tilde{A}_k = (\tilde{a}_{ij}^{(k)})_{n\times n}$ under the following condition:

$$a_{ij}^{U(k)} \geq a_{ij}^{L(k)} > 0, \quad a_{ij}^{L(k)} + a_{ji}^{U(k)} = 1, \quad a_{ji}^{L(k)} + a_{ij}^{U(k)} = 1,$$
$$a_{ii}^{L(k)} = a_{ii}^{U(k)} = \frac{1}{2}, \quad i, j = 1, 2, \ldots, n. \tag{54}$$

Then, we utilize the weighted UPH operator to develop an approach to group decision making based on uncertain preference relations, which involves the following steps.

Approach III.

Step 1: Calculate the supports

$$\mathrm{Sup}(\tilde{a}_{ij}^{(k)}, \tilde{a}_{ij}^{(l)}) = 1 - \frac{d\big(\tilde{a}_{ij}^{(k)}, \tilde{a}_{ij}^{(l)}\big)}{\sum_{l=1, l\neq k}^{m} d\big(\tilde{a}_{ij}^{(k)}, \tilde{a}_{ij}^{(l)}\big)}, \; l = 1, 2, \ldots, m \tag{55}$$

which satisfy the support condition 1)-3) in Section 4. Here, without loss of generality, we let

$$d\big(\tilde{a}_{ij}^{(k)}, \tilde{a}_{ij}^{(l)}\big) = \frac{1}{2}\big(\big|a_{ij}^{L(l)} - a_{ij}^{L(k)}\big| + \big|a_{ij}^{U(l)} - a_{ij}^{U(k)}\big|\big). \tag{56}$$

Especially, if $\sum_{l=1, l\neq k}^{m} d(\tilde{a}_{ij}^{(k)}, \tilde{a}_{ij}^{(l)}) = 0$, then we stipulate $\mathrm{Sup}(\tilde{a}_{ij}^{(k)}, \tilde{a}_{ij}^{(l)}) = 1$.

Step 2: Utilize the weights w_k $(k = 1, 2, \ldots, m)$ of the decision makers d_k $(k = 1, 2, \ldots, m)$ to calculate the weighted support $T'(\tilde{a}_{ij}^{(k)})$ of the uncertain preference value $\tilde{a}_{ij}^{(k)}$ by the other uncertain preference values $\tilde{a}_{ij}^{(l)}$ $(l = 1, 2, \ldots, m$, and $l \neq k)$

$$T'(\tilde{a}_{ij}^{(k)}) = \sum_{l=1, l\neq k}^{m} w_l \mathrm{Sup}(\tilde{a}_{ij}^{(k)}, \tilde{a}_{ij}^{(l)}) \tag{57}$$

and calculate the weights $\dot{v}_{ij}^{(k)}$ $(k = 1, 2, \ldots, m)$ associated with the uncertain preference values $\tilde{a}_{ij}^{(k)}$ $(k = 1, 2, \ldots, m)$

$$\dot{v}_{ij}^{(k)} = \frac{w_k\left(1 + T'(\tilde{a}_{ij}^{(k)})\right)}{\sum_{k=1}^{m} w_k\left(1 + T'(\tilde{a}_{ij}^{(k)})\right)}, \; k = 1, 2, \ldots, m \tag{58}$$

where $\dot{v}_{ij}^{(k)} \geq 0$, $k = 1, 2, \ldots, m$, and $\sum_{k=1}^{m} \dot{v}_{ij}^{(k)} = 1$.

Step 3: Utilize the weighted UPH operator to aggregate all the individual uncertain preference relations $\tilde{A}_k = (\tilde{a}_{ij}^{(k)})_{n\times n}$ $(k = 1, 2, \ldots, m)$ into the collective uncertain preference relation $\tilde{A} = (\tilde{a}_{ij})_{n\times n}$, where

$$\tilde{a}_{ij} = [a_{ij}^{l}, a_{ij}^{U}] = \mathrm{UPH}_w(\tilde{a}_{ij}^{(1)}, \tilde{a}_{ij}^{(2)}, \ldots, \tilde{a}_{ij}^{(m)})$$
$$= \frac{1}{\sum_{k=1}^{m} \frac{\dot{v}_{ij}^{(k)}}{\tilde{a}_{ij}^{(k)}}}, \; i, j = 1, 2, \ldots, n. \tag{59}$$

Step 4: Utilize the uncertain NRAM (UNRAM) given by

$$\tilde{v}_i = \frac{\sum_{j=1}^{n} \tilde{a}_{ij}}{\sum_{i=1}^{n} \sum_{j=1}^{n} \tilde{a}_{ij}}, \; i = 1, 2, \ldots, n \tag{60}$$

to derive the uncertain priority vector $\tilde{v} = (\tilde{v}_1, \tilde{v}_2, \ldots, \tilde{v}_n)^T$ of $\tilde{A} = (\tilde{a}_{ij})_{n\times n}$.

Step 5: Compare each pair of the uncertain priority weights $\tilde{v}_i$ $(i = 1, 2, \ldots, n)$ by using the possibility degree formula (36) and construct a possibility degree matrix $P = (p_{ij})_{n\times n}$, where $p_{ij} = p(\tilde{v}_i \geq \tilde{v}_j)$, $i, j = 1, 2, \ldots, n$, which satisfy $p_{ij} \geq 0$ $p_{ij} + p_{ji} = 1$, $p_{ii} = 0.5$, $i, j = 1, 2, \ldots, n$. Summing all the elements in each line of the matrix P, we get

$$p_i = \sum_{j=1}^{n} p_{ij}, \quad i = 1, 2, \ldots, n. \tag{61}$$

Then we rank the uncertain priority weights $\tilde{v}_i$ $(i = 1, 2, \ldots, n)$ in descending order in accordance with p_i $(i = 1, 2, \ldots, n)$.

Step 6: Rank all alternatives x_i $(i = 1, 2, \ldots, n)$ in accordance with the descending order of the uncertain priority weights $\tilde{v}_i$ $(i = 1, 2, \ldots, n)$.

In the case where the information about the weights of decision makers is unknown, then we utilize the UPOWH operator to develop an approach to group decision making based on uncertain preference relations, which can be described as follows.

Approach IV.

Step 1: Calculate the supports

$$\mathrm{Sup}\big(\tilde{a}_{ij}^{\mathrm{index}(k)}, \tilde{a}_{ij}^{\mathrm{index}(l)}\big) = 1 - \frac{d\big(\tilde{a}_{ij}^{\mathrm{index}(k)}, \tilde{a}_{ij}^{\mathrm{index}(l)}\big)}{\sum_{l=1, l\neq k}^{m} d\big(\tilde{a}_{ij}^{\mathrm{index}(k)}, \tilde{a}_{ij}^{\mathrm{index}(l)}\big)}, \; l = 1, 2, \ldots, m \tag{62}$$

which indicates the support of lth largest uncertain preference value $\tilde{a}_{ij}^{\mathrm{index}(l)}$ for the kth largest uncertain preference value $\tilde{a}_{ij}^{\mathrm{index}(k)}$ of $\tilde{a}_{ij}^{(s)}$ $(s = 1, 2, \ldots, m)$ (here, we can use Step 5 of Approach III to rank uncertain preference values). Especially, if $\sum_{l=1, l\neq k}^{m} d(\tilde{a}_{ij}^{\mathrm{index}(k)}, \tilde{a}_{ij}^{\mathrm{index}(l)}) = 0$, then we stipulate $\mathrm{Sup}(\tilde{a}_{ij}^{\mathrm{index}(k)}, \tilde{a}_{ij}^{\mathrm{index}(l)}) = 1$.

Step 2: Calculate the support $T(\tilde{a}_{ij}^{\mathrm{index}(k)})$ of the kth largest uncertain preference value $\tilde{a}_{ij}^{\mathrm{index}(k)}$ by the other uncertain preference values $\tilde{a}_{ij}^{(l)}$ $(l = 1, 2, \ldots, m$, and $l \neq k)$

$$T(\tilde{a}_{ij}^{\mathrm{index}(k)}) = \sum_{l=1, l\neq k}^{m} \mathrm{Sup}(\tilde{a}_{ij}^{\mathrm{index}(k)}, \tilde{a}_{ij}^{\mathrm{index}(l)}) \tag{63}$$

and by (48), calculate the weight $\dot{u}_{ij}^{(k)}$ associated with the kth largest uncertain preference value $\tilde{a}_{ij}^{\mathrm{index}(k)}$, where

$$\dot{u}_{ij}^{(k)} = g\left(\frac{\dot{R}_{ij}^{(k)}}{TV'_{ij}}\right) - g\left(\frac{\dot{R}_{ij}^{(k-1)}}{TV'_{ij}}\right), \; \dot{R}_{ij}^{(k)} = \sum_{l=1}^{k} V_{ij}^{\mathrm{index}(l)},$$

$$TV'_{ij} = \sum_{l=1}^{m} V_{ij}^{\text{index}(l)},\ V_{ij}^{\text{index}(l)} = 1 + T(\tilde{a}_{ij}^{\text{index}(l)}) \tag{64}$$

where $\dot{u}_{ij}^{(k)} \geq 0$, $k = 1, 2, \ldots, m$, and $\sum_{k=1}^{m} \dot{u}_{ij}^{(k)} = 1$, and g is the BUM function described in Section 2.

Step 3: Utilize the UPOWH operator to aggregate all the individual uncertain preference relations $\tilde{A}_k = (\tilde{a}_{ij}^{(k)})_{n\times n}$ $(k = 1, 2, \ldots, m)$ into the collective uncertain preference relation $\tilde{A} = (\tilde{a}_{ij})_{n\times n}$, where

$$\begin{aligned} \tilde{a}_{ij} &= [a_{ij}^L, a_{ij}^U] = \text{UPOWH}(\tilde{a}_{ij}^{(1)}, \tilde{a}_{ij}^{(1)}, \ldots, \tilde{a}_{ij}^{(m)}) \\ &= \frac{1}{\sum_{k=1}^{m} \frac{\dot{u}_{ij}^{(k)}}{\tilde{a}_{ij}^{\text{index}(k)}}},\ i, j = 1, 2, \ldots, n. \end{aligned} \tag{65}$$

Step 4: For this step, see Approach III.
Step 5: For this step, see Approach III.
Step 6: For this step, see Approach III.

6 Illustrative example

Four university students share a house, where they intend to have broadband Internet connection installed (adapted from [20, 29]). There are four options available to choose from, which are provided by three Internet service providers.

1) Option 1 (x_1): 1 Mbps broadband;
2) Option 2 (x_2): 2 Mbps broadband;
3) Option 3 (x_3): 3 Mbps broadband;
4) Option 4 (x_4): 8 Mbps broadband.

Since the Internet service and its monthly bill will be shared among the four students d_k $(k = 1, 2, 3, 4)$ (whose weight vector $w = (0.3, 0.3, 0.2, 0.2)^T$), they decide to perform a group decision analysis. Suppose that the students reveal their preference relations for the options independently and anonymously and construct the following preference relations, respectively:

$$A_1 = \begin{pmatrix} 0.5 & 0.4 & 0.5 & 0.8 \\ 0.6 & 0.5 & 0.8 & 0.9 \\ 0.5 & 0.2 & 0.5 & 0.6 \\ 0.2 & 0.1 & 0.4 & 0.5 \end{pmatrix},\quad A_2 = \begin{pmatrix} 0.5 & 0.8 & 0.7 & 0.4 \\ 0.2 & 0.5 & 0.6 & 0.6 \\ 0.3 & 0.4 & 0.5 & 0.8 \\ 0.6 & 0.4 & 0.2 & 0.5 \end{pmatrix}$$
$$A_3 = \begin{pmatrix} 0.5 & 0.4 & 0.7 & 0.6 \\ 0.6 & 0.5 & 0.3 & 0.7 \\ 0.3 & 0.7 & 0.5 & 0.6 \\ 0.4 & 0.3 & 0.4 & 0.5 \end{pmatrix},\quad A_4 = \begin{pmatrix} 0.5 & 0.7 & 0.7 & 0.5 \\ 0.3 & 0.5 & 0.4 & 0.4 \\ 0.3 & 0.6 & 0.5 & 0.9 \\ 0.5 & 0.6 & 0.1 & 0.5 \end{pmatrix}.$$

Since the weights of students are given, we then utilize Approach I to find the decision result.

We first utilize (27) to calculate the supports $\text{Sup}(a_{ij}^{(k)}, a_{ij}^{(l)})$ $(i, j, k, l = 1, 2, 3, 4, k \neq l)$, which are contained in the matrices $S^{kl} = (S^{kl}(a_{ij}^{(k)}, a_{ij}^{(l)}))_{4\times 4}$

$(k = 1, 2, 3, 4)$, respectively

$$S^{12} = \begin{pmatrix} 1 & 0.429 & 0.667 & 0.556 \\ 0.429 & 1 & 0.818 & 0.700 \\ 0.667 & 0.818 & 1 & 0.600 \\ 0.556 & 0.700 & 0.600 & 1 \end{pmatrix}, \quad S^{13} = \begin{pmatrix} 1 & 1 & 0.667 & 0.778 \\ 1 & 1 & 0.545 & 0.800 \\ 0.667 & 0.545 & 1 & 1 \\ 0.778 & 0.800 & 1 & 1 \end{pmatrix}$$

$$S^{14} = \begin{pmatrix} 1 & 0.571 & 0.667 & 0.667 \\ 0.571 & 1 & 0.636 & 0.500 \\ 0.667 & 0.636 & 1 & 0.400 \\ 0.667 & 0.500 & 0.400 & 1 \end{pmatrix}, \quad S^{21} = \begin{pmatrix} 1 & 0.556 & 0 & 0.429 \\ 0.556 & 1 & 0.714 & 0.500 \\ 0 & 0.714 & 1 & 0.600 \\ 0.429 & 0.500 & 0.600 & 1 \end{pmatrix}$$

$$S^{23} = \begin{pmatrix} 1 & 0.556 & 1 & 0.714 \\ 0.556 & 1 & 0.571 & 0.833 \\ 1 & 0.571 & 1 & 0.600 \\ 0.714 & 0.833 & 0.600 & 1 \end{pmatrix}, \quad S^{24} = \begin{pmatrix} 1 & 0.889 & 1 & 0.857 \\ 0.889 & 1 & 0.714 & 0.667 \\ 1 & 0.714 & 1 & 0.800 \\ 0.857 & 0.667 & 0.800 & 1 \end{pmatrix}$$

$$S^{31} = \begin{pmatrix} 1 & 1 & 0 & 0.600 \\ 1 & 1 & 0.444 & 0.667 \\ 0 & 0.444 & 1 & 1 \\ 0.600 & 0.667 & 1 & 1 \end{pmatrix}, \quad S^{32} = \begin{pmatrix} 1 & 0.429 & 1 & 0.600 \\ 0.429 & 1 & 0.667 & 0.833 \\ 1 & 0.667 & 1 & 0.600 \\ 0.600 & 0.833 & 0.600 & 1 \end{pmatrix}$$

$$S^{34} = \begin{pmatrix} 1 & 0.571 & 1 & 0.800 \\ 0.571 & 1 & 0.889 & 0.500 \\ 1 & 0.889 & 1 & 0.400 \\ 0.800 & 0.500 & 0.400 & 1 \end{pmatrix}, \quad S^{41} = \begin{pmatrix} 1 & 0.571 & 0 & 0.400 \\ 0.571 & 1 & 0.429 & 0.500 \\ 0 & 0.429 & 1 & 0.571 \\ 0.400 & 0.500 & 0.571 & 1 \end{pmatrix}$$

$$S^{42} = \begin{pmatrix} 1 & 0.857 & 1 & 0.800 \\ 0.857 & 1 & 0.714 & 0.800 \\ 1 & 0.714 & 1 & 0.857 \\ 0.800 & 0.800 & 0.857 & 1 \end{pmatrix}, \quad S^{43} = \begin{pmatrix} 1 & 0.571 & 1 & 0.800 \\ 0.571 & 1 & 0.857 & 0.700 \\ 1 & 0.857 & 1 & 0.571 \\ 0.800 & 0.70 & 0.571 & 1 \end{pmatrix}.$$

Then, we utilize the weight vector $w = (0.3, 0.3, 0.2, 0.2)^T$ of the students d_k $(k = 1, 2, 3, 4)$ and (28) to calculate the weighted supports $T'(a_{ij}^{(k)})$ $(i, j, k = 1, 2, 3, 4)$ of the preference values $a_{ij}^{(k)}$ $(i, j, k = 1, 2, 3, 4)$, which are contained in the matrices $T'_k = (T'(a_{ij}^{(k)}))_{4\times 4}$ $(k = 1, 2, 3, 4)$, respectively

$$T'_1 = \begin{pmatrix} 0.700 & 0.443 & 0.467 & 0.456 \\ 0.443 & 0.700 & 0.482 & 0.470 \\ 0.467 & 0.482 & 0.700 & 0.460 \\ 0.456 & 0.470 & 0.460 & 0.700 \end{pmatrix}, \quad T'_2 = \begin{pmatrix} 0.700 & 0.456 & 0.400 & 0.443 \\ 0.456 & 0.700 & 0.471 & 0.450 \\ 0.400 & 0.471 & 0.700 & 0.460 \\ 0.443 & 0.450 & 0.460 & 0.700 \end{pmatrix}$$

$$T'_3 = \begin{pmatrix} 0.800 & 0.543 & 0.500 & 0.520 \\ 0.543 & 0.800 & 0.511 & 0.550 \\ 0.500 & 0.511 & 0.800 & 0.560 \\ 0.520 & 0.550 & 0.560 & 0.800 \end{pmatrix}, \quad T'_4 = \begin{pmatrix} 0.800 & 0.543 & 0.500 & 0.520 \\ 0.543 & 0.800 & 0.514 & 0.530 \\ 0.500 & 0.514 & 0.800 & 0.543 \\ 0.520 & 0.530 & 0.543 & 0.800 \end{pmatrix}$$

and then utilize (29) to calculate the weights $v_{ij}^{(k)}$ $(i, j, k = 1, 2, 3, 4)$ associated with the preference values $a_{ij}^{(k)}$ $(i, j, k = 1, 2, 3, 4)$, which are contained in the matrices $V_k = (v_{ij}^{(k)})_{4\times 4}$ $(k = 1, 2, 3, 4)$, respectively

$$V_1 = \begin{pmatrix} 0.293 & 0.291 & 0.301 & 0.296 \\ 0.291 & 0.293 & 0.298 & 0.295 \\ 0.301 & 0.298 & 0.293 & 0.293 \\ 0.295 & 0.295 & 0.293 & 0.293 \end{pmatrix}, \quad V_2 = \begin{pmatrix} 0.293 & 0.294 & 0.288 & 0.293 \\ 0.293 & 0.293 & 0.296 & 0.292 \\ 0.287 & 0.296 & 0.293 & 0.293 \\ 0.293 & 0.292 & 0.293 & 0.293 \end{pmatrix}$$

$$V_3 = \begin{pmatrix} 0.207 & 0.207 & 0.205 & 0.206 \\ 0.208 & 0.207 & 0.203 & 0.208 \\ 0.206 & 0.203 & 0.207 & 0.208 \\ 0.206 & 0.208 & 0.208 & 0.207 \end{pmatrix}, \quad V_4 = \begin{pmatrix} 0.207 & 0.208 & 0.206 & 0.206 \\ 0.208 & 0.207 & 0.203 & 0.205 \\ 0.206 & 0.203 & 0.207 & 0.206 \\ 0.206 & 0.205 & 0.206 & 0.207 \end{pmatrix}.$$

Based on this, we utilize the weighted PH operator (30) to aggregate all the individual preference relations $A_k = (a_{ij}^{(k)})_{4\times 4}$ $(k = 1, 2, 3, 4)$ into the collective preference relation

$$A = \begin{pmatrix} 0.5000 & 0.5237 & 0.6248 & 0.5383 \\ 0.3344 & 0.5000 & 0.4878 & 0.6157 \\ 0.3411 & 0.3499 & 0.5000 & 0.6992 \\ 0.3460 & 0.2121 & 0.2093 & 0.5000 \end{pmatrix}.$$

After this, we utilize the NRAM (31) to derive the priority vector of A

$$v = (0.3003, 0.2661, 0.2596, 0.1740)^T.$$

Using this, we get the ranking of the options as follows:

$$x_1 \succ x_2 \succ x_3 \succ x_4.$$

7 Conclusions

In this paper, based on the PA operator, we have developed several new non-linear weighted harmonic aggregation operators including the PH operator, weighted PH operator, POWH operator, UPH operator, weighted UPH operator and UPOWH operator. We have studied some desired properties of the developed operators, such as commutativity, idempotency and boundedness. The fundamental idea of these operators is that the weight of each input argument depends on the other input arguments and allows argument values to support each other in the harmonic aggregation process. Moreover, we have applied the developed operators to aggregate all individual preference (or uncertain preference) relations into collective preference (or uncertain preference) under various group decision making environment and then developed some group decision making approaches. The merit of the developed approaches is that they can take all the decision arguments and their relationships into account. In the future, we will develop several applications of the developed aggregation operators in other fields, such as pattern recognition, supply chain management and image processing.

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J. COMPUTATIONAL ANALYSIS AND APPLICATIONS, VOL. 15, NO.6, 1138-1149, 2013, COPYRIGHT 2013 EUDOXUS PRESS, LLC

MULTIPLICATIONAL COMBINATIONS AND A GENERAL SCHEME OF SINGLE-STEP ITERATIVE METHODS FOR MULTIPLE ROOTS

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ABSTRACT. In this paper, a general form of single-step iterative methods for multiple roots of nonlinear equations is derived under a number of assumptions of optimization. Definition of multiplicational combinations and their properties are used upon the optimization procedure. Among all, we construct a family of iterative methods with nine parameters and simplest terms, and we obtain 23 simplest iterative methods within the family, those including all existing methods of single-step scheme. Numerical comparisons between the methods also present interesting and noteworthy results.

1. INTRODUCTION

Solving nonlinear equations is one of the most basic problems of mathematics, yet it is often greatly complicated. Therefore, to develop methods to obtain roots of a nonlinear equation $f(x) = 0$ has become crucial, especially with advance of computational technology.

Newton's method, defined by

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)} \tag{1}$$

makes use of an approximated root to obtain a new approximation with less error. This classical method, however, ceases to be efficient when a multiple root of f is to be obtained. In such cases, one may solve a nonlinear equation $u(x) = 0$ where $u(x) = f(x)/f'(x)$ instead of $f(x) = 0$, since $u(x)$ has multiple roots of $f(x)$ as its simple roots, see [1, p.126]. When multiplicity m of the desired root of f is known, one may use the modified Newton's method,

$$x_{n+1} = x_n - m\frac{f_n}{f'_n} \tag{2}$$

where $f_n^{(i)}$ denotes $f^{(i)}(x_n)$ instead of the original Newton's method (1).

The modified Newton's method for multiple roots is quadratically convergent. More advanced iterative algorithms with cubic or higher order of convergence are actively being developed, in

Keywords: Newton's method, Iterative methods, Single-step methods, Nonlinear equations, Cubic order, Multiple roots.

2010 MSC: 65H05.

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order to improve the computational efficiency. One widely known cubically convergent example is Halley's method(HM), namely,

$$x_{n+1} = x_n - \frac{f_n}{\frac{m+1}{2m} f'_n - \frac{f_n f''_n}{2f'_n}}, \tag{3}$$

see [2].

The Euler-Chebyshev method(ECM),

$$x_{n+1} = x_n - \frac{m(3-m)}{2} \frac{f_n}{f'_n} - \frac{m^2}{2} \frac{{f_n}^2 f''_n}{{f'_n}^3} \tag{4}$$

is also of cubic convergence, see [1].

Osada in [3] and Chun and Neta in [4], developed other cubically convergent iterative methods,

$$x_{n+1} = x_n - \frac{1}{2} m(m+1) \frac{f_n}{f'_n} + \frac{1}{2}(m-1)^2 \frac{f'_n}{f''_n}, \tag{5}$$

and

$$x_{n+1} = x_n - \frac{2m^2 {f_n}^2 f''_n}{m(3-m) f_n f'_n f''_n + (m-1)^2 {f'_n}^3}, \tag{6}$$

OM and CNM in short, respectively.

Biazar and Ghanbari in [5] assumed a form of Newton-like methods with four parameters as follows:

$$x_{n+1} = x_n - \frac{A f_n {f'_n}^2 f''_n + B {f'_n}^4 + C {f_n}^2 {f''_n}^2}{{f'_n}^3 f''_n + D f_n f'_n {f''_n}^2}. \tag{7}$$

From the error equation of the assumed method, parameters are controlled to make the method cubically convergent. A new method thereby introduced is

$$x_{n+1} = x_n - \frac{f'_n}{\frac{m+3}{2(m-1)} f''_n - \frac{m(m+1)}{2(m-1)^2} \frac{f_n {f''_n}^2}{{f'_n}^2}}, \tag{8}$$

which is to be referred to as Biazar and Ghanbari's method(BGM).

In Section 2.1, we start with basic but essential definitions. We also define multiplicational combinations with restricted derivatives of f, and write a general expression for them. Then, a Newton-like method with nine parameters is constructed under a number of assumptions. In Section 2.2, we derive the error equation of the method and solve for parameters to obtain a cubic convergence. In such a way, we derive a number of Newton-like methods, some of which are introduced previously. Section 3 contains numerical comparisons between the methods introduced or derived.

2. Development of methods

2.1. **Construction of the scheme.** Before we begin, the order of convergence and multiple roots must be defined clearly.

Definition 1. *(See* [6]*) With α a real number, and n a non-negative integer, if a real sequence $\{x_n\}$ converges to α and for n large enough there exist constants $c \geq 0$ and $p \geq 0$ that satisfy*

$$| x_{n+1} - \alpha | \leq c | x_n - \alpha |^p, \tag{9}$$

then the maximum of p is said to be an order of convergence of $\{x_n\}$ to α.

Definition 2. *(See* [7, p.79]*) A root α of an equation $f(x) = 0$ is said to have the multiplicity m if and only if $f(\alpha) = 0, f'(\alpha) = 0, f''(\alpha) = 0, \dots, f^{(m-1)}(\alpha) = 0$ and $f^{(m)}(\alpha) \neq 0$. In this case, f can be written as*

$$f(x) = (x - \alpha)^m g(x), \tag{10}$$

with $g(\alpha) \neq 0$.

Now, as a preparation for the rest of the section, we define a new concept of multiplicational combinations.

Definition 3. *Let f be a two times differentiable function. With any integers a,b, and c such that $a + b + c = 0$,*

$$F_{k,-c} = f^a f'^b f''^c \tag{11}$$

is a multiplicational combination of f, f', and f'', with differential order k=b+2c.

Multiplicational combinations acquire an important property that will be used importantly for the discussion followed.

Theorem 1. *If $F_{k,-c}$ is a multiplicational combination of f, f', and f'', with differential order k,*

$$F_{k,-c} = F_{k,s} = (\frac{f'}{f})^k (\frac{f'^2}{ff''})^s, \tag{12}$$

for some integer $s = -c$. The converse is also true.

Proof. Let $F_{k,-c} = f^a f'^b f''^c$ for integers a, b, and c. By Definition 3, $a+b+c = 0$ and $b+2c = k$. Solving the system gives $a = -k + c$ and $b = k - 2c$. Thus

$$F_{k,-c} = f^{-k+c} f'^{k-2c} f''^c = (\frac{f'}{f})^k (\frac{f'^2}{ff''})^{-c}. \tag{13}$$

Letting $s = -c$, we have

$$F_{k,s} = (\frac{f'}{f})^k (\frac{f'^2}{ff''})^s. \tag{14}$$

If $u = (\frac{f'}{f})^k (\frac{f'^2}{ff''})^s$,

$$u = f^{-k-s} f'^{k+2s} f''^{-s} = F_{k,s}, \tag{15}$$

and thus is a multiplicational combination of f, f', and f'', with differential order k. This completes the proof. $\square$

Single-step iterative methods are generally expressed as $x_{n+1} = x_n - g(x_n)$, where $g(x_n)$ denotes an iteration function of x_n. For computational efficiency, we only consider $g(x_n)$'s that consist of f_n, f_n', f_n'' and a finite number of fundamental arithmetic operations between them. With the assumption, $g(x_n)$ can be written as follows:

$$g(x_n) = \frac{\sum_{a,b,c} {f_n}^a {f_n'}^b {f_n''}^c \theta(a,b,c)}{\sum_{a,b,c} {f_n}^a {f_n'}^b {f_n''}^c \phi(a,b,c)}, \tag{16}$$

where θ and ϕ symbolize the linear combination of ${f_n}^a {f_n'}^b {f_n''}^c$'s in the numerator and the denominator, respectively. It is reasonable to assume that all terms included in the sum are required to have the same arithmetic order, namely, $a+b+c$. Thus, by an appropriate division, both the numerator and the denominator each reduces to a linear combination of multiplicational combinations. Then by Theorem 1,

$$g(x_n) = \frac{\sum_{k,s} (\frac{f'}{f})^k (\frac{f'^2}{ff''})^s \theta(k,s)}{\sum_{k,s} (\frac{f'}{f})^k (\frac{f'^2}{ff''})^s \phi(k,s)}. \tag{17}$$

Here, for optimization(see Remark 1), we assume that the numerator and the denominator each consists of multiplicational combinations of uniform differential order. That is, for integers k_1 and k_2,

$$g(x_n) = \frac{(\frac{f'}{f})^{k_1} \sum_s (\frac{f'^2}{ff''})^s \theta(s)}{(\frac{f'}{f})^{k_2} \sum_s (\frac{f'^2}{ff''})^s \phi(s)}. \tag{18}$$

Theorem 2. *For an iteration function defined by (18), if $x_{n+1} = x_n - g(x_n)$ is cubically convergent to α, the root of $f(x) = 0$ with multiplicity m, it is required that $k_1 - k_2 = -1$.*

Proof. Taylor's expansion for f about a multiple root α of $f(x) = 0$ with multiplicity m gives

$$f(x_n) = f^{(m)}(\alpha)(c_0 e_n^m + c_1 e_n^{m+1} + c_2 e_n^{m+2} + \cdots), \tag{19}$$

$$f'(x_n) = f^{(m)}(\alpha)\{m c_0 e_n^{m-1} + (m+1) c_1 e_n^m + (m+2) c_2 e_n^{m+1} + \cdots\}, \tag{20}$$

and

$$f''(x_n) = f^{(m)}(\alpha)\{m(m-1) c_0 e_n^{m-2} + (m+1) m c_1 e_n^{m-1} + (m+2)(m+1) c_2 e_n^m, \tag{21}$$

where c_n's and e_n are defined as follows:

$$c_n = \frac{1}{(m+n)!} \frac{f^{(m+n)}(\alpha)}{f^{(m)}(\alpha)}, e_n = x_n - \alpha. \tag{22}$$

Then,

$$\frac{f_n}{f_n'} = \frac{1}{m} e_n - \frac{1}{m^2} \frac{c_1}{c_0} e_n^2 + \Big(\frac{m+1}{m^3} \frac{c_1^2}{c_0^2} - \frac{2}{m^2} \frac{c_2}{c_0}\Big) e_n^3 + \cdots, \tag{23}$$

$$\frac{f_n'^2}{f_n f_n''} = \frac{m}{m-1} - \frac{2}{(m-1)^2} \frac{c_1}{c_0} e_n + \Big(\frac{3m^2+1}{m(m-1)^3} \frac{c_1^2}{c_0^2} - \frac{6}{(m-1)^2} \frac{c_2}{c_0}\Big) e_n^2 + \cdots, \tag{24}$$

and thus

$$g(x_n) = \big(e_n + O(e_n^2)\big)^{k_2 - k_1} \big(1 + O(e_n)\big) = e_n^{k_2 - k_1} + O(e_n^{k_2 - k_1 + 1}). \tag{25}$$

For cubic convergence, we require $e_{n+1} = O(e_n^3)$ and thus,

$$g(x_n) = e_n + O(e_n^3). \tag{26}$$

From (25) and (26), $k_2 - k_1 = 1$, which completes the proof. □

Thereby the iteration function $g(x_n)$ reduces to its final form,

$$g(x_n) = (\frac{f_n}{f'_n})\frac{\sum_s F_{0,s}\theta(s)}{\sum_s F_{0,s}\phi(s)}. \tag{27}$$

There are infinitely many $F_{0,s}$'s, however, writing from the simplest terms, five examples of multiplicational combinations of zeroth differential order can be written as

$$1, \frac{f'^2}{ff''}, (\frac{f'^2}{ff''})^{-1}, (\frac{f'^2}{ff''})^2, (\frac{f'^2}{ff''})^{-2}, \cdots . \tag{28}$$

Therefore, we construct a Newton-like method with nine parameters as follows:

$$x_{n+1} = x_n - (\frac{f_n}{f'_n})\left(\frac{A + B(\frac{f'^2_n}{f_n f''_n}) + C(\frac{f'^2_n}{f_n f''_n})^{-1} + D(\frac{f'^2_n}{f_n f''_n})^2 + E(\frac{f'^2_n}{f_n f''_n})^{-2}}{1 + F(\frac{f'^2_n}{f_n f''_n}) + G(\frac{f'^2_n}{f_n f''_n})^{-1} + H(\frac{f'^2_n}{f_n f''_n})^2 + I(\frac{f'^2_n}{f_n f''_n})^{-2}}\right) \tag{29}$$

2.2. **Solving for parameters.** During the last section, (29) was derived to be the simplest possible form for the cubic order methods. Now we will find which among the form actually acquire the desired order.

Theorem 3. *Let α be an exact root of f and its multiplicity be m. Let n be an integer with $n \geq 0$, x_n an approximation after n iterations. Then the Newton-like method defined by (29) is cubically convergent if and only if*

$$X\begin{pmatrix} A & B & C & D & E & F & G & H & I \end{pmatrix}^T = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \tag{30}$$

with

$$X = \begin{pmatrix} \frac{1}{m} & \frac{1}{m-1} & \frac{m-1}{m^2} & \frac{m}{(m-1)^2} & \frac{(m-1)^2}{m^3} & -\frac{m}{m-1} & -\frac{m-1}{m} & -\frac{m^2}{(m-1)^2} & -\frac{(m-1)^2}{m^2} \\ \frac{1}{m^2} & \frac{m+1}{m(m-1)^2} & \frac{m-3}{m^3} & \frac{m+3}{(m-1)^3} & \frac{(m-1)(m-5)}{m^4} & -\frac{2}{(m-1)^2} & \frac{2}{m^2} & -\frac{4m}{(m-1)^3} & \frac{4(m-1)}{m^3} \end{pmatrix} \tag{31}$$

is satisfied.

Proof. We use the Taylor's expansion (19) through (21) of f about α and definition (22) to obtain expressions for the nine terms included in (29).

$$\frac{f_n}{f'_n} = \frac{1}{m}e_n - \frac{1}{m^2}\frac{c_1}{c_0}e_n^2 + \Big(\frac{m+1}{m^3}\frac{c_1^2}{c_0^2} - \frac{2}{m^2}\frac{c_2}{c_0}\Big)e_n^3 + \cdots \tag{32}$$

$$\frac{f'_n}{f''_n} = \frac{1}{m-1}e_n - \frac{m+1}{m(m-1)^2}\frac{c_1^2}{c_0^2}e_n^2 - \Big(\frac{(m+1)^2}{m(m-1)^3}\frac{c_1^2}{c_0^2} - \frac{2(m+2)}{m(m-1)^2}\frac{c_2}{c_0}\Big)e_n^3 + \cdots \tag{33}$$

$$\frac{f_n^2 f''_n}{f_n'^3} = \frac{m-1}{m^2}e_n - \frac{m-3}{m^3}\frac{c_1}{c_0}e_n^2 + \Big(\frac{m^2-3m-6}{m^4}\frac{c_1^2}{c_0^2} - \frac{2(m-4)}{m^3}\frac{c_2}{c_0}\Big)e_n^3 + \cdots \tag{34}$$

$$\frac{f_n'^3}{f_n^2 f''_n} = \frac{m}{(m-1)^2}e_n - \frac{m+3}{(m-1)^3}\frac{c_1}{c_0}e_n^2 + \Big(\frac{(m+2)(m+3)}{(m-1)^4}\frac{c_1^2}{c_0^2} - \frac{2(m+5)}{(m-1)^3}\frac{c_2}{c_0}\Big)e_n^3 + \cdots \tag{35}$$

$$\begin{aligned}\frac{f_n^3 f_n''^2}{f_n'^5} ={}& \frac{(m-1)^2}{m^3}e_n - \frac{(m-1)(m-5)}{m^4}\frac{c_1}{c_0}e_n^2 \\ &+\Big(\frac{m^3-7m^2-5m+15}{m^5}\frac{c_1^2}{c_0^2} - \frac{2(m-1)(m-7)}{m^4}\frac{c_2}{c_0}\Big)e_n^3 + \cdots\end{aligned} \tag{36}$$

$$\frac{f_n'^2}{f_n f''_n} = \frac{m}{m-1} - \frac{2}{(m-1)^2}\frac{c_1}{c_0}e_n + \Big(\frac{3m^2+1}{m(m-1)^3}\frac{c_1^2}{c_0^2} - \frac{6}{(m-1)^2}\frac{c_2}{c_0}\Big)e_n^2 + \cdots \tag{37}$$

$$\frac{f_n f''_n}{f_n'^2} = \frac{m-1}{m} + \frac{2}{m^2}\frac{c_1}{c_0}e_n + \Big(-\frac{3m+1}{m^3}\frac{c_1^2}{c_0^2} + \frac{6}{m^2}\frac{c_2}{c_0}\Big)e_n^2 + \cdots \tag{38}$$

$$\frac{f_n'^4}{f_n^2 f_n''^2} = \frac{m^2}{(m-1)^2} - \frac{4m}{(m-1)^3}\frac{c_1}{c_0}e_n + \Big(\frac{6(m^2+1)}{(m-1)^4}\frac{c_1^2}{c_0^2} - \frac{12m}{(m-1)^3}\frac{c_2}{c_0}\Big)e_n^2 + \cdots \tag{39}$$

$$\frac{f_n^2 f_n''^2}{f_n'^4} = \frac{(m-1)^2}{m^2} + \frac{4(m-1)}{m^3}\frac{c_1}{c_0}e_n + \Big(-\frac{2(3m^2-5)}{m^4}\frac{c_1^2}{c_0^2} - \frac{12(m-1)}{m^3}\frac{c_2}{c_0}\Big)e_n^2 + \cdots \tag{40}$$

From these equations, an error equation of (29) is easily derived:

$$e_{n+1} = e_n - K_1 e_n - K_2 {e_n}^2 + O(e_n^3) \tag{41}$$

where

$$\begin{aligned}K_1 = \Big(&\frac{1}{m}A + \frac{1}{m-1}B + \frac{m-1}{m^2}C + \frac{m}{(m-1)^2}D + \frac{(m-1)^2}{m^3}E \\ &- \frac{m}{m-1}F - \frac{m-1}{m}G - \frac{m^2}{(m-1)^2}H - \frac{(m-1)^2}{m^2}I\Big)\end{aligned} \tag{42}$$

and

$$\begin{aligned}K_2 = \Big(&\frac{1}{m^2}A + \frac{m+1}{m(m-1)^2}B + \frac{m-3}{m^3}C + \frac{m+3}{(m-1)^3}D + \frac{(m-1)(m-5)}{m^4}E \\ &- \frac{2}{(m-1)^2}F + \frac{2}{m^2}G - \frac{4m}{(m-1)^3}H + \frac{4(m-1)}{m^3}I\Big).\end{aligned} \tag{43}$$

The condition for (29) to be cubically convergent is $K_1 = 1$ and $K_2 = 0$, which is equivalent to (30). This completes the proof.

□

Any combinations of parameters satisfying (30) would yield a cubic order Newton-like iterative method. However, a combination with all parameters activated will lead to a very complicated method, resulting in a relatively high computational cost. For this reason, it would be the best to let as many parameters as possible be zero, leaving only two of them non-zero. Noting that A, B, C, D, E cannot be all zero at the same time, there are 30 combinations in each of which all parameters except for two of them are zero. Nevertheless, it can be observed that 7 pairs are equivalent, by multiplying an appropriate power of $\frac{f'^2}{ff''}$ to both the numerator and the denominator. Thereby we obtain 23 unique cubic order methods among the family of (29).

Letting all parameters but A and B be zero, and solving (30) gives

$$A = \frac{m(m+1)}{2}, B = -\frac{(m-1)^2}{2}, \tag{44}$$

yielding a method

$$x_{n+1} = x_n - \frac{m(m+1)}{2}\frac{f_n}{f'_n} + \frac{(m-1)^2}{2}\frac{f'_n}{f''_n}. \tag{45}$$

Similarly, 22 other methods obtained are displayed in Table 1. In the left column are combinations of non-zero parameters, and by solving (30) for the parameters, we obtain iterative methods displayed in the right column.

parameters	iterative method obtained
A,C	$x_{n+1} = x_n - \frac{m(3-m)}{2}\frac{f_n}{f'_n} - \frac{m^2}{2}\frac{f_n^2 f''}{f_n'^3} \quad (46)$
A,D	$x_{n+1} = x_n - \frac{m(m+3)}{4}\frac{f_n}{f'_n} + \frac{(m-1)^3}{4m}\frac{f_n'^3}{f_n f_n''^2} \quad (47)$
A,E	$x_{n+1} = x_n + \frac{m(m-5)}{4}\frac{f_n}{f'_n} - \frac{m^3}{4(m-1)}\frac{f_n^3 f_n''^2}{f_n'^5} \quad (48)$
A,F	$x_{n+1} = x_n + \frac{2m^2 f_n^2 f_n''}{m(m-3)f_n f'_n f''_n - (m-1)^2 f_n'^3} \quad (49)$
A,G	$x_{n+1} = x_n - \frac{2m f_n f'_n}{(m+1)f_n'^2 - m f_n f''_n} \quad (50)$
A,H	$x_{n+1} = x_n + \frac{4m^3 f_n^3 f_n''^2}{m^2(m-5)f_n^2 f'_n f_n''^2 - (m-1)^3 f_n'^5} \quad (51)$
A,I	$x_{n+1} = x_n - \frac{4m(m-1)f_n f_n'^3}{(m-1)(m+3)f_n'^4 - m^2 f_n^2 f_n''^2} \quad (52)$

Table 1. Non-zero parameters and corresponding iterative methods.

parameters	iterative method obtained	
B,C	$x_{n+1} = x_n + \frac{(m-1)(m-3)}{4}\frac{f'_n}{f''_n} - \frac{m^2(m+1)}{4(m-1)}\frac{f_n^2 f''_n}{f'^3_n}$	(53)
B,D	$x_{n+1} = x_n - \frac{(m-1)(m+3)}{2}\frac{f'_n}{f''_n} + \frac{(m-1)^2(m+1)}{2m}\frac{f'^3_n}{f_n f''^2_n}$	(54)
B,E	$x_{n+1} = x_n + \frac{(m-1)(m-5)}{6}\frac{f'_n}{f''_n} - \frac{m^3(m+1)}{6(m-1)^2}\frac{f_n^3 f''^2_n}{f'^5_n}$	(55)
B,G	$x_{n+1} = x_n - \frac{2(m-1)^2 f'^3_n}{(m-1)(m+3)f'^2_n f''_n - m(m+1)f_n f''^2_n}$	(56)
B,H	$x_{n+1} = x_n + \frac{4m^2(m-1)f_n^2 f'_n f''_n}{m^2(m-3)f_n^2 f''^2_n - (m-1)^2(m+1)f'^4_n}$	(57)
B,I	$x_{n+1} = x_n - \frac{4(m-1)^3 f'^5_n}{(m-1)^2(m+5)f'^4_n f''_n - m^2(m+1)f_n^2 f''^3_n}$	(58)
C,D	$x_{n+1} = x_n - \frac{m^2(m+3)}{6(m-1)}\frac{f_n^2 f''_n}{f'^3_n} + \frac{(m-1)^2(m-3)}{6m}\frac{f'^3_n}{f_n f''^2_n}$	(59)
C,E	$x_{n+1} = x_n + \frac{m^2(m-5)}{2(m-1)}\frac{f_n^2 f''_n}{f'^3_n} - \frac{m^2(m-3)}{2(m-1)^2}\frac{f_n^3 f''^2_n}{f'^5_n}$	(60)
C,F	$x_{n+1} = x_n + \frac{2m^3 f_n^3 f''^2_n}{m(m-1)(m-5)f_n f'^3_n f''_n - (m-1)^2(m-3)f'^5_n}$	(61)
C,H	$x_{n+1} = x_n + \frac{4m^4 f_n^4 f''^3_n}{m^2(m-1)(m-7)f_n^2 f'^3_n f''^2_n - (m-1)^3(m-3)f'^7_n}$	(62)
D,E	$x_{n+1} = x_n + \frac{(m-1)^2(m-5)}{8m}\frac{f'^3_n}{f_n f''^2_n} - \frac{m^3(m+3)}{8(m-1)^2}\frac{f_n^3 f''^2_n}{f'^5_n}$	(63)
D,G	$x_{n+1} = x_n - \frac{2(m-1)^3 f'^5_n}{m(m-1)(m+5)f_n f'^2_n f''^2_n - m^2(m+3)f_n^2 f''^3_n}$	(64)
D,I	$x_{n+1} = x_n - \frac{4(m-1)^4 f'^7_n}{m(m-1)^2(m+7)f_n f'^4_n f''^2_n - m^3(m+3)f_n^3 f''^4_n}$	(65)
E,F	$x_{n+1} = x_n + \frac{2m^4 f_n^4 f''^3_n}{m(m-1)^2(m-7)f_n f'^5_n f''_n - (m-1)^3(m-5)f'^7_n}$	(66)
E,H	$x_{n+1} = x_n + \frac{4m^5 f_n^5 f''^4_n}{m^2(m-1)^2(m-9)f_n^2 f'^5_n f''^2_n - (m-1)^4(m-5)f'^9_n}$	(67)

Table 1. (continued)

Method (45) is Osada's method(OM) introduced in (5), (46) is Euler-Chebyshev method(ECM) introduced in (4), (48) is Chun and Neta's method(CNM) introduced in (6), (49) is Halley's method(HM) introduced in (3), and (56) is Biazar and Ghanbari's method(BGM) introduced in (8). Moreover, since these methods are constructed by allowing only two of nine parameters to be non-zero, more can be constructed from (29) by setting various combinations of non-zero parameters, though an excess of non-zero terms would corrupt the computational efficiency.

An efficiency index of an iterative method is defined by $p^{1/d}$ where p denotes the order of convergence of an iterative method, and d denotes the number of function evaluations required per each iteration, which is very reasonable considering the definition of the order of convergence. The efficiency index of methods (45) through (67) is $3^{1/3} = 1.442$, which is higher than the Newton's method (2) or optimal fourth-order iterative methods, with efficiency index $2^{1/2} = 4^{1/4} = 1.414$. Note that the third-ordered methods (45) through (67) require one functional and two derivative evaluations per iteration.

Remark 1. *Summing multiplicational combinations of uniform differential order k preserves the expansion form of* $e_n^k(p_1 + p_2\frac{c_1}{c_0}e_n + (p_3\frac{c_1^2}{c_0^2} + p_4\frac{c_2}{c_0})e_n^2 + O(e_n^3))$*, where* p_i*'s are constants. While the error equation must be an identity of* c_i*'s and* e_n*, it is optimal to reduce as many terms of* c_i*'s and* e_n *as possible in order to keep the method simple. In fact, all existing single-step methods of cubic convergence are included within (27), or in fact, within (29).*

Remark 2. *The condition for (29) to converge with fourth order, simultaneously derived, is equivalent to an impossible system of equations. Therefore we consider it to be impossible to construct a fourth-order iterative method of single-step scheme, with three or less function evaluations. This limits the efficiency of single-step iterative methods for multiple roots.*

3. Numerical Comparisons

In this Section, numerical comparisons between cubically convergent methods of family (29) are presented. Test functions used for root-finding are displayed in Table 2, along with each of their approximate root and their multiplicity, and values used as initial points for each test function.

test function	approximate root	multiplicity	initial	value
$f_1(x) = (x^3 + 4x^2 - 10)^3$	1.36523	m=3	2	1
$f_2(x) = (\sin^2 x - x^2 + 1)^2$	1.40449	m=2	2.3	2
$f_3(x) = (x^2 - e^x - 3x + 2)^5$	0.25753	m=5	-1	1
$f_4(x) = (\cos x - x)^3$	0.73909	m=3	1.7	1
$f_5(x) = ((x-1)^3 - 1)^6$	2	m=6	3	2.3
$f_6(x) = (xe^{x^2} - \sin^2 x + 3\cos x + 5)^4$	-1.20765	m=4	-2	-1
$f_7(x) = (\sin x - x/2)^2$	1.89549	m=2	1.7	2

Table 2. Test functions, approximate roots, their multiplicity, and initial values used.

Displayed in Table 3 are the number of iterations required to reach $| f(x_n) | \leq 10^{-128}$ for each method and for each test function and an initial value. In the parentheses are the absolute value of $f(x_n)$ after such iterations. Average numbers of iterations required for these cases are also displayed for each method. All computations were done using Mathematica, inserting inputs with significant figures large enough. Here * denotes where the approximation does not converge into the exact root.

From the result, we consider (52) to be the most powerful iterative method among the family, and (50), (56), or (63) are also of considerable quality. It is interesting that though (64)and (65) often fail to converge into the root either temporarily or permanently, other methods have similar speed of convergence, differing by no more than 1 in average number of iterations. In fact, all methods in the comparison required the same number of iterations in two cases, namely, $f_3(x), x_0 = 1$ and $f_4(x), x_0 = 1$.

4. CONCLUSION

Reduced from the most primitive form of iteration functions, a general single-step iterative scheme is constructed under a number of assumptions while maintaining simplicity. Considering only a finite number of multiplicational combinations, 23 cubically convergent iterative methods, those we consider to be the simplest among the scheme, are derived by the method of undetermined coefficients in the error equation. They include all existing single-step iterative methods. The multiplicational combination-based approach allows construction of more methods with consistency, within the same scheme. The numerical comparisons show the quality of the derived methods, and it can be observed from the comparisons that few of these methods have higher quality than the others, though not of significant difference.

	$f_1(x)$		$f_2(x)$		$f_3(x)$		$f_4(x)$
methods	$x_0 = 2$	$x_0 = 1$	$x_0 = 2.3$	$x_0 = 2$	$x_0 = -1$	$x_0 = 1$	$x_0 = 1.7$
(45)(OM)	5(8e-322)	5(2e-258)	6(3e-343)	5(7e-153)	4(1e-267)	4(7e-286)	5(2e-364)
(46)(ECM)	5(5e-371)	5(7e-374)	5(2e-142)	5(1e-190)	4(2e-278)	4(2e-279)	5(5e-378)
(47)	5(3e-304)	5(7e-210)	6(9e-317)	5(3e-141)	4(2e-264)	4(3e-289)	5(9e-359)
(48)(CNM)	4(2e-133)	4(1e-149)	5(2e-169)	5(1e-227)	4(8e-287)	4(2e-276)	5(4e-386)
(49)(HM)	5(5e-371)	5(7e-374)	6(4e-377)	5(6e-168)	4(5e-300)	4(2e-273)	5(5e-378)
(50)	4(1e-154)	4(1e-179)	5(8e-172)	5(3e-231)	4(1e-358)	4(1e-267)	4(5e-131)
(51)	5(6e-342)	5(1e-321)	6(3e-341)	5(5e-152)	4(8e-287)	4(2e-276)	5(7e-371)
(52)	4(7e-195)	4(7e-294)	5(2e-266)	5(1e-335)	3(8e-180)	4(8e-265)	4(1e-134)
(53)	5(5e-371)	5(7e-374)	5(7e-166)	5(1e-222)	4(3e-275)	4(5e-281)	5(5e-378)
(54)	5(7e-262)	6(3e-324)	6(5e-225)	6(3e-302)	4(6e-261)	4(1e-296)	5(5e-344)
(55)	4(4e-149)	4(3e-268)	5(7e-173)	5(3e-253)	4(8e-287)	4(2e-276)	4(6e-131)
(56)(BGM)	4(3e-146)	4(1e-131)	5(3e-240)	5(2e-144)	4(3e-309)	4(2e-259)	4(5e-156)
(57)	5(5e-371)	5(7e-374)	6(3e-336)	5(2e-149)	4(2e-309)	4(6e-272)	5(5e-378)
(58)	5(7e-323)	5(3e-325)	5(5e-131)	6(2e-143)	4(2e-272)	4(1e-255)	4(3e-169)
(59)	5(5e-371)	5(7e-374)	5(3e-208)	5(1e-289)	4(1e-272)	4(1e-282)	5(5e-378)
(60)	5(5e-371)	5(7e-374)	6(8e-333)	5(1e-147)	4(8e-287)	4(2e-276)	5(5e-378)
(61)	5(5e-371)	5(7e-374)	5(1e-164)	5(9e-221)	4(8e-287)	4(2e-276)	5(5e-378)
(62)	5(5e-371)	5(7e-374)	5(4e-205)	5(6e-283)	4(6e-282)	4(6e-278)	5(5e-378)
(63)	4(1e-174)	4(5e-164)	5(1e-157)	4(5e-147)	4(8e-287)	4(2e-276)	4(2e-134)
(64)	5(1e-194)	5(3e-228)	17(4e-357)	74(3e-164)	4(3e-236)	4(1e-249)	4(6e-135)
(65)	6(2e-370)	5(3e-185)	*	6(2e-187)	4(4e-215)	4(6e-245)	5(6e-383)
(66)	4(2e-147)	4(2e-160)	5(1e-152)	5(1e-268)	4(8e-287)	4(2e-276)	4(7e-131)
(67)	4(1e-171)	5(8e-390)	5(9e-153)	5(4e-299)	4(8e-287)	4(2e-276)	4(2e-134)

Table 3. Numbers of iterations for test functions and initial points given in Table 1, with $\mid f(x_n) \mid$ after such iterations.

	$f_4(x)$	$f_5(x)$		$f_6(x)$		$f_7(x)$		
	$x_0 = 1$	$x_0 = 3$	$x_0 = 2.3$	$x_0 = -2$	$x_0 = -1$	$x_0 = 1.7$	$x_0 = 2$	average
(45)	4(1e-237)	5(4e-258)	4(9e-238)	6(1e-141)	5(3e-317)	5(3e-227)	4(2e-157)	4.79
(46)	4(6e-247)	5(3e-286)	4(1e-253)	6(5e-200)	4(4e-150)	5(2e-333)	4(6e-177)	4.64
(47)	4(1e-233)	5(3e-246)	4(5e-231)	7(4e-375)	5(1e-261)	5(2e-187)	4(4e-151)	4.86
(48)	4(2e-252)	5(2e-303)	4(3e-263)	6(5e-248)	4(1e-184)	4(2e-146)	4(7e-195)	4.43
(49)	4(6e-247)	5(5e-351)	4(2e-288)	6(8e-255)	4(3e-181)	5(3e-276)	4(1e-165)	4.71
(50)	4(4e-259)	4(7e-140)	4(4e-326)	5(6e-196)	4(5e-361)	4(1e-139)	4(1e-195)	4.21
(51)	4(6e-242)	5(2e-324)	4(1e-274)	6(7e-196)	4(1e-152)	5(8e-236)	4(3e-157)	4.71
(52)	4(8e-267)	4(6e-158)	4(6e-354)	5(2e-289)	4(5e-259)	4(4e-192)	4(2e-243)	4.14
(53)	4(6e-247)	5(2e-277)	4(1e-248)	6(8e-189)	4(7e-140)	4(1e-178)	4(1e-193)	4.57
(54)	4(2e-223)	5(6e-227)	4(6e-220)	7(2e-268)	5(5e-167)	6(2e-221)	4(5e-130)	5.07
(55)	4(9e-259)	5(5e-300)	4(3e-261)	6(7e-265)	4(8e-205)	5(3e-280)	4(1e-198)	4.43
(56)	4(2e-308)	4(4e-215)	3(6e-146)	7(6e-354)	4(4e-160)	5(1e-228)	4(1e-152)	4.36
(57)	4(6e-247)	5(2e-370)	4(3e-298)	6(3e-286)	4(3e-192)	5(3e-245)	4(8e-157)	4.71
(58)	4(2e-315)	4(1e-252)	3(7e-175)	7(4e-180)	4(6e-132)	5(1e-185)	4(3e-134)	4.57
(59)	4(6e-247)	5(3e-269)	4(6e-244)	6(2e-179)	4(1e-129)	5(3e-378)	4(9e-230)	4.64
(60)	4(6e-247)	5(5e-307)	4(3e-265)	6(1e-230)	4(4e-171)	5(5e-248)	4(2e-156)	4.71
(61)	4(6e-247)	5(1e-319)	4(5e-272)	6(3e-213)	4(8e-161)	4(1e-135)	4(4e-193)	4.57
(62)	4(6e-247)	5(1e-306)	4(4e-265)	6(6e-199)	4(4e-151)	5(2e-331)	4(3e-228)	4.64
(63)	4(5e-266)	5(9e-297)	4(2e-259)	6(1e-281)	4(4e-242)	5(7e-186)	4(1e-159)	4.36
(64)	4(1e-372)	4(1e-310)	4(9e-389)	*	5(9e-297)	6(4e-354)	5(3e-328)	10.85
(65)	4(7e-285)	4(2e-201)	4(1e-319)	*	5(1e-242)	6(3e-270)	5(7e-294)	4.83
(66)	4(1e-258)	5(8e-311)	4(2e-267)	6(1e-246)	4(1e-186)	6(1e-381)	4(3e-163)	4.5
(67)	4(8e-266)	5(7e-307)	4(3e-265)	6(6e-263)	4(2e-211)	5(3e-292)	4(5e-145)	4.5

Table 3. (continued)

ACKNOWLEDGEMENT

This work was supported by the Individual Research Program of Seoul Science High School in 2011. The authors wish to thank Mr. Googhin Kim and Mr. Changbum Chun for their helpful comments.

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J. COMPUTATIONAL ANALYSIS AND APPLICATIONS, VOL. 15, NO.6, 1150-1157, 2013, COPYRIGHT 2013 EUDOXUS PRESS, LLC

COMPACT DIFFERENCES OF VOLTERRA COMPOSITION OPERATORS FROM BERGMAN-TYPE SPACES TO BLOCH-TYPE SPACES

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Abstract. This paper characterizes the metrically compactness of differences of Volterra composition operators from the weighted Bergman-type space A^p_u, $0 < p < \infty$, to the Bloch-type space B^∞_v of analytic functions on the unit disk $\mathbb{D}$ in terms of inducing symbols φ_1, $\varphi_2 : \mathbb{D} \to \mathbb{D}$ and ψ_1, $\psi_2 : \mathbb{D} \to \mathbb{C}$.

1. Introduction

Let $\mathbb{D}$ be the open unit disk in the complex plane, $H(\mathbb{D})$ the space of all analytic functions on $\mathbb{D}$, and $H^\infty(\mathbb{D}) = H^\infty$ the space of all bounded analytic functions on $\mathbb{D}$ with the supremum norm $\|f\|_\infty = \sup_{z\in\mathbb{D}} |f(z)|$.

Let $dA(z) = \frac{1}{\pi}dxdy$ be the normalized Lebesgue measure on $\mathbb{D}$. A positive continuous function u on $[0, 1)$ is normal, if there exist positive numbers s and t, $0 < s < t$, such that $u(r)/(1-r)^s$ is decreasing on $[0, 1)$ and $\lim_{r\to1} \mu(r)/(1-r)^s = 0$; $u(r)/(1 - r)^t$ is increasing on $[0, 1)$ and $\lim_{r\to1} u(r)/(1 - r)^t = \infty$. For $0 < p < \infty$ and the normal function u, the Bergman-type space $A^p_u(\mathbb{D}) = A^p_u$ consists of all $f \in H(\mathbb{D})$ such that

$$\|f\|^p_{p,u} = \int_{\mathbb{D}} |f(z)|^p \frac{u^p(|z|)}{1 - |z|} dA(z) < \infty.$$

When $p \geq 1$, the Bergman-type space with the norm $\| \cdot \|_{p,u}$ becomes a Banach space. If $p \in (0, 1)$, it is a Fréchet space with the translation invariant metric

$$d(f, g) = \|f - g\|^p_{p,u}.$$

Let v be a positive continuous function on $\mathbb{D}$ (*weight*). The weighted-type space $H^\infty_v(\mathbb{D}) = H^\infty_v$ consists of all $f \in H(\mathbb{D})$ such that

$$\|f\|_{H^\infty_v} = \sup_{z\in\mathbb{D}} v(z)|f(z)| < \infty.$$

It is known that H^∞_v is a Banach space. The Bloch-type space $B^\infty_v(\mathbb{D}) = B^\infty_v$ consists of all $f \in H(\mathbb{D})$ such that

$$\|f\|_v = \sup_{z\in\mathbb{D}} v(z)|f'(z)| < \infty.$$

Various kinds of weights and related weighted-type spaces and Bloch-type spaces have been studied, e.g., in [1, 2, 4, 10, 11, 12].

2000 *Mathematics Subject Classification.* Primary 47B38; Secondary 47B33, 47B37.

Key words and phrases. Volterra composition operator, Bergman-type space, weighted-type space, Bloch-type space, metrically bounded operator, metrically compact operator.

Let φ be an analytic self-map of $\mathbb{D}$ and ψ be an analytic function on $\mathbb{D}$. For $f \in H(\mathbb{D})$ the Volterra composition operator $V_{\varphi,\psi}$ is defined by

$$V_{\varphi,\psi} f(z) = \int_0^z (f \circ \varphi)(\xi)(\psi \circ \varphi)'(\xi) d\xi, \quad z \in \mathbb{D}.$$

As a kind of integral-type operator, the Volterra composition operators have been studied in [7, 14, 17].

Let X and Y be topological vector spaces whose topologies are given by translation-invariant metrics d_X and d_Y, respectively, and $L : X \to Y$ be a linear operator. It is said that L is *metrically bounded* if there exists a positive constant K such that

$$d_Y(Lf, 0) \le K d_X(f, 0)$$

for all $f \in X$. When X and Y are Banach spaces, the metrically boundedness coincides with the usual definition of bounded operators between Banach spaces. Recall that $L : X \to Y$ is *metrically compact* if it maps bounded sets into relatively compact sets. If X and Y are Banach spaces then metrically compactness becomes usual compactness. For some results in this topic see [3, 5, 9, 16, 18, 19].

Let φ_1, φ_2 be nonconstant analytic self-maps of $\mathbb{D}$ and $\psi_1, \psi_2 \in H(\mathbb{D})$. Differences of Voterra composition operators on $H(\mathbb{D})$ are defined as follows

$$(V_{\varphi_1,\psi_1} - V_{\varphi_2,\psi_2})(f)(z) = \int_0^z \big((f \circ \varphi_1)(\xi)(\psi_1 \circ \varphi_1)'(\xi) - (f \circ \varphi_2)(\xi)(\psi_2 \circ \varphi_1)'(\xi)\big) d\xi, z \in \mathbb{D}.$$

Differences of composition operators was studied first on the Hardy space $H^2(\mathbb{D})$ in [3]. Recently Nieminen [13] has characterized the compactness of difference of weighted composition operators $W_{\varphi_1,\psi_1} - W_{\varphi_2,\psi_2}$ on weighted-type space given by standard weights. Lindström and wolf [9] have generalized Nieminen's result to more general weights v and u and found an expression for the essential norm $\|W_{\varphi_1,\psi_1} - W_{\varphi_2,\psi_2}\|_{e, H_v^\infty \to H_u^\infty}$, where $\max\{\|\varphi_1\|_\infty, \|\varphi_2\|_\infty\} = 1$.

Here we continue this line of research and investigate the metrically compactness of differences of Volterra composition operators acting from the weighted Bergman-type space A_u^p to the Bloch-type space B_v^∞ on the open unit disk. These results extend the corresponding results on the single Volterra composition operators (see, for example, [7, 14, 17]).

For $w \in \mathbb{D}$, let σ_w be the Möbius transformation of $\mathbb{D}$ defined by $\sigma_w(z) = (w - z)/(1 - \overline{w} z)$. Note that the pseudo-hyperbolic metric $\rho(z, w) = |\sigma_w(z)|$.

Throughout this paper, constants are denoted by C, they are positive and may differ from one occurrence to the other. The notation $a \asymp b$ means that there is a positive constant C such that $a/C \le b \le Ca$.

2. Auxiliary results

The proof of the following lemma is standard, so it will be omitted (see, e.g., Lemma 3 in [15]).

Lemma 1. *Assume that $p > 0$, u is a normal function on $[0, 1)$, v is a weight on $\mathbb{D}$, φ_1, φ_2 are analytic self-maps of $\mathbb{D}$, ψ_1, ψ_2 are analytic functions on $\mathbb{D}$ and the operator $V_{\varphi_1,\psi_1} - V_{\varphi_2,\psi_2} : A_u^p \to B_v^\infty$ is metrically bounded. Then the operator $V_{\varphi_1,\psi_1} - V_{\varphi_2,u_2} : A_u^p \to B_v^\infty$ is metrically compact if and only if for every bounded* *sequence* $(f_n)_{n\in\mathbb{N}}$ *in* A^p_u *such that* $f_n \to 0$ *uniformly on every compact subset of* $\mathbb{D}$ *as* $n \to \infty$ *it follows that*

$$\lim_{n\to\infty} \|(V_{\varphi_1,\psi_1} - V_{\varphi_2,\psi_2})f_n\|_v = 0.$$

The following lemma was proved in [8].

Lemma 2. *There exists a constant* $C > 0$ *independent of* $f \in A^p_u$ *such that*

$$|f(z)| \le \frac{C\|f\|_{p,u}}{u(|z|)(1-|z|^2)^{1/p}}. \tag{1}$$

Lemma 3. *Let* $p > 0$, u *is a normal function on* $[0,1)$, v *is a weight on* $\mathbb{D}$, φ *is an analytic self-map of* $\mathbb{D}$ *and* ψ *is an analytic function on* $\mathbb{D}$. *Then the operator* $V_{\varphi,\psi} : A^p_u \to B^\infty_v$ *is metrically bounded if and only if*

$$\sup_{z\in\mathbb{D}} \frac{v(z)|\varphi'(z)||\psi'(z)|}{u(|\varphi(z)|)(1-|\varphi(z)|^2)^{1/p}} < \infty. \tag{2}$$

Proof. Suppose that $V_{\varphi,\psi} : A^p_u \to B^\infty_v$ is metrically bounded. For a fixed $w \in \mathbb{D}$, setting

$$f_w(z) = \frac{(1-|\varphi(w)|^2)^{t+1}}{u(|\varphi(w)|)(1-\overline{\varphi(w)}z)^{1/p+t+1}},$$

then it is easy to show $f_w \in A^p_u$ and $\|f_w\|_{p,u} \le C$. Thus

$$\begin{aligned} C\|V_{\varphi,\psi}\| &\ge \|V_{\varphi,\psi}f_w\|_v = \sup_{z\in\mathbb{D}} v(z)|\varphi'(z)||\psi'(z)||f_w(\varphi(z))| \\ &\ge v(w)|\varphi'(w)||\psi'(w)||f_w(\varphi(w))| \\ &= \frac{v(w)|\varphi'(w)||\psi'(w)|}{u(|\varphi(w)|)(1-|\varphi(w)|^2)^{1/p}}. \end{aligned}$$

So, we prove that (2) holds.

If (2) holds, by Lemma 2, then we have

$$\begin{aligned} \|V_{\varphi,\psi}f\|_v &= \sup_{z\in\mathbb{D}} v(z)|\varphi'(z)||\psi'(z)||f(\varphi(z))| \\ &\le C\sup_{z\in\mathbb{D}} \frac{v(z)|\varphi'(z)||\psi'(z)|}{u(|\varphi(z)|)(1-|\varphi(z)|^2)^{1/p}}\|f\|_{p,u}. \end{aligned}$$

It follows that $V_{\varphi,\psi} : A^p_u \to B^\infty_v$ is metrically bounded. □

The next lemma shows that $H^\infty \subseteq A^p_u$.

Lemma 4. *Assume that* $p > 0$ *and* u *is a normal function on* $[0,1)$. *Then* $H^\infty \subseteq A^p_u$.

Proof. For $f \in H^\infty$, we assume that $|f(z)| \le M$ for all $z \in \mathbb{D}$. Then by the definition of the normal function and the Beta function,

$$\begin{aligned} \|f\|^p_{p,u} &= \int_{\mathbb{D}} |f(z)|^p \frac{u^p(|z|)}{1-|z|} dA(z) \le M \int_{\mathbb{D}} \frac{u^p(|z|)}{1-|z|} dA(z) \\ &= M \int_{\mathbb{D}} \frac{u^p(|z|)}{(1-|z|)^{ps}} (1-|z|)^{ps-1} dA(z) \\ &= \frac{M}{\pi} \int_0^{2\pi} \int_0^1 \frac{u^p(r)}{(1-r)^{ps}} (1-r)^{ps-1} r dr d\theta \end{aligned}$$

$$\leq \quad 2Mu^p(0)B(2,ps),$$

where $B(2,ps)$ is the Beta function. Thus we prove that $f \in A_u^p$. $\square$

The following lemma is very useful in the proof of the main result.

Lemma 5. *Assume that u is a normal function on $[0,1)$ such that u is continuously differentiable. Then there exists a constant $C>0$ such that*

$$\Big|u(|z|)(1-|z|^2)^{1/p}f(z)-u(|w|)(1-|w|^2)^{1/p}f(w)\Big| \leq C\|f\|_{p,u}\rho(z,w) \tag{3}$$

for all $f \in A_u^p$ and for all z,w in $\mathbb{D}$.

Proof. By Lemma 3 we have that if $f \in A_u^p$, then $f \in H^\infty_{u(|z|)(1-|z|^2)^{1/p}}$ and moreover $\|f\|_{u(|z|)(1-|z|^2)^{1/p}} \leq C\|f\|_{p,u}$. By the definition of normal function, it follows that

$$\frac{u(|z|)(1-|z|^2)^{1/p}}{(1-|z|)^{1/p+t}}$$

is increasing on $[0,1)$, where t is the positive number in the definition of normal function. Then by the proof in [9], we obtain that $u(|z|)(1-|z|^2)^{1/p}$ satisfies the following so-called Lusky condition (which is due to Lusky [11])

$$\inf_{n\in\mathbb{N}} \frac{u(1-2^{-n-1})(1-(1-2^{-n-1})^2)^{1/p}}{u(1-2^{-n})(1-(1-2^{-n})^2)^{1/p}} > 0.$$

Therefore, by the Lemma 1 in [9], for each $f \in H^\infty_{u(|z|)(1-|z|^2)^{1/p}}$ and $z, V \in \mathbb{D}$ there exists a $C>0$ such that

$$\begin{aligned}\Big|u(|z|)(1-|z|^2)^{1/p}f(z)-u(|w|)(1-|w|^2)^{1/p}f(w)\Big| &\leq C\|f\|_{u(|z|)(1-|z|^2)^{1/p}}\rho(z,w)\\ &\leq C\|f\|_{p,u}\rho(z,w).\end{aligned}$$

From this inequality estimate (3) follows. $\square$

3. Main results

In this section we formulate and prove the main result of this paper.

Theorem 1. *Assume that $p>0$, u is a normal function on $[0,1)$ such that u is continuously differentiable, v is a weight on $\mathbb{D}$, φ_1, φ_2 are nonconstant analytic self-maps of $\mathbb{D}$, ψ_1, ψ_2 are analytic functions on $\mathbb{D}$ and V_{φ_1,ψ_1}, $V_{\varphi_2,\psi_2}: A_u^p \to B_v^\infty$ are metrically bounded operators. Then the operator $V_{\varphi_1,\psi_1}-V_{\varphi_2,\psi_2}: A_u^p \to B_v^\infty$ is metrically compact if and only if the following conditions hold:*

(a)

$$\lim_{|\varphi_1(z)|\to 1} \frac{v(z)|\varphi_1'(z)||\psi_1'(z)|}{u(|\varphi_1(z)|)(1-|\varphi_1(z)|^2)^{\frac{1}{p}}}\,\rho(\varphi_1(z),\varphi_2(z))=0;$$

(b)

$$\lim_{|\varphi_2(z)|\to 1} \frac{v(z)|\varphi_2'(z)||\psi_2'(z)|}{u(|\varphi_2(z)|)(1-|\varphi_2(z)|^2)^{\frac{1}{p}}}\,\rho(\varphi_1(z),\varphi_2(z))=0;$$

(c)

$$\lim_{\min\{|\varphi_1(z)|,|\varphi_2(z)|\}\to 1} v(z)\Big|\frac{\varphi_1'(z)\psi_1'(z)}{u(|\varphi_1(z)|)(1-|\varphi_1(z)|^2)^{\frac{1}{p}}}-\frac{\varphi_2'(z)\psi_2'(z)}{u(|\varphi_2(z)|)(1-|\varphi_2(z)|^2)^{\frac{1}{p}}}\Big|=0.$$

Proof. Suppose that the operator $V_{\varphi_1,\psi_1} - V_{\varphi_2,\psi_2} : A^p_u \to B^\infty_v$ is metrically compact. If $\|\varphi_1\|_\infty < 1$, then *(a)* vacuously holds. Hence assume that $\|\varphi_1\|_\infty = 1$. Suppose to the contrary that *(a)* is not true. Then there exists a sequence $(z_n)_{n\in\mathbb{N}}$ such that $|\varphi_1(z_n)| \to 1$ as $n \to \infty$ and

$$\delta := \lim_{n\to\infty} \frac{v(z_n)|\varphi_1'(z_n)||\psi_1'(z_n)|}{u(|\varphi_1(z_n)|)(1-|\varphi_1(z_n)|^2)^{1/p}}\,\rho(\varphi_1(z_n),\varphi_2(z_n)) > 0. \tag{4}$$

Since $|\varphi_1(z_n)| \to 1$ as $n \to \infty$, we can use the proof of Theorem 3.1 in [6] to find functions $f_n \in H^\infty$, $n \in \mathbb{N}$, such that

$$\sum_{n=1}^{\infty} |f_n(z)| \le 1, \quad \text{for all } z \in \mathbb{D}, \tag{5}$$

and

$$f_n(\varphi_1(z_n)) > 1 - \frac{1}{2^n}, \quad n \in \mathbb{N}. \tag{6}$$

Since $f_n \in H^\infty$, by Lemma 4 we have that $f_n \in A^p_u$ and $\|f_n\|_{p,u} \le C$ for all $n \in \mathbb{N}$. Note that form (6) it follows that $\lim\limits_{n\to\infty} |f_n(\varphi_1(z_n))| = 1$. Now, we define

$$k_n(z) = \frac{(1-|\varphi(z_n)|^2)^{t+1}}{u(|\varphi(z_n)|)(1-\overline{\varphi(z_n)}z)^{1/p+t+1}}, \quad n \in \mathbb{N}.$$

By the proof of Theorem 3.1 in [8], we obtain that that $\sup_{n\in\mathbb{N}} \|k_n\|_{p,u} \le C$. Put $g_n(z) = f_n(z)\sigma_{\varphi_2(z_n)}(z)k_n(z), n \in \mathbb{N}$. Then clearly $g_n \in A^p_u$ with $\sup_{n\in\mathbb{N}} \|g_n\|_{p,u} \le C$ and $g_n \to 0$ uniformly on compact subsets of $\mathbb{D}$ as $n \to \infty$. Since $V_{\varphi_1,\psi_1} - V_{\varphi_2,\psi_2} : A^p_u \to B^\infty_v$ is metrically compact, by Lemma 1 we get

$$\lim_{n\to\infty} \|(V_{\varphi_1,\psi_1} - V_{\varphi_2,\psi_2})g_n\|_v = 0. \tag{7}$$

On the other hand, from the definition of the space B^∞_v, the definition of functions g_n and by using (6), we have that

$$\begin{aligned}\|(V_{\varphi_1,\psi_1} - V_{\varphi_2,\psi_2})g_n\|_v \ge& v(z_n)\big|\varphi_1'(z_n)\psi_1'(z_n)g_n(\varphi_1(z_n)) - \varphi_2'(z_n)\psi_2'(z_n)g_n(\varphi_2(z_n))\big|\\ =& v(z_n)\big|\varphi_1'(z_n)\psi_1'(z_n)f_n(\varphi_1(z_n))\sigma_{\varphi_2(z_n)}(\varphi_1(z_n))k_n(\varphi_1(z_n))\big|\\ \ge& \frac{v(z_n)|\varphi_1'(z_n)||\psi_1'(z_n)|\rho(\varphi_1(z_n),\varphi_2(z_n))}{u(|\varphi_1(z_n)|)(1-|\varphi_1(z_n)|^2)^{\frac{1}{p}}}\big(1-\frac{1}{2^n}\big).\end{aligned} \tag{8}$$

Letting $n \to \infty$ in (8) and using (4), we obtain

$$\lim_{n\to\infty} \|(V_{\varphi_1,\psi_1} - V_{\varphi_2,\psi_2})g_n\|_v \ge \lim_{n\to\infty} \frac{v(z_n)|\varphi_1'(z_n)||\psi_1'(z_n)|\rho(\varphi_1(z_n),\varphi_2(z_n))}{u(|\varphi_1(z_n)|)(1-|\varphi_1(z_n)|^2)^{\frac{1}{p}}} = \delta > 0,$$

which contradicts (7). This shows that

$$\lim_{n\to\infty} \frac{v(z_n)|\varphi_1'(z_n)||\psi_1'(z_n)|}{u(|\varphi_1(z_n)|)(1-|\varphi_1(z_n)|^2)^{\frac{1}{p}}}\rho(\varphi_1(z_n),\varphi_2(z_n)) = 0,$$

for every sequence $(z_n)_{n\in\mathbb{N}}$ such that $|\varphi_1(z_n)| \to 1$ as $n \to \infty$, which implies *(a)*.

Condition *(b)* is proved similarly. Hence we omit it.

Now, we prove *(c)*. Suppose to the contrary that *(c)* does not hold. Then there is a sequence $(z_n)_{n\in\mathbb{N}}$ such that $\min\{|\varphi_1(z_n)|, |\varphi_2(z_n)|\} \to 1$ as $n \to \infty$ and

$$\beta := \lim_{n\to\infty} v(z_n)\Big|\frac{\varphi_1'(z_n)\psi_1'(z_n)}{u(|\varphi_1(z_n)|)(1-|\varphi_1(z_n)|^2)^{\frac{1}{p}}} - \frac{\varphi_2'(z_n)\psi_2'(z_n)}{u(|\varphi_2(z_n)|)(1-|\varphi_2(z_n)|^2)^{\frac{1}{p}}}\Big|. \tag{9}$$

We may also assume that there is the following limit

$$l := \lim_{n\to\infty} \rho(\varphi_1(z_n), \varphi_2(z_n)) \geq 0. \tag{10}$$

Assume that $l > 0$. Then we have that for sufficiently large n, say $n \geq n_0$

$$\begin{aligned} 0 <\frac{\beta}{2} \leq v(z_n)\Big|&\frac{\varphi_1'(z_n)\psi_1'(z_n)}{u(|\varphi_1(z_n)|)(1-|\varphi_1(z_n)|^2)^{\frac{1}{p}}} - \frac{\varphi_2'(z_n)\psi_2'(z_n)}{u(|\varphi_2(z_n)|)(1-|\varphi_2(z_n)|^2)^{\frac{1}{p}}}\Big| \\ \leq\frac{2}{l}\Big(&\frac{v(z_n)|\varphi_1'(z_n)||\psi_1'(z_n)|}{u(|\varphi_1(z_n)|)(1-|\varphi_1(z_n)|^2)^{\frac{1}{p}}} + \frac{v(z_n)|\varphi_2'(z_n)||\psi_2'(z_n)|}{u(|\varphi_2(z_n)|)(1-|\varphi_2(z_n)|^2)^{\frac{1}{p}}}\Big)\rho(\varphi_1(z_n), \varphi_2(z_n)). \end{aligned} \tag{11}$$

Letting $n \to \infty$ in (11) and using *(a)* and *(b)*, we arrive at a contradiction. Thus, we can assume that $l = 0$. Let the sequences of functions $(f_n)_{n\in\mathbb{N}}$ and $(k_n)_{n\in\mathbb{N}}$ be defined as above. Set

$$h_n(z) = f_n(z)k_n(z), \quad n \in \mathbb{N}.$$

Then $\sup_{n\in\mathbb{N}} \|h_n\|_{p,u} \leq C$ and $h_n \to 0$ uniformly on compact subsets of $\mathbb{D}$ as $n \to \infty$. Hence by Lemma 1

$$\lim_{n\to\infty} \|(V_{\varphi_1,\psi_1} - V_{\varphi_2,\psi_2})h_n\|_v = 0. \tag{12}$$

Since $V_{\varphi_2,\psi_2} : A^p_u \to B^\infty_v$ is metrically bounded, then by Lemma 3 we have that

$$M := \sup_{z\in\mathbb{D}} \frac{v(z)|\varphi_2'(z)||\psi_2'(z)|}{u(|\varphi_2(z)|)(1-|\varphi_2(z)|^2)^{1/p}} < \infty. \tag{13}$$

We have

$$\begin{aligned} \|(V_{\varphi_1,\psi_1} - V_{\varphi_2,\psi_2})h_n\|_v &\geq v(z_n)\big|\varphi_1'(z_n)\psi_1'(z_n)h_n(\varphi_1(z_n)) - \varphi_2'(z_n)\psi_2'(z_n)h_n(\varphi_2(z_n))\big| \\ &= v(z_n)\big|\varphi_1'(z_n)\psi_1'(z_n)f_n(\varphi_1(z_n))k_n(\varphi_1(z_n)) - \varphi_2'(z_n)\psi_2'(z_n)f_n(\varphi_2(z_n))k_n(\varphi_2(z_n))\big| \\ &\geq v(z_n)\Big|\frac{\varphi_1'(z_n)\psi_1'(z_n)f_n(\varphi_1(z_n))}{u(|\varphi_1(z_n)|)(1-|\varphi_1(z_n)|^2)^{1/p}} - \frac{\varphi_2'(z_n)\psi_2'(z_n)f_n(\varphi_1(z_n))}{u(|\varphi_2(z_n)|)(1-|\varphi_2(z_n)|^2)^{1/p}}\Big| \\ &\quad - v(z_n)\Big|\frac{\varphi_2'(z_n)\psi_2'(z_n)f_n(\varphi_1(z_n))}{u(|\varphi_2(z_n)|)(1-|\varphi_2(z_n)|^2)^{1/p}} - \varphi_2'(z_n)\psi_2'(z_n)f_n(\varphi_2(z_n))k_n(\varphi_2(z_n))\Big| \\ &\geq v(z_n)\Big|\frac{\varphi_1'(z_n)\psi_1'(z_n)}{u(|\varphi_1(z_n)|)(1-|\varphi_1(z_n)|^2)^{1/p}} - \frac{\varphi_2'(z_n)\psi_2'(z_n)}{u(|\varphi_2(z_n)|)(1-|\varphi_2(z_n)|^2)^{1/p}}\Big|(1-\frac{1}{2^n}) \\ &\quad - \frac{v(z_n)|\varphi_2'(z_n)||\psi_2'(z_n)|}{u(|\varphi_2(z_n)|)(1-|\varphi_2(z_n)|^2)^{1/p}}\Big|u(|\varphi_1(z_n)|)(1-|\varphi_1(z_n)|^2)^{1/p}h_n(\varphi_1(z_n)) \\ &\quad - u(|\varphi_2(z_n)|)(1-|\varphi_2(z_n)|^2)^{1/p}h_n(\varphi_2(z_n))\Big|. \end{aligned} \tag{14}$$

From (13), applying Lemma 5 to the functions h_n with the points $z = \varphi_1(z_n)$ and $w = \varphi_2(z_n)$, and by using the fact $\sup_{n\in\mathbb{N}} \|h_n\|_{p,u} \leq C$, we get

$$\begin{aligned} &\frac{v(z_n)|\varphi_2'(z_n)||\psi_2'(z_n)|}{u(|\varphi_2(z_n)|)(1-|\varphi_2(z_n)|^2)^{1/p}}\Big|u(|\varphi_1(z_n)|)(1-|\varphi_1(z_n)|^2)^{1/p}h_n(\varphi_1(z_n)) - u(|\varphi_2(z_n)|) \\ &(1-|\varphi_2(z_n)|^2)^{1/p}h_n(\varphi_2(z_n))\Big| \leq CM\rho(\varphi_1(z_n), \varphi_2(z_n)). \end{aligned} \tag{15}$$

Using (15) in (14), then letting $n \to \infty$ is such obtained inequality and using (12) we obtain that $\beta = 0$, which is a contradiction. This proves *(c)*.

Now we assume that conditions *(a)*-*(c)* hold. Assume $(f_n)_{n\in\mathbb{N}}$ is a bounded sequence in A^p_u such that $f_n \to 0$ uniformly on compact subsets of $\mathbb{D}$. To prove that $V_{\varphi_1,\psi_1}-V_{\varphi_2,\psi_2}:A^p_u\to B^\infty_v$ is a metrically compact operator, in view of Lemma 1, it is enough to show that $\|(V_{\varphi_1,\psi_1}-V_{\varphi_2,\psi_2})f_n\|_v\to 0$ as $n\to\infty$. Suppose to the contrary that this is not true. Then for some $\varepsilon>0$ there is a subsequence $(f_{n_k})_{k\in\mathbb{N}}$ of $(f_n)_{n\in\mathbb{N}}$ such that $\|(V_{\varphi_1,\psi_1}-V_{\varphi_2,\psi_2})f_{n_k}\|_v\geq 2\varepsilon>0$ for every $k\in\mathbb{N}$. We may assume that $(f_{n_k})_{k\in\mathbb{N}}$ is $(f_n)_{n\in\mathbb{N}}$. Then there is a sequence $(z_n)_{n\in\mathbb{N}}$ in $\mathbb{D}$ such that

$$v(z_n)\big|\varphi_1'(z_n)\psi_1'(z_n)f_n(\varphi_1(z_n))-\varphi_2'(z_n)\psi_2'(z_n)f_n(\varphi_2(z_n))\big|\geq\varepsilon>0,\quad n\in\mathbb{N}. \tag{16}$$

We may also assume that the sequences $(\varphi_1(z_n))_{n\in\mathbb{N}}$ and $(\varphi_2(z_n))_{n\in\mathbb{N}}$ converge. If it were $\max\{|\varphi_1(z_n)|,|\varphi_2(z_n)|\}\to q<1$, then from (16), since for the test function $f(z)\equiv 1\in A^p_u$ (by Lemma 4), from the boundedness of the operators $V_{\varphi_i,\psi_i}:A^p_u\to B^\infty_v$, $i=1,2$, we have that $\psi_1\circ\varphi_1,\psi_2\circ\varphi_2\in B^\infty_v$ and since $f_n(\varphi_i(z_n))\to 0$ as $n\to\infty$, $i=1,2$, we would obtain a contradiction. Hence $\max\{|\varphi_1(z_n)|,|\varphi_2(z_n)|\}\to 1$ as $n\to\infty$. We can suppose that $|\varphi_1(z_n)|\to 1$ and $\varphi_2(z_n)\to z_0$ as $n\to\infty$. Also, we can suppose that limit in (10) exists. Assume that $l>0$. Then by *(a)* and *(b)*, we get

$$\lim_{|\varphi_1(z_n)|\to 1}\frac{v(z_n)|\varphi_1'(z_n)||\psi_1'(z_n)|}{u(|\varphi_1(z_n)|)(1-|\varphi_1(z_n)|^2)^{1/p}}\,\rho(\varphi_1(z_n),\varphi_2(z_n))=0 \tag{17}$$

and

$$\lim_{|\varphi_2(z_n)|\to 1}\frac{v(z_n)|\varphi_2'(z_n)||\psi_2'(z_n)|}{u(|\varphi_2(z_n)|)(1-|\varphi_2(z_n)|^2)^{1/p}}\,\rho(\varphi_1(z_n),\varphi_2(z_n))=0. \tag{18}$$

From (16) and Lemma 2, it follows that

$$\begin{aligned}0<&\varepsilon\leq\frac{v(z_n)|\varphi_1'(z_n)||\psi_1'(z_n)|}{u(|\varphi_1(z_n)|)(1-|\varphi_1(z_n)|^2)^{\frac1p}}\Big|u(|\varphi_1(z_n)|)(1-|\varphi_1(z_n)|^2)^{\frac1p}f_n(\varphi_1(z_n))\Big|\\&+\frac{v(z_n)|\varphi_2'(z_n)||\psi_2'(z_n)|}{u(|\varphi_2(z_n)|)(1-|\varphi_2(z_n)|^2)^{\frac1p}}\Big|u(|\varphi_2(z_n)|)(1-|\varphi_2(z_n)|^2)^{\frac1p}f_n(\varphi_2(z_n))\Big|\\\leq&C\Big(\frac{v(z_n)|\varphi_1'(z_n)||\psi_1'(z_n)|}{u(|\varphi_1(z_n)|)(1-|\varphi_1(z_n)|^2)^{\frac1p}}+\frac{v(z_n)|\varphi_2'(z_n)||\psi_2'(z_n)|}{u(|\varphi_2(z_n)|)(1-|\varphi_2(z_n)|^2)^{\frac1p}}\Big)\|f_n\|_{p,u}.\end{aligned} \tag{19}$$

Letting $n\to\infty$ in (19) and using (18) we obtain a contradiction. Thus, we conclude that $l=0$ which implies that $|\varphi_2(z_n)|\to 1$ as $n\to\infty$. From (16), Lemmas 2, 3 and 5, and using *(a)* and *(b)* we have

$$\begin{aligned}&0<\varepsilon\leq v(z_n)\big|\varphi_1'(z_n)\psi_1'(z_n)f(\varphi_1(z_n))-\varphi_2'(z_n)\psi_2'(z_n)f(\varphi_2(z_n))\big|\\&\leq\frac{v(z_n)|\varphi_1'(z_n)||\psi_1'(z_n)|}{u(|\varphi_1(z_n)|)(1-|\varphi_1(z_n)|^2)^{\frac1p}}\Big|u(|\varphi_1(z_n)|)(1-|\varphi_1(z_n)|^2)^{\frac1p}f(\varphi_1(z_n))\\&\quad-u(|\varphi_2(z_n)|)(1-|\varphi_2(z_n)|^2)^{\frac1p}f(\varphi_2(z_n))\Big|+v(z_n)\Big|\frac{\varphi_1'(z_n)\psi_1'(z_n)}{u(|\varphi_1(z_n)|)(1-|\varphi_1(z_n)|^2)^{\frac1p}}\\&\quad-\frac{\varphi_2'(z_n)\psi_2'(z_n)}{u(|\varphi_2(z_n)|)(1-|\varphi_2(z_n)|^2)^{\frac1p}}\Big|u(|\varphi_2(z_n)|)(1-|\varphi_2(z_n)|^2)^{\frac1p}|f(\varphi_2(z_n))|\\&\quad\leq C\frac{v(z_n)|\varphi_1'(z_n)||\psi_1'(z_n)|}{u(|\varphi_1(z_n)|)(1-|\varphi_1(z_n)|^2)^{\frac1p}}\|f_n\|_{p,u}\rho(\varphi_1(z_n),\varphi_2(z_n))+v(z_n)\\&\quad\times\Big|\frac{\varphi_1'(z_n)\psi_1'(z_n)}{u(|\varphi_1(z_n)|)(1-|\varphi_1(z_n)|^2)^{\frac1p}}-\frac{\varphi_2'(z_n)\psi_2'(z_n)}{u(|\varphi_2(z_n)|)(1-|\varphi_2(z_n)|^2)^{\frac1p}}\Big|\|f_n\|_{p,u}\\&\quad\longrightarrow 0,\end{aligned}$$

as $n \to \infty$, which is a contradiction. The proof is complete. $\square$

Acknowledgments. This work is supported by the Introduction of Talent Project of SUSE (Grant No.2011RC13).

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J. COMPUTATIONAL ANALYSIS AND APPLICATIONS, VOL. 15, NO.6, 1158-1164, 2013, COPYRIGHT 2013 EUDOXUS PRESS, LLC

SOME NEW ERROR INEQUALITIES FOR A TAYLOR-LIKE FORMULA

WENJUN LIU AND QILIN ZHANG

Abstract. Some new error inequalities for a Taylor-like formula are established. Sharp bounds are given when n is an odd and even integer, respectively.

1. Introduction

Error analysis for the Taylor and generalized Taylor formulas has been extensively studied in recent years. The approach from an inequalities point of view to estimate the error terms has been used in these studies (see [1]-[18] and the references therein). In [19], by appropriately choosing the Peano kernel

$$G_n(x) = \begin{cases} \dfrac{1}{n!}\left(x-\dfrac{3a+t}{4}\right)^{n-1}\left[x+\dfrac{(n-3)a-(n+1)t}{4}\right], & x\in\left[a,\dfrac{a+t}{2}\right], \\ \dfrac{1}{n!}\left(x-\dfrac{a+3t}{4}\right)^{n-1}\left[x+\dfrac{(n-3)t-(n+1)a}{4}\right], & x\in\left(\dfrac{a+t}{2},t\right], \end{cases} \tag{1}$$

a Taylor-like formula was derived as follows.

Lemma 1. ([19]) *Let $f:[a,t]\to\mathbb{R}$ be a function such that $f^{(n)}$ is absolutely continuous. Then*

$$\begin{aligned} f(t) =& f(a)-\sum_{k=1}^{n}\frac{(-1)^k(t-a)^k}{4^k k!}(1+k)\left[f^k(t)-(-1)^k f^k(a)\right] \\ &-\sum_{k=2}^{n}\frac{(-1)^k(t-a)^k}{4^k k!}(1-k)[1-(-1)^k]f^k\left(\frac{a+t}{2}\right)+R(f). \end{aligned} \tag{2}$$

By introducing the notations

$$\begin{aligned} F_n(t,a) =& f(a)-\sum_{k=1}^{n}\frac{(-1)^k(t-a)^k}{4^k k!}(1+k)\left[f^k(t)-(-1)^k f^k(a)\right] \\ &-\sum_{k=2}^{n}\frac{(-1)^k(t-a)^k}{4^k k!}(1-k)[1-(-1)^k]f^k\left(\frac{a+t}{2}\right), \end{aligned}$$

the following error inequalities were derived in [19].

Theorem 1. *Let $f:[a,t]\to\mathbb{R}$ be a function such that $f^{(n)}$ is absolutely continuous. If there exist real numbers γ_n,Γ_n such that $\gamma_n\le f^{(n+1)}(x)\le\Gamma_n$, $x\in[a,t]$, then*

$$|f(t)-F_n(t,a)|\le\frac{\Gamma_n-\gamma_n}{(n+1)!}\frac{2n+2}{4^{n+1}}(t-a)^{n+1},\quad \textit{if } n \textit{ is odd} \tag{3}$$

2010 *Mathematics Subject Classification.* 26D10, 41A58, 41A80.

Key words and phrases. Taylor-like formula, approximation error, inequality, sharp bound.

and

$$|f(t) - F_n(t,a)| \leq \frac{1}{n!\,4^n} \|f^{(n+1)}\|_\infty (t-a)^{n+1}, \quad \textit{if } n \textit{ is even.} \tag{4}$$

If there exists a real number γ_n such that $\gamma_n \leq f^{(n+1)}(x)$, $x \in [a,t]$, then

$$|f(t) - F_n(t,a)| \leq \left[\frac{f^{(n)}(t) - f^{(n)}(a)}{t-a} - \gamma_n\right] \frac{n+1}{n!\,4^n}(t-a)^{n+1}, \quad \textit{if } n \textit{ is odd.} \tag{5}$$

If there exists a real number Γ_n such that $f^{(n+1)}(x) \leq \Gamma_n$, $x \in [a,t]$, then

$$|f(t) - F_n(t,a)| \leq \left[\Gamma_n - \frac{f^{(n)}(t) - f^{(n)}(a)}{t-a}\right] \frac{n+1}{n!\,4^n}(t-a)^{n+1}, \quad \textit{if } n \textit{ is odd.} \tag{6}$$

The purpose of this paper is to establish some new error inequalities for the above Taylor-like formula. Especially, sharp bounds will be given when n is an odd and even integer, respectively.

2. Main results

The following lemma is needed in the proof of our main results.

Lemma 2. *The Peano kernels $G_n(t)$, satisfy*

$$\int_a^t G_n(x)dx = \begin{cases} 0, & n \textit{ odd}, \\ \dfrac{2}{(n+1)!\,4^n}(t-a)^{n+1}, & n \textit{ even}, \end{cases} \tag{7}$$

$$\int_a^t |G_n(x)|dx = \frac{1}{n!\,4^n}(t-a)^{n+1}, \tag{8}$$

$$\max_{x\in[a,t]} |G_n(x)| = \frac{n+1}{n!\,4^n}(t-a)^n, \tag{9}$$

$$\int_a^t G_n^2(x)dx = \frac{2n^3+n^2+2n-1}{(2n+1)(2n-1)(n!)^2 4^{2n}}(t-a)^{2n+1}, \tag{10}$$

$$\max_{x\in[a,t]} \left| G_{2m}(x) - \frac{1}{t-a}\int_a^t G_{2m}(x)dx \right| = \frac{4m^2+4m-1}{(2m+1)!\,4^{2m}}(t-a)^{2m}. \tag{11}$$

Proof. The proof of (7)-(9) were given in [19]. (10) can be obtained by a direct calculation.

From (7), we have

$$\begin{aligned}
&\max_{x\in[a,t]} \left| G_{2m}(x) - \frac{1}{t-a}\int_a^t G_{2m}(x)dx \right| = \max_{x\in[a,t]} \left| G_{2m}(x) - \frac{2(t-a)^{2m}}{(2m+1)!\,4^{2m}} \right| \\
=&\max\Bigg\{ \max_{x\in\left[a,\frac{a+t}{2}\right]} \left| \frac{1}{(2m)!}\left(x - \frac{3a+t}{4}\right)^{2m-1} \left[x + \frac{(2m-3)a-(2m+1)t}{4}\right] - \frac{2(t-a)^{2m}}{(2m+1)!\,4^{2m}} \right|, \\
&\max_{x\in\left[\frac{a+t}{2},t\right]} \left| \frac{1}{(2m)!}\left(x - \frac{a+3t}{4}\right)^{2m-1} \left[x + \frac{(2m-3)t-(2m+1)a}{4}\right] - \frac{2(t-a)^{2m}}{(2m+1)!\,4^{2m}} \right| \Bigg\} \\
=&\frac{(t-a)^{2m}}{(2m-1)!\,4^{2m}} \max\left\{ \left|\frac{1}{2m} + 1 - \frac{2}{2m+1}\right|, \left|\frac{1}{2m} - 1 - \frac{2}{2m+1}\right| \right\} \\
=&\frac{4m^2+4m-1}{(2m+1)!\,4^{2m}}(t-a)^{2m}.
\end{aligned}$$

Thus, (11) is obtained. □

We first establish two new error inequalities for $f^{(n+1)} \in L^1[a,b]$ and $f^{(n+1)} \in L^2[a,b]$, respectively.

Theorem 2. *Let $f:[a,t]\to\mathbb{R}$ be a function such that $f^{(n)}$ is absolutely continuous on $[a,t]$. If $f^{(n+1)}\in L^1[a,t]$, then we have*

$$|f(t)-F_n(t,a)| \le \frac{n+1}{n!\,4^n}\|f^{(n+1)}\|_1 (t-a)^n, \tag{12}$$

where $\|f^{(n+1)}\|_1 := \int_a^t |f^{(n+1)}(x)|dx$ is the usual Lebesgue norm on $L^1[a,t]$.

Proof. By using the identity (2), we have

$$|f(t)-F_n(t,a)| = \left|\int_a^t G_n(x)f^{(n+1)}(x)dx\right| \le \max_{x\in[a,t]}|G_n(x)|\int_a^t |f^{(n+1)}(x)|dx. \tag{13}$$

Consequently, the inequality (12) follows from (13) and (9). □

Theorem 3. *Let $f:[a,t]\to\mathbb{R}$ be a function such that $f^{(n)}$ is absolutely continuous on $[a,t]$. If $f^{(n+1)}\in L^2[a,t]$, then we have*

$$|f(t)-F_n(t,a)| \le \frac{\sqrt{2n^3+n^2+2n-1}}{\sqrt{(2n+1)(2n-1)}\,n!\,4^n}\|f^{(n+1)}\|_2 (t-a)^{n+\frac{1}{2}}, \tag{14}$$

where $\|f^{(n+1)}\|_2 := \left(\int_a^t |f^{(n+1)}(x)|^2dx\right)^{\frac{1}{2}}$ is the usual Lebesgue norm on $L^2[a,t]$.

Proof. By using the identity (2), we have

$$|f(t)-F_n(t,a)| = \left|\int_a^t G_n(x)f^{(n+1)}(x)dx\right| \le \|f^{(n+1)}\|_2\|G_n\|_2. \tag{15}$$

Consequently, the inequality (14) follows from (15) and (10). □

Then, if $f^{(n+1)}$ is integrable and bounded and n is an even integer, we prove two perturbed error inequalities.

Theorem 4. *Let $f:[a,b]\to\mathbb{R}$ be such that $f^{(n+1)}$ is integrable with $\gamma_n \le f^{(n+1)}(x)\le \Gamma_n$ for all $x\in[a,t]$, where $\gamma_n,\Gamma_n\in R$ are constants. If n is an even integer ($n=2m$), we have*

$$\begin{aligned}&\left|f(t)-F_{2m}(t,a)-\frac{2(t-a)^{2m+1}}{(2m+1)!\,4^{2m}}\frac{f^{(2m)}(t)-f^{(2m)}(a)}{t-a}\right|\\ &\le \left[\frac{f^{(2m)}(t)-f^{(2m)}(a)}{t-a}-\gamma_{2m}\right]\frac{4m^2+4m-1}{(2m+1)!\,4^{2m}}(t-a)^{2m+1},\end{aligned} \tag{16}$$

$$\begin{aligned}&\left|f(t)-F_{2m}(t,a)-\frac{2(t-a)^{2m+1}}{(2m+1)!\,4^{2m}}\frac{f^{(2m)}(t)-f^{(2m)}(a)}{t-a}\right|\\ &\le \left[\Gamma_{2m}-\frac{f^{(2m)}(t)-f^{(2m)}(a)}{t-a}\right]\frac{4m^2+4m-1}{(2m+1)!\,4^{2m}}(t-a)^{2m+1}.\end{aligned} \tag{17}$$

Proof. By (7) and (2), we can obtain

$$\begin{aligned}
&\left|f(t)-F_{2m}(t,a)-\frac{2(t-a)^{2m+1}}{(2m+1)!\,4^{2m}}\frac{f^{(2m)}(t)-f^{(2m)}(a)}{t-a}\right| \\
=&\left|\int_a^t\left[G_{2m}(x)-\frac{1}{t-a}\int_a^t G_{2m}(x)dx\right][f^{(2m+1)}(x)-C]dx\right|, \qquad (18)
\end{aligned}$$

where $C\in R$ is a constant.

If we choose $C=\gamma_{2m}$, we have

$$\begin{aligned}
&\left|f(t)-F_{2m}(t,a)-\frac{2(t-a)^{2m+1}}{(2m+1)!\,4^{2m}}\frac{f^{(2m)}(t)-f^{(2m)}(a)}{t-a}\right| \\
\le&\max_{x\in[a,t]}\left|G_{2m}(x)-\frac{1}{t-a}\int_a^t G_{2m}(x)dx\right|\int_a^t|f^{(2m+1)}(x)-\gamma_{2m}|dx, \qquad (19)
\end{aligned}$$

and hence the inequality (16) follows from (19) and (11).

Similarly we can prove that the inequality (17) holds. □

Next, we derive two sharp bounds when n is an odd and even integer, respectively.

Theorem 5. *Let $f:[a,t]\to\mathbb{R}$ be a function such that $f^{(n)}$ is absolutely continuous on $[a,t]$ and $f^{(n+1)}\in L^2[a,t]$, where n is an odd integer. Then we have*

$$|f(t)-F_n(t,a)|\le\frac{\sqrt{2n^3+n^2+2n-1}}{\sqrt{(2n+1)(2n-1)}\,n!\,4^n}\sqrt{\sigma(f^{(n+1)})}(t-a)^{n+\frac12}, \qquad (20)$$

where $\sigma(\cdot)$ is defined by $\sigma(f)=\|f\|_2^2-\frac{1}{t-a}\left(\int_a^t f(x)dx\right)^2$. Inequality (20) is sharp in the sense that the constant $\frac{\sqrt{2n^3+n^2+2n-1}}{\sqrt{(2n+1)(2n-1)}\,n!\,4^n}$ cannot be replaced by a smaller one.

Proof. From (2), (7) and (10), we can easily get

$$\begin{aligned}
|f(t)-F_n(t,a)|=&\left|\int_a^t G_n(x)\left[f^{(n+1)}(x)-\frac{1}{t-a}\int_a^t f^{(n+1)}(x)dx\right]dx\right| \\
\le&\left(\int_a^t G_n^2(x)dx\right)^{\frac12}\left(\int_a^t\left[f^{(n+1)}(x)-\frac{1}{t-a}\int_a^t f^{(n+1)}(x)dx\right]^2dx\right)^{\frac12} \\
=&\left(\frac{2n^3+n^2+2n-1}{(2n+1)(2n-1)(n!)^2 4^{2n}}(t-a)^{2n+1}\right)^{\frac12}\left(\|f^{(n+1)}\|_2^2-\frac{[f^{(n)}(t)-f^{(n)}(a)]^2}{t-a}\right)^{\frac12} \\
=&\frac{\sqrt{2n^3+n^2+2n-1}}{\sqrt{(2n+1)(2n-1)}\,n!\,4^n}\sqrt{\sigma(f^{(n+1)})}(t-a)^{n+\frac12}.
\end{aligned}$$

To prove the sharpness of (20), we suppose that (20) holds with a constant $C>0$ as

$$|f(t)-F_n(t,a)|\le C\sqrt{\sigma(f^{(n+1)})}(t-a)^{n+\frac12}. \qquad (21)$$

We may find a function $f:[a,t]\to\mathbb{R}$ such that $f^{(n)}$ is absolutely continuous on $[a,t]$ as

$$f^{(n)}(x)=\begin{cases}\dfrac{1}{(n+1)!}\left(x-\dfrac{3a+t}{4}\right)^n\left[x+\dfrac{(n-2)a-(n+2)t}{4}\right], & x\in\left[a,\dfrac{a+t}{2}\right],\\ \dfrac{1}{(n+1)!}\left(x-\dfrac{a+3t}{4}\right)^n\left[x+\dfrac{(n-2)t-(n+2)a}{4}\right], & x\in\left(\dfrac{a+t}{2},t\right]\end{cases}$$

It follows that

$$f^{(n+1)}(x)=G_n(x). \tag{22}$$

It's easy to find that the left-hand side of the inequality (21) becomes

$$L.H.S.(21)=\frac{2n^3+n^2+2n-1}{(2n+1)(2n-1)(n!)^2 4^{2n}}(t-a)^{2n+1}, \tag{23}$$

and the right-hand side of the inequality (21) is

$$R.H.S.(21)=\frac{\sqrt{2n^3+n^2+2n-1}}{\sqrt{(2n+1)(2n-1)}\,n!\,4^n}C(t-a)^{2n+1}. \tag{24}$$

It follows from (21), (23) and (24) that

$$C\geq\frac{\sqrt{2n^3+n^2+2n-1}}{\sqrt{(2n+1)(2n-1)}\,n!\,4^n},$$

which prove that the constant $\frac{\sqrt{2n^3+n^2+2n-1}}{\sqrt{(2n+1)(2n-1)}\,n!\,4^n}$ is the best possible in (20). ☐

Theorem 6. *Let $f:[a,t]\to\mathbb{R}$ be a function such that $f^{(n)}$ is absolutely continuous on $[a,t]$ and $f^{(n+1)}\in L^2[a,t]$, where n is an even integer ($n=2m$). Then we have*

$$\left|f(t)-F_{2m}(t,a)-\frac{2(t-a)^{2m+1}}{(2m+1)!\,4^{2m}}\frac{f^{(2m)}(t)-f^{(2m)}(a)}{t-a}\right| \tag{25}$$

$$\leq\frac{1}{(2m)!\,4^{2m}}\sqrt{\frac{1}{4m+1}+\frac{4m^2}{4m-1}-\frac{4}{(2m+1)^2}}\sqrt{\sigma(f^{(2m+1)})}(t-a)^{2m+\frac{1}{2}}. \tag{26}$$

Inequality (25) is sharp in the sense that the constant $\frac{1}{(2m)!\,4^{2m}}\sqrt{\frac{1}{4m+1}+\frac{4m^2}{4m-1}-\frac{4}{(2m+1)^2}}$ cannot be replaced by a smaller one.

Proof. From (2), (7) and (10), we can easily obtain

$$\begin{aligned}&\left|f(t)-F_{2m}(t,a)-\frac{2(t-a)^{2m+1}}{(2m+1)!\,4^{2m}}\frac{f^{(2m)}(t)-f^{(2m)}(a)}{t-a}\right|\\
=&\left|\int_a^t G_{2m}(x)f^{(2m+1)}(x)dx-\frac{1}{t-a}\int_a^t G_{2m}(x)dx\int_a^t f^{(2m+1)}(x)dx\right|\\
=&\frac{1}{2(t-a)}\left|\int_a^t\int_a^t[G_{2m}(x)-G_{2m}(y)][f^{(2m+1)}(x)-f^{(2m+1)}(y)]dxdy\right|\\
\leq&\frac{1}{2(t-a)}\left(\int_a^t\int_a^t[G_{2m}(x)-G_{2m}(y)]^2dxdy\right)^{\frac{1}{2}}\left(\int_a^t\int_a^t[f^{(2m+1)}(x)-f^{(2m+1)}(y)]^2dxdy\right)^{\frac{1}{2}}\\
=&\left(\int_a^t G_{2m}^2(x)dx-\frac{1}{t-a}\left[\int_a^t G_{2m}(y)dy\right]^2\right)^{\frac{1}{2}}\left(\int_a^t[f^{(2m)}(x)]^2dx-\frac{1}{t-a}\left[\int_a^t f^{(2m)}(y)dy\right]^2\right)^{\frac{1}{2}}\end{aligned}$$

$$=\frac{1}{(2m)!\,4^{2m}}\sqrt{\frac{1}{4m+1}+\frac{4m^2}{4m-1}-\frac{4}{(2m+1)^2}}\sqrt{\sigma(f^{(2m+1)})}(t-a)^{2m+\frac{1}{2}}.$$

To prove the sharpness of (25), we suppose that (25) holds with a constant $C>0$ as

$$\begin{aligned}&\left|f(t)-F_{2m}(t,a)-\frac{2(t-a)^{2m+1}}{(2m+1)!\,4^{2m}}\frac{f^{(2m)}(t)-f^{(2m)}(a)}{t-a}\right|\\ \leq &C\sqrt{\sigma(f^{(2m+1)})}(t-a)^{2m+\frac{1}{2}}.\end{aligned}\tag{27}$$

We may find a function $f:[a,b]\to R$ such that $f^{(2m)}$ is absolutely continuous on $[a,t]$ as

$$\begin{aligned}&f^{(n)}(x)\\ =&\begin{cases}\dfrac{1}{(2m+1)!}\left(x-\dfrac{3a+t}{4}\right)^{2m}\left[x+\dfrac{(2m-2)a-(2m+2)t}{4}\right]-\dfrac{2(t-a)^{2m+1}}{2(2m+1)!\,4^{2m}}, & x\in\left[a,\dfrac{a+t}{2}\right],\\ \dfrac{1}{(2m+1)!}\left(x-\dfrac{a+3t}{4}\right)^{2m}\left[x+\dfrac{(2m-2)t-(2m+2)a}{4}\right]+\dfrac{2(t-a)^{2m+1}}{2(2m+1)!\,4^{2m}}, & x\in\left(\dfrac{a+t}{2},t\right].\end{cases}\end{aligned}$$

It follows that

$$f^{(2m+1)}(x)=G_{2m}(x).\tag{28}$$

It's easy to find that the left-hand side of the inequality (27) becomes

$$L.H.S.(27)=\frac{1}{((2m)!)^2\,4^{4m}}\left[\frac{1}{4m+1}+\frac{4m^2}{4m-1}-\frac{4}{(2m+1)^2}\right](t-a)^{4m+1},\tag{29}$$

and the right-hand side of the inequality (27) is

$$R.H.S.(27)=\frac{1}{(2m)!\,4^{2m}}\sqrt{\frac{1}{4m+1}+\frac{4m^2}{4m-1}-\frac{4}{(2m+1)^2}}C(t-a)^{4m+1}.\tag{30}$$

It follows from (27), (29) and (30) that

$$C\geq\frac{1}{(2m)!\,4^{2m}}\sqrt{\frac{1}{4m+1}+\frac{4m^2}{4m-1}-\frac{4}{(2m+1)^2}},$$

which prove that the constant $\frac{1}{(2m)!\,4^{2m}}\sqrt{\frac{1}{4m+1}+\frac{4m^2}{4m-1}-\frac{4}{(2m+1)^2}}$ is the best possible in (25). $\square$

Remark 1. We note that some applications of the classical or perturbed Taylor's formula with the integral remainder in numerical analysis, for special means and some usual mappings have been given in [7]. The interested reader can also apply the results we obtained here in these mentioned fields.

ACKNOWLEDGEMENTS

This work was supported by the National Natural Science Foundation of China (Grant No. 40975002, 11126289) and the Natural Science Foundation of the Jiangsu Higher Education Institutions (Grant No. 09KJB110005).

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J. COMPUTATIONAL ANALYSIS AND APPLICATIONS, VOL. 15, NO.6, 1165-1175, 2013, COPYRIGHT 2013 EUDOXUS PRESS, LLC

ADDITIVE FUNCTIONAL INEQUALITIES IN GENERALIZED QUASI-BANACH SPACES

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Abstract. In this paper, we investigate the Hyers-Ulam stability of the following function inequalities

$$\begin{aligned} \|af(x)+bf(y)+cf(z)\| &\leq \left\|Kf\left(\frac{ax+by+cz}{K}\right)\right\| \quad (0<|K|<|a+b+c|), \\ \|af(x)+bf(y)+Kf(z)\| &\leq \left\|Kf\left(\frac{ax+by}{K}+z\right)\right\| \quad (0<K<|a+b+K|) \end{aligned}$$

in generalized quasi-Banach spaces.

1. Introduction and preliminaries

The stability problem of functional equations originated from a question of Ulam [1] in 1940, concerning the stability of group homomorphisms. Let $(G_1, .)$ be a group and let $(G_2, *)$ be a metric group with the metric $d(.,.)$. Given $\epsilon > 0$, does there exist a $\delta 0$, such that if a mapping $h : G_1 \to G_2$ satisfies the inequality $d(h(x.y), h(x) * h(y)) < \delta$ for all $x, y \in G_1$, then there exists a homomorphism $H : G_1 \to G_2$ with $d(h(x), H(x)) < \epsilon$ for all $x \in G_1$? In the other words, Under what condition does there exists a homomorphism near an approximate homomorphism? The concept of stability for functional equation arises when we replace the functional equation by an inequality which acts as a perturbation of the equation. In 1941, Hyers [2] gave the first affirmative answer to the question of Ulam for Banach spaces. Let $f : E \to E'$ be a mapping between Banach spaces such that

$$\|f(x+y) - f(x) - f(y)\| \leq \delta$$

for all $x, y \in E$, and for some $\delta > 0$. Then there exists a unique additive mapping $T : E \to E'$ such that

$$\|f(x) - T(x)\| \leq \delta$$

for all $x \in E$. Moreover, if $f(tx)$ is continuous in $t \in \mathbb{R}$ for each fixed $x \in E$, then T is $\mathbb{R}$-linear. In 1978, Th.M. Rassias [3] proved the following theorem.

Theorem 1.1. *Let $f : E \to E'$ be a mapping from a normed vector space E into a Banach space E' subject to the inequality*

$$\|f(x+y) - f(x) - f(y)\| \leq \epsilon(\|x\|^p + \|y\|^p) \tag{1.1}$$

2010 *Mathematics Subject Classification.* Primary 39B62, 39B52, 46B25.

Key words and phrases. Hyers-Ulam stability; additive functional inequality; generalized quasi-Banach space; additive mapping.

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for all $x, y \in E$, where ϵ and p are constants with $\epsilon > 0$ and $p < 1$. Then there exists a unique additive mapping $T : E \to E'$ such that

$$\|f(x) - T(x)\| \leq \frac{2\epsilon}{2 - 2^p}\|x\|^p \tag{1.2}$$

for all $x \in E$. If $p < 0$ then inequality (1.1) holds for all $x, y \neq 0$, and (1.2) for $x \neq 0$. Also, if the function $t \mapsto f(tx)$ from $\mathbb{R}$ into E' is continuous in $t \in \mathbb{R}$ for each fixed $x \in E$, then T is $\mathbb{R}$-linear.

In 1991, Gajda [4] answered the question for the case $p > 1$, which was raised by Th.M. Rassias. On the other hand, J.M. Rassias [5] generalized the Hyers-Ulam stability result by presenting a weaker condition controlled by a product of different powers of norms.

Theorem 1.2. ([6, 7]) *If it is assumed that there exist constants $\Theta \geq 0$ and $p_1, p_2 \in \mathbb{R}$ such that $p = p_1 + p_2 \neq 1$, and $f : E \to E'$ is a mapping from a norm space E into a Banach space E' such that the inequality*

$$\|f(x + y) - f(x) - f(y)\| \leq \Theta\|x\|^{p_1}\|y\|^{p_2}$$

for all $x, y \in E$, then there exists a unique additive mapping $T : E \to E'$ such that

$$\|f(x) - T(x)\| \leq \frac{\Theta}{2 - 2^p}\|x\|^p,$$

for all $x \in E$. If, in addition, $f(tx)$ is continuous in $t \in \mathbb{R}$ for each fixed $x \in E$, then T is $\mathbb{R}$-linear

More generalizations and applications of the Hyers-Ulam stability to a number of functional equations and mappings can be found in [8]–[22].

In [23], Park et al. investigated the following inequalities

$$\|f(x) + f(y) + f(z)\| \leq \left\|2f\left(\frac{x + y + z}{2}\right)\right\|,$$

$$\|f(x) + f(y) + f(z)\| \leq \|f(x + y + z)\|,$$

$$\|f(x) + f(y) + 2f(z)\| \leq \left\|2f\left(\frac{x + y}{2} + z\right)\right\|$$

in Banach spaces. Recently, Cho et al. [24] investigated the following functional inequality

$$\|f(x) + f(y) + f(z) \leq \left\|Kf\left(\frac{x + y + z}{K}\right)\right\| \qquad (0 < |K| < |3|)$$

in non-Archimedean Banach spaces. Lu and Park [25] investigated the following functional inequality

$$\left\|\sum_{i=1}^{N} f(x_i)\right\| \leq \left\|Kf\left(\frac{\sum_{i=1}^{N}(x_i)}{K}\right)\right\| \qquad (0 < |K| \leq N)$$

in Fréchet spaces.

In [26], we investigated the following functional inequalities

$$\|f(x) + f(y) + f(z)\| \leq \left\|Kf\left(\frac{x + y + z}{K}\right)\right\| \qquad (0 < |K| < 3), \tag{1.3}$$

$$\|f(x)+f(y)+Kf(z)\| \le \left\|Kf\left(\frac{x+y}{K}+z\right)\right\| \qquad (0<K\neq 2) \tag{1.4}$$

and proved the Hyers-Ulam stability of the functional inequalities (1.3) and (1.4) in Banach spaces.

We consider the following functional inequalities

$$\|af(x)+bf(y)+cf(z)\| \le \left\|Kf\left(\frac{ax+by+cz}{K}\right)\right\| \quad (0<|K|<|a+b+c|), \tag{1.5}$$

$$\|af(x)+bf(y)+Kf(z)\| \le \left\|Kf\left(\frac{ax+by}{K}+z\right)\right\| \quad (0<K<|a+b+K|), \tag{1.6}$$

where a,b,c are nonzero real numbers.

Now, we recall some basic facts concerning quasi-Banach spaces and some preliminary results.

Definition 1.3. ([27, 28]) Let X be a linear space. A quasi-norm is a real-valued function on X satisfying the following:

(1) $\|x\|\ge 0$ for all $x\in X$ and $\|x\|=0$ if and only if $x=0$.
(2) $\|\lambda x\| = |\lambda|\|x\|$ for all $\lambda\in\mathbb{R}$ and all $x\in X$.
(3) There is a constant $C\ge 1$ such that $\|x+y\|\le C(\|x\|+\|y\|)$ for all $x,y\in X$.

The pair $(X,\|\cdot\|)$ is called a *quasi-normed space* if $\|\cdot\|$ is a quasi-norm on X.

A *quasi-Banach space* is a complete quasi-normed space.

Baak [29] generalized the concept of quasi-normed spaces.

Definition 1.4. ([29]) Let X be a linear space. A **generalized quasi-norm** is a real-valued function on X satisfying the following:

(1) $\|x\|\ge 0$ for all $x\in X$ and $\|x\|=0$ if and only if $x=0$.
(2) $\|\lambda x\| = |\lambda|\cdot\|x\|$ for all $\lambda\in\mathbb{R}$ and all $x\in X$.
(3) There is a constant $C\ge 1$ such that $\|\sum_{j=1}^{\infty} x_j\| \le \sum_{j=1}^{\infty} C\|x_j\|$ for all $x_1,x_2,\dots\in X$ with $\sum_{j=1}^{\infty} x_j\in X$.

The pair $(X,\|\cdot\|)$ is called a *generalized quasi-normed space* if $\|\cdot\|$ is a generalized quasi-norm on X. The smallest possible C is called the *modulus of concavity* of $\|\cdot\|$.

A *generalized quasi-Banach space* is a complete generalized quasi-normed space.

In this paper, we show that the Hyers-Ulam stability of the functional inequalities (1.5) and (1.6) in generalized quasi-Banach spaces.

Throughout this paper, assume that X is a generalized quasi-normed vector space with generalized quasi-norm $\|\cdot\|$ and that $(Y,\|\cdot\|)$ is a generalized quasi-Banach space. Let C be the modulus of concavity of $\|\cdot\|$.

2. Hyers-Ulam stability of the functional inequality (1.5)

Throughout this section, assume that K is a real number with $0<|K|<|a+b+c|$.

Proposition 2.1. *Let $f: X\to Y$ be a mapping such that*

$$\|af(x)+bf(y)+cf(z)\| \le \left\|Kf\left(\frac{ax+by+cz}{K}\right)\right\| \tag{2.1}$$

for all $x,y,z\in X$. Then the mapping $f: X\to Y$ is additive.

Proof. Letting $x = y = z = 0$ in (2.1), we get

$$\|(a+b+c)f(0)\| \leq \|Kf(0)\|.$$

So $f(0) = 0$.

Letting $z = 0$ and $y = -\frac{b}{a}x$ in (2.1), we get

$$\left\|af(x) + bf\left(-\frac{a}{b}x\right)\right\| \leq \|Kf(0)\| = 0$$

for all $x \in X$. So $f(x) = -\frac{b}{a}f(-\frac{a}{b}x)$ for all $x \in X$.

Replacing x by $-x$ and letting $y = 0$ and $z = \frac{a}{c}x$ in (2.1), we get

$$\left\|af(-x) + cf\left(\frac{a}{c}x\right)\right\| \leq \|Kf(0)\| = 0$$

for all $x \in X$. So $f(-x) = -\frac{c}{a}f(\frac{a}{c}x)$ for all $x \in X$. Then we get

$$\begin{aligned}
\|f(x) + f(-x)\| &= \left\|-\frac{b}{a}f\left(-\frac{a}{b}x\right) - \frac{c}{a}f\left(\frac{a}{c}x\right)\right\| \\
&= \frac{1}{|a|}\left\|af(0) + bf\left(-\frac{a}{b}x\right) + cf\left(\frac{a}{c}x\right)\right\| \\
&\leq \frac{1}{|a|}\left\|Kf\left(\frac{a\cdot 0 - b\frac{a}{b}x + c\frac{a}{c}x}{K}\right)\right\| = 0
\end{aligned}$$

Thus $f(x) = -f(-x)$.

$$\begin{aligned}
\|f(x) + f(y) - f(x+y)\| &= \|f(x) + f(y) + f(-x-y)\| \\
&= \left\|-\frac{a}{a}f(-\frac{a}{a}x) - \frac{b}{a}f(-\frac{a}{b}y) - \frac{c}{a}f(\frac{ax+ay}{c})\right\| \\
&= \frac{1}{|a|}\left\|af(-\frac{a}{a}x) + bf(-\frac{a}{b}y) + cf(\frac{ax+ay}{c})\right\| \\
&= \frac{1}{|a|}\left\|Kf\left(\frac{a\cdot(-\frac{a}{a}x) + b\cdot(-\frac{a}{b}x) + c\cdot\frac{a(x+y)}{c}}{K}\right)\right\| = 0.
\end{aligned}$$

Thus

$$f(x+y) = f(x) + f(y)$$

for all $x, y \in X$, as desired. □

Theorem 2.2. *Assume that a mapping $f : X \to Y$ satisfies the inequality*

$$\|af(x) + bf(y) + cf(z)\| \leq \left\|Kf\left(\frac{ax+by+cz}{K}\right)\right\| + \phi(x,y,z), \tag{2.2}$$

where $\phi : X^3 \to [0,\infty)$ satisfies $\phi(0,0,0) = 0$ and

$$\widetilde{\phi}(x,y,z) := \sum_{j=0}^{\infty}\left(\frac{c}{a}\right)^j \phi\left(\left(\frac{a}{c}\right)^j y, \left(\frac{a}{c}\right)^j z, \left(\frac{a}{c}\right)^j x\right) < \infty$$

for all $x, y, z \in X$. Then there exists a unique additive mapping $A : X \to Y$ such that

$$\|A(x) - f(x)\| \leq \frac{C^2}{|a|}\left[\widetilde{\phi}\left(x, -\frac{a}{b}x, 0\right) + \widetilde{\phi}\left(0, -\frac{a}{b}x, \frac{a}{c}x\right)\right] \tag{2.3}$$

for all $x \in X$.

Proof. Letting $x = y = z = 0$ in (2.2), we get $\|(a+b+c)f(0)\| \le \|Kf(0)\| + \phi(0,0,0) = \|Kf(0)\|$. So $f(0) = 0$.

Letting $y = 0$ and $z = -\frac{a}{c}x$ in (2.2), we get

$$\left\|af(x) + cf\left(-\frac{a}{c}x\right)\right\| \le \phi\left(x, 0, -\frac{a}{c}x\right)$$

for all $x \in X$. So $\left\|f(x) + \frac{c}{a}f(-\frac{a}{c}x)\right\| \le \frac{1}{|a|}\phi\left(x, 0, -\frac{a}{c}x\right)$ for all $x \in X$.

Letting $y = -\frac{a}{b}x$ and $z = 0$ in (2.2), we obtain

$$\left\|f(x) + \frac{b}{a}f\left(-\frac{a}{b}x\right)\right\| \le \frac{1}{|a|}\phi\left(x, -\frac{a}{b}x, 0\right)$$

for all $x \in X$. So

$$\begin{aligned} \left\|f(x) - \frac{c}{a}f\left(\frac{a}{c}x\right)\right\| &= \left\|f(x) + \frac{b}{a}f\left(-\frac{ax}{b}\right) - \frac{b}{a}f\left(-\frac{ax}{b}\right) - \frac{c}{a}f\left(\frac{a}{c}x\right)\right\| \\ &\le C\left(\left\|f(x) + \frac{b}{a}f\left(-\frac{ax}{b}\right)\right\| + \left\|\frac{b}{a}f\left(-\frac{ax}{b}\right) + \frac{c}{a}f\left(\frac{a}{c}x\right)\right\|\right) \qquad (2.4) \\ &\le \frac{C}{|a|}\left[\phi\left(x, -\frac{ax}{b}, 0\right) + \phi\left(0, -\frac{ax}{b}, \frac{ax}{c}\right)\right] \end{aligned}$$

for all $x \in X$.

It follows from (2.4) that

$$\begin{aligned} &\left\|(\frac{c}{a})^l f\left((\frac{a}{c})^l x\right) - (\frac{c}{a})^m f\left((\frac{a}{c})^m x\right)\right\| \\ &\le C\sum_{j=l}^{m-1}\left\|(\frac{c}{a})^j f\left((\frac{a}{c})^j x\right) - (\frac{c}{a})^{j+1} f\left((\frac{a}{c})^{j+1} x\right)\right\| \\ &\le \frac{C^2}{|a|}\sum_{j=l}^{m-1}\left(\frac{c}{a}\right)^j\left[\phi\left(\left(\frac{a}{c}\right)^j x, -\frac{a}{b}\left(\frac{a}{c}\right)^j x, 0\right) + \phi\left(0, -\frac{a}{b}\left(\frac{a}{c}\right)^j x, \left(\frac{a}{c}\right)^{j+1} x\right)\right] \end{aligned}$$

for all nonnegative integers m and l with $m > l$ and all $x \in X$. It means that the sequence $\{(\frac{c}{a})^n f((\frac{a}{c})^n x)\}$ is a Cauchy sequence for all $x \in X$. Since Y is complete, the sequence $\{(\frac{c}{a})^n f((\frac{a}{c})^n x)\}$ converges. We define the mapping $A : X \to Y$ by $A(x) = \lim_{n\to\infty}\{(\frac{c}{a})^n f((\frac{a}{c})^n x)\}$ for all $x \in X$. Moreover, letting $l = 0$ and passing the limit $m \to \infty$, we get (2.3).

Next, we show that $A : X \to Y$ is an additive mapping.

$$\begin{aligned}
\|A(x)+A(-x)\| &= \lim_{n\to\infty}(\frac{c}{a})^n \left\|f\left(\frac{a^n x}{c^n}\right)+f\left(\frac{-a^n x}{c^n}\right)\right\| \\
&\le C\lim_{n\to\infty}(\frac{c}{a})^n \left[\left\|f\left(\frac{a^n x}{c^n}\right)+\frac{b}{a}f\left(-\frac{a}{b}\cdot\frac{a^n x}{c^n}\right)\right\|\right. \\
&+ \left\|f\left(-\frac{a^n x}{c^n}\right)+\frac{c}{a}f\left(\frac{a}{c}\cdot\frac{a^n x}{c^n}\right)\right\| \\
&+ \left.\left\|\frac{b}{a}f\left(-\frac{a}{b}\cdot\frac{a^n x}{c^n}\right)+\frac{c}{a}f\left(\frac{a}{c}\cdot\frac{a^n x}{c^n}\right)\right\|\right] \\
&\le C\frac{1}{|a|}\lim_{n\to\infty}(\frac{c}{a})^n\left[\phi\left(\frac{a^n x}{c^n},-\frac{a}{b}\frac{a^n x}{c^n},0\right)+\phi\left(-\frac{a^n x}{c^n},0,\frac{a^{n+1}x}{c^{n+1}}\right)\right. \\
&+ \left.\phi\left(0,-\frac{a}{b}\frac{a^n x}{c^n},\frac{a^{n+1}x}{c^{n+1}}\right)\right] \\
&= 0
\end{aligned}$$

and so $A(-x) = -A(x)$ for all $x \in X$.

$$\begin{aligned}
&\|A(x)+A(y)-A(x+y)\|| = \lim_{n\to\infty}(\frac{c}{a})^n\left\|f\left(\frac{a^n x}{c^n}\right)+f\left(\frac{a^n y}{c^n}\right)-f\left(\frac{a^n(x+y)}{c^n}\right)\right\| \\
&= C\lim_{n\to\infty}(\frac{c}{a})^n\left[\left\|f\left(\frac{a^n x}{c^n}\right)+\frac{b}{a}f\left(-\frac{a}{b}\frac{a^n x}{c^n}\right)\right\|\right. \\
&+\left\|f\left(\frac{a^n y}{c^n}\right)+\frac{c}{a}f\left(-\frac{a^{n+1}y}{c^{n+1}}\right)\right\| \\
&+\left.\left\|f\left(\frac{a^n(x+y)}{c^n}\right)+\frac{b}{a}f\left(-\frac{a}{b}\frac{a^n x}{c^n}\right)+\frac{c}{a}f\left(-\frac{a^{n+1}y}{c^{n+1}}\right)\right\|\right] \\
&\le C\frac{1}{|a|}\lim_{n\to\infty}(\frac{c}{a})^n\left[\phi\left(\frac{a^n x}{c^n},-\frac{a}{b}(\frac{a^n x}{c^n}),0\right)+\phi\left(\frac{a^n y}{c^n},0,-\frac{a}{c}(\frac{a^n x}{c^n})\right)\right. \\
&+\left.\phi\left(\frac{a^n(x+y)}{c^n},-\frac{a}{b}(\frac{a^n x}{c^n}),-\frac{a}{c}(\frac{a^n x}{c^n})\right)\right] \\
&= 0
\end{aligned}$$

for all $x, y \in X$. Thus the mapping $A : X \to Y$ is additive.

Now, we prove the uniqueness of A. Assume that $T : X \to Y$ is another additive mapping satisfying (2.3). Then we obtain

$$\begin{aligned}
\|A(x)-T(x)\| &= (\frac{c}{a})^n\left\|A\left((\frac{a}{c})^n x\right)-T\left((\frac{a}{c})^n x\right)\right\| \\
&\le C\cdot(\frac{c}{a})^n\left[\left\|A\left((\frac{a}{c})^n x\right)-f\left((\frac{a}{c})^n x\right)\right\|\right. \\
&+ \left.\left\|T\left((\frac{a}{c})^n x\right)-f\left((\frac{a}{c})^n x\right)\right\|\right] \\
&\le 2C\frac{C^2}{|a|}\left[\widetilde{\phi}\left(x,-\frac{a}{b}x,0\right)+\widetilde{\phi}(0,-\frac{a}{b}x,\frac{a}{c}x)\right]
\end{aligned}$$

which tends to zero as $n \to \infty$ for all $x \in X$. Then we can conclude that $A(x) = T(x)$ for all $x \in X$. This complete the proof. $\square$

Corollary 2.3. *Let p and θ be positive real numbers with $p > 1$. Let $f : X \to Y$ be a mapping satisfying*

$$\|af(x) + bf(y) + cf(z)\| \leq \left\|Kf\left(\frac{ax+by+cz}{K}\right)\right\| + \theta(\|x\|^p + \|y\|^p + \|z\|^p)$$

for all $x, y, z \in X$. Then there exists a unique additive mapping $A : X \to Y$ such that

$$\|f(x) - A(x)\| \leq \frac{C}{|a|} \cdot \frac{c^p + a^p}{c^p - c(a+b)^{p-1}} \theta \|x\|^p$$

for all $x \in X$.

3. Hyers-Ulam stability of the functional inequality (1.6)

Throughout this section, assume that K, a, b are nonzero real numbers with $0 < K \neq 2$ and $|a + b + K| \geq K$.

Proposition 3.1. *Let $f : X \to Y$ be a mapping such that*

$$\|af(x) + bf(y) + Kf(z)\| \leq \left\|Kf\left(\frac{ax+by}{K} + z\right)\right\| \tag{3.1}$$

for all $x, y, z \in X$. Then the mapping $f : X \to Y$ is additive.

Proof. Letting $x = y = z = 0$ in (3.1), we get

$$\|(K + a + b)f(0)\| \leq \|Kf(0)\|.$$

So $f(0) = 0$.

Letting $y = -\frac{a}{b}x$ and $z = 0$ in (3.1), we get

$$\left\|af(x) + bf\left(-\frac{a}{b}x\right)\right\| \leq \|Kf(0)\| = 0$$

for all $x \in X$. So $f(x) = -\frac{b}{a}f(-\frac{a}{b}x)$ for all $x \in X$.

Replacing x by $-x$ and letting $y = 0$ and $z = \frac{a}{K}x$ in (3.1), we get

$$\left\|af(-x) + Kf\left(\frac{a}{K}x\right)\right\| \leq \|Kf(0)\| = 0$$

for all $x \in X$. So $f(-x) = -\frac{K}{a}f(\frac{a}{K}x)$ for all $x \in X$.

Thus we get

$$\|f(x) + f(-x)\| = \frac{1}{|a|}\left\|bf\left(-\frac{a}{b}x\right) + Kf\left(\frac{a}{K}x\right)\right\| \leq \frac{1}{|a|}\|f(0)\| = 0$$

for all $x \in X$. So $f(-x) = -f(x)$ for all $x \in X$.

Letting $z = \frac{-x-y}{K}$ in (3.1), we get

$$\begin{aligned}\left\|af(x) + bf(y) - Kf\left(\frac{ax+by}{K}\right)\right\| &= \left\|af(x) + bf(y) + Kf\left(\frac{-ax-by}{K}\right)\right\| \\ &\leq \|Kf(0)\| = 0\end{aligned}$$

for all $x, y \in X$. Thus

$$Kf\left(\frac{ax+by}{K}\right) = af(x) + bf(y) \tag{3.2}$$

for all $x, y \in X$. Letting $y = 0$ in (3.2), we get $f(x) = \frac{a}{K} f\left(\frac{Kx}{a}\right)$ for all $x \in X$. Letting $x = 0$ in (3.2), we get $f(y) = \frac{b}{K} f\left(\frac{Ky}{b}\right)$. So

$$\begin{aligned}\|f(x) + f(y) - f(x+y)\| &= \left\|\frac{a}{K} f\left(\frac{Kx}{a}\right) + \frac{b}{K} f\left(\frac{Ky}{b}\right) + f(-x-y)\right\| \\ &= \frac{1}{|K|}\left\|af\left(\frac{Kx}{a}\right) + bf\left(\frac{Ky}{b}\right) + Kf(-x-y)\right\| = 0\end{aligned}$$

for all $x, y \in X$, as desired. □

Theorem 3.2. *Assume that a mapping $f : X \to Y$ satisfies the inequality*

$$\|af(x) + bf(y) + Kf(z)\| \leq \left\|Kf\left(\frac{ax+by}{K} + z\right)\right\| + \phi(x, y, z), \tag{3.3}$$

where $\phi : X^3 \to [0, \infty)$ satisfies $\phi(0, 0, 0) = 0$ and

$$\widetilde{\phi}(x, y, z) := \sum_{j=1}^{\infty} \left|\left(\frac{a}{K}\right)^j\right| \phi\left(\left(\frac{K}{a}\right)^j x, \left(\frac{K}{a}\right)^j y, \left(\frac{K}{a}\right)^j z\right) < \infty$$

for all $x, y, z \in X$. Then there exists a unique additive mapping $A : X \to Y$ such that

$$\|A(x) - f(x)\| \leq \frac{C^2}{|K|}\left[\widetilde{\phi}\left(0, -\frac{K}{a}x, x\right) + \widetilde{\phi}\left(\frac{K}{a}x, -\frac{K}{b}x, 0\right)\right] \tag{3.4}$$

for all $x \in X$.

Proof. Letting $x = y = z = 0$ in (3.3), we get $\|(K + a + b)f(0)\| \leq \|Kf(0)\| + \phi(0, 0, 0) = \|Kf(0)\|$. So $f(0) = 0$.

Letting $x = 0, y = -\frac{Kx}{b}, z = x$ in (3.3), we obtain

$$\left\|af(0) + bf\left(-\frac{K}{b}x\right) + Kf(x)\right\| \leq \phi\left(0, -\frac{K}{b}x, x\right)$$

for all $x \in X$.

Letting $y = 0, z = -\frac{Kx}{a}$in (3.3), we obtain

$$\left\|af(x) + bf(0) + Kf\left(\frac{-ax}{K}\right)\right\| \leq \phi\left(x, 0, -\frac{ax}{K}\right)$$

for all $x \in X$.

Letting $x = \frac{Kx}{a}, y = -\frac{Kx}{b}, z = 0$ in (3.3), we get

$$\left\|af\left(\frac{Kx}{a}\right) + bf\left(-\frac{Kx}{b}\right) + Kf(0)\right\| \leq \phi\left(\frac{Kx}{a}, -\frac{Kx}{b}, 0\right)$$

for all $x \in X$. So

$$\begin{aligned}&\left\|f(x) - \frac{a}{K} f\left(\frac{K}{a}x\right)\right\| \\ &\leq C\left[\left\|f(x) + \frac{b}{K} f\left(-\frac{Kx}{b}\right)\right\| + \left\|\frac{b}{K} f\left(-\frac{K}{b}x\right) + \frac{a}{K} f\left(\frac{K}{a}x\right)\right\|\right] \\ &\leq \frac{C}{|K|}\left[\phi\left(0, -\frac{K}{b}x, x\right) + \phi\left(\frac{K}{a}x, -\frac{K}{b}x, 0\right)\right]\end{aligned} \tag{3.5}$$

for all $x \in X$. It follows from (3.5) that

$$
\begin{aligned}
&\left\|\left(\frac{a}{K}\right)^l f\left(\left(\frac{K}{a}\right)^l x\right)-\left(\frac{a}{K}\right)^m f\left(\left(\frac{K}{a}\right)^m x\right)\right\| \\
&\leq C \sum_{j=l}^{m-1}\left\|\left(\frac{a}{K}\right)^j f\left(\left(\frac{K}{a}\right)^j x\right)-\left(\frac{a}{K}\right)^{j+1} f\left(\left(\frac{K}{a}\right)^{j+1} x\right)\right\| \\
&\leq C^2 \sum_{j=l}^{m-1}\left|(\frac{a}{K})^j\right|\left[\left\|f(\left(\left(\frac{K}{a}\right)^j x\right)+\frac{b}{K} f\left(-\frac{K}{b}\left(\left(\frac{K}{a}\right)^j x\right)\right))\right\|\right. \\
&\left.+\left\|\frac{b}{K} f\left(-\frac{K}{b}\left(\left(\frac{K}{a}\right)^j x\right)\right)+\frac{a}{K} f\left(\frac{K}{a}\left(\left(\frac{K}{a}\right)^j x\right)\right)\right\|\right] \\
&\leq \frac{C^2}{|K|} \sum_{j=l}^{m-1}\left|(\frac{a}{K})^j\right|\left[\phi\left(0,-\frac{K}{a}\left(\frac{K}{a}\right)^j x,\left(\frac{K}{a}\right)^j x\right)+\phi\left(\frac{K}{a}\left(\frac{K}{a}\right)^j x,-\frac{K}{b}\left(\frac{K}{a}\right)^j x, 0\right)\right]
\end{aligned}
$$

for all nonnegative integers m and l with $m > l$ and all $x \in X$. It means that the sequence $\{(\frac{a}{K})^n f((\frac{K}{a})^n x)\}$ is a Cauchy sequence for all $x \in X$. Since Y is complete, the sequence $\{(\frac{a}{K})^n f((\frac{K}{a})^n x)\}$ converges. So we may define the mapping $A : X \to Y$ by $A(x) = \lim_{n\to\infty}((\frac{a}{K})^n f((\frac{K}{a})^n x))$ for all $x \in X$.

Moreover, by letting $l = 0$ and passing the limit $m \to \infty$, we get (3.4).

Now, we show that A is additive.

$$
\begin{aligned}
&\|A(x)+A(y)-A(x+y)\| \\
&=\lim_{n \rightarrow \infty}\left|\frac{a}{K}\right|^n\left\|f((\frac{K}{a})^n x)+f((\frac{K}{a})^n y)-f((\frac{K}{a})^n(x+y))\right\| \\
&\leq C \lim_{n \rightarrow \infty}\left|\frac{a}{K}\right|^n\left[\left\|f\left(\left(\frac{K}{a}\right)^n x\right)+\frac{b}{K} f\left(-\frac{K}{b}\left(\frac{K}{a}\right)^n x\right)\right\|\right. \\
&+\left\|f\left(\left(\frac{K}{a}\right)^n y\right)+\frac{a}{K} f\left(-\frac{K}{a}\left(\frac{K}{a}\right)^n y\right)\right\| \\
&\left.+\left\|\frac{a}{K} f\left(-\frac{K}{a}\left(\frac{K}{a}\right)^n y\right)+\frac{b}{K} f\left(-\frac{K}{b}\left(\frac{K}{a}\right)^n x\right)+f\left(\left(\frac{K}{a}\right)^n(x+y)\right)\right\|\right] \\
&\leq C \lim_{n \rightarrow \infty}\left|\frac{a}{K}\right|^n\left[\phi\left(0,-\frac{K}{b}\left(\frac{K}{a}\right)^n x,\left(\frac{K}{a}\right)^n x\right)\right. \\
&+\phi\left(-\frac{K}{a}\left(\frac{K}{a}\right)^n y, 0,\left(\frac{K}{a}\right)^n y\right) \\
&\left.+\phi\left(-\frac{K}{a}\left(\frac{K}{a}\right)^n y,-\frac{K}{b}\left(\frac{K}{a}\right)^n x,\left(\frac{K}{a}\right)^n(x+y)\right)\right] \\
&=0
\end{aligned}
$$

for all $x, y \in X$. So the mapping $A : X \to Y$ is an additive mapping.

Now, we show that the uniqueness of A. Assume that $T : X \to Y$ is another additive mapping satisfying (3.4). Then we get

$$
\begin{aligned}
&\|A(x) - T(x)\| = \lim_{n\to\infty} \left|\frac{a}{K}\right|^n \left\| A\left|\left(\frac{K}{a}\right|^n x\right) - T\left(\left(\frac{K}{a}\right)^n x\right)\right\| \\
&\le C \lim_{n\to\infty} \left|\frac{a}{K}\right|^n \left[\left\| A\left(\left(\frac{K}{a}\right)^n x\right) - f\left(\left(\frac{K}{a}\right)^n x\right)\right\| + \left\| T\left(\left(\frac{K}{a}\right)^n x\right) - f\left(\left(\frac{K}{a}\right)^n x\right)\right\|\right] \\
&\le 2C\frac{C^2}{|K|} \lim_{n\to\infty} \left[\widetilde{\phi}\left(0, -\frac{K}{a}\left(\frac{K}{a}\right)^n x, \left(\frac{K}{a}\right)^n x)\right) + \widetilde{\phi}\left(\frac{K}{a}\left(\frac{K}{a}\right)^n x, -\frac{K}{b}\left(\frac{K}{a}\right)^n x, 0\right)\right] \\
&= 0
\end{aligned}
$$

for all $x \in X$. Thus we may conclude that $A(x) = T(x)$ for all $x \in X$. This proves the uniqueness of A. So the mapping $A : X \to Y$ is a unique additive mapping satisfying (3.4). $\square$

Corollary 3.3. *Let p, θ and K be positive real numbers with $p > 1$ and $|a + b + K| > K$. Let $f : X \to Y$ be a mapping satisfying*

$$\|af(x) + bf(y) + Kf(z)\| \le \left\| Kf\left(\frac{ax + by}{K} + z\right)\right\| + \theta(\|x\|^p + \|y\|^p + \|z\|^p)$$

for all $x, y, z \in X$. Then there exists a unique additive mapping $A : X \to Y$ such that

$$\|f(x) - A(x)\| \le \frac{\frac{1}{K}\left(\frac{a}{K}\right)^p + \frac{3a}{K}}{\left(\frac{a}{K}\right)^p - \frac{a}{K}} \theta\|x\|^p$$

for all $x \in X$.

Acknowledgments

C. Park was supported by Basic Science Research Program through the National Research Foundation of Korea funded by the Ministry of Education, Science and Technology (NRF-2012R1A1A2004299). D. Y. Shin was supported by Basic Science Research Program through the National Research Foundation of Korea funded by the Ministry of Education, Science and Technology (NRF-2010-0021792).

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Volume 15, Number 7 November 2013
ISSN:1521-1398 PRINT,1572-9206 ONLINE

Journal of Computational Analysis and Applications

EUDOXUS PRESS,LLC

Journal of Computational Analysis and Applications
ISSNno.'s:1521-1398 PRINT,1572-9206 ONLINE

SCOPE OF THE JOURNAL

An international publication of Eudoxus Press, LLC(eight times annually)

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COMPOSITION OPERATORS FROM HARDY SPACE TO n-TH WEIGHTED-TYPE SPACE OF ANALYTIC FUNCTIONS ON THE UPPER HALF-PLANE

ZHI-JIE JIANG AND ZUO-AN LI

ABSTRACT. Motivated by some recent results on composition operators, the boundedness of composition operator from the Hardy space to the n-th weighted-type space on the half plane $\mathbb{H} = \{z \in \mathbb{C} : \mathrm{Im}z > 0\}$ is characterized.

1. INTRODUCTION

Let $\mathbb{H} = \{z \in \mathbb{C} : \mathrm{Im}z > 0\}$ be the upper half plane in the complex plane $\mathbb{C}$ and $H(\mathbb{H})$ the space of all analytic functions in $\mathbb{H}$. For $p > 0$, the Hardy space $H^p(\mathbb{H})$ consists of all $f \in H(\mathbb{H})$ such that

$$\|f\|^p_{H^p(\mathbb{H})} = \sup_{y>0} \int_{-\infty}^{+\infty} |f(x+iy)|^p dx < \infty.$$

When $p \geq 1$, the Hardy space with the norm $\|\cdot\|_{H^p(\mathbb{H})}$ becomes a Banach space(even a Hilbert space when $p = 2$), and when $0 < p < 1$,

$$d(f,g) = \|f - g\|^p_{H^p(\mathbb{H})}$$

defines a Fréchet space distance on $H^p(\mathbb{H})$. For some details of this space and some operators on it see, e.g. [2], [3], [10] and [12].

Let $\mu(z)$ be a positive continuous function on a domain $X \subseteq \mathbb{C}$, and $n \in \mathbb{N}_0$ be fixed. The n-th weighted-type space on X, denoted by $\mathcal{W}^{(n)}_\mu(X)$ consists of all $f \in H(X)$ such that

$$b_{\mathcal{W}^{(n)}_\mu(X)}(f) := \sup_{z \in X} \mu(z)|f^{(n)}(z)| < \infty.$$

For $n = 0$ the space is called the weighted-type space $\mathcal{A}_\mu(X)$, for $n = 1$ the Bloch-type $\mathcal{B}_\mu(X)$, and for $n = 2$ the Zygmund-type space $\mathcal{Z}_\mu(X)$. Some information of these spaces on the unit disc and some operators on them can be found, e.g., in [5], [8], [9], [11], [14] and [16]. This considerable interest in Zygmund-type spaces, as well as a necessity for unification of weighted-type, Bloch-type and Zygmund-type spaces, motivated us to define the n-th weighted-type space.

The quantity $b_{\mathcal{W}^{(n)}_\mu(X)}(f)$ is a seminorm on the n-th weighted-type space $\mathcal{W}^{(n)}_\mu(X)$ and a norm on $\mathcal{W}^{(n)}_\mu(X)/\mathbb{P}_{n-1}$, where $\mathbb{P}_{n-1}$ is the set of all polynomials whose degrees are less than or equal to $n - 1$. A natural norm on the n-th weighted-type

2000 *Mathematics Subject Classification.* Primary 47B38; Secondary 47B33, 47B37.

Key words and phrases. Hardy space, upper half plane, n-th weighted-type space, composition operator, boundedness.

space $\mathcal{W}_\mu^{(n)}(X)$ is defined as follows

$$\|f\|_{\mathcal{W}_\mu^{(n)}(X)} = \sum_{j=0}^{n-1} |f^{(j)}(a)| + b_{\mathcal{W}_\mu^{(n)}(X)}(f),$$

where a is an element in X. Under this norm this space becomes a Banach space. For $X = \mathbb{H}$, we obtain the space $\mathcal{W}_\mu^{(n)}(\mathbb{H})$ on which the following norm can be introduced by

$$\|f\|_{\mathcal{W}_\mu^{(n)}(\mathbb{H})} := \sum_{j=0}^{n-1} |f^{(j)}(i)| + \sup_{z\in\mathbb{H}} \mu(z)|f^{(n)}(z)|,$$

and for $X = \mathbb{D}$, the unit disc we get the space $\mathcal{W}_\mu^{(n)}(\mathbb{D})$, and a norm on it is introduced by

$$\|f\|_{\mathcal{W}_\mu^{(n)}(\mathbb{D})} := \sum_{j=0}^{n-1} |f^{(n)}(0)| + \sup_{z\in\mathbb{D}} \mu(z)|f^{(n)}(z)|.$$

Let φ be an analytic self-map of X. The composition operator induced by φ is defined on $H(X)$ by

$$C_\varphi f(z) = f(\varphi(z)), z \in X.$$

A natural problem is to characterize the bounded or compact composition operator between two given spaces of analytic functions in terms of function theoretic properties of the induced symbol φ.

During the past few decades, composition operators have been studied extensively on spaces of analytic functions on the unit disc or the unit ball. One can consult [1] and [13] for the general theory of these operators. As a consequence of the Littlewood's subordination theorem, it is well known that every composition operator is bounded on Hardy spaces and weighted Bergman spaces of the unit disc. However, when people consider the Hardy space or the Bergman space on the upper half plane, they find that the situation is entirely different. There do exist unbounded composition operators on these spaces. Matache [10] proved that there didn't exist compact composition operators on Hardy spaces of the upper half plane. Shapiro and Smith [12] also showed that there were no compact composition operators on Bergman spaces of the upper half plane. Because of these facts of composition operators, many authors recently have begun to investigate them on spaces of analytic functions on the upper half plane. The present author in [5] characterized the boundedness of composition operators from the weighted Bergman spaces to the weighted-type, Bloch-type and Zymund-type spaces with the weight $\mu(z) = \mathrm{Im}z$ on the upper half plane. In [16], Stević generalized the result of [14].

In [6], the present author characterized the boundedness of composition operator from the weighted Bergman space to n-th weighted-type space with $\mu(z) = \mathrm{Im}z$ and $n = 4$. Motivated by [5], [6], [14] and [16], here we characterize the boundedness of composition operator from the Hardy space to the n-th weighted-type space on the upper half plane. On the one hand, this paper can be regarded as a generalization of results in [14] and [16]; on the other hand, it also can be regarded as a continuation of investigations of composition operators see, e.g. [4]-[12],[14]-[16].

Let Y be a Banach space. Recall that the norm of the composition operator is defined by

$$\|C_\varphi\|_{H^p(\mathbb{H})\to Y} := \sup_{\|f\|_{H^p(\mathbb{H})}\le 1} \|C_\varphi f\|_Y.$$

It is easy to see that this quantity is finite if and only if the operator $C_\varphi : H^p(\mathbb{H}) \to Y$ is bounded.

Throughout this paper, constants are denoted by C, they are positive and may differ from one occurrence to the other. The notation $a \asymp b$ means that there is a positive constant C such that $a/C \le b \le Ca$.

2. Main results

In this section, we first quote and prove several auxiliary lemmas. The first lemma was proved in [16].

Lemma 2.1. *Suppose that $p \ge 1$, $n \in \mathbb{N}$ and $w \in \mathbb{H}$, then the function*

$$f_{w,n}(z) = \frac{(\mathrm{Im} w)^{n-\frac{1}{p}}}{(z-\overline{w})^n}$$

belongs to $H^p(\mathbb{H})$ and $\sup_{w\in\mathbb{H}} \|f_{w,n}\|_{H^p(\mathbb{H})} \le \pi^{\frac{1}{p}}$.

Lemma 2.2. *Suppose that $p \ge 1$, then there exists a positive constant C independent of f such that*

$$|f^{(n)}(z)| \le C \frac{\|f\|_{H^p(\mathbb{H})}}{(\mathrm{Im} z)^{n+\frac{1}{p}}}.$$

Proof. For each $f \in H^p(\mathbb{H})$, it follows from Cauchy's integral formula that

$$f(z) = \frac{1}{2\pi i}\int_{-\infty}^{+\infty} \frac{f(t)}{t-z}dt. \tag{1}$$

Differentiating in (1) under the integral sign n times , we have

$$f^{(n)}(z) = \frac{n!}{2\pi}\int_{-\infty}^{+\infty} \frac{f(t)}{(t-z)^{n+1}}dt.$$

Then

$$|f^{(n)}(z)| \le \frac{n!}{2\pi}\int_{-\infty}^{+\infty} \frac{f(t)}{[(t-x)^2+y^2]^{(n+1)/2}}dt. \tag{2}$$

By using the change $t-x=sy$, we have

$$\int_{-\infty}^{+\infty} \frac{y^n}{[(t-x)^2+y^2]^{(n+1)/2}}dt = \int_{-\infty}^{+\infty} \frac{ds}{(1+s^2)^{(n+1)/2}} =: c_n < \infty. \tag{3}$$

From (3) and applying Jensen's inequality in (2), we get

$$\begin{aligned}
|f^{(n)}(z)|^p &\le d_n \int_{-\infty}^{+\infty} \frac{|f(t)|^p}{y^{np}} \frac{y^n}{[(t-x)^2+y^2]^{(n+1)/2}}dt \\
&\le d_n \int_{-\infty}^{+\infty} \frac{|f(t)|^p}{y^{np+1}}dt \\
&= d_n \frac{\|f\|^p_{H^p(\mathbb{H})}}{y^{np+1}},
\end{aligned}$$

where $d_n = (c_n n!/2\pi)^p$, from which the desired result is obtained.

The following lemma was proved in [15].

Lemma 2.3. *Suppose that $a > 0$ and*

$$D_n(a) = \begin{vmatrix} 1 & 1 & \cdots & 1 \\ a & a+1 & \cdots & a+n+1 \\ \cdots & \cdots & \cdots & \cdots \\ \prod_{j=0}^{n-2}(a+j) & \prod_{j=0}^{n-2}(a+j+1) & \cdots & \prod_{j=0}^{n-2}(a+j+n-1) \end{vmatrix},$$

then $D_n(a) = \prod_{j=1}^{n-1} j!$

Before we formulate and prove the main result of this paper, we will need the following classical Faàdi Bruno's formula

$$(f \circ \varphi)^{(n)}(z) = \sum \frac{n!}{k_1! \cdots k_n!} f^{(k)}(\varphi(z)) \prod_{j=1}^{n} \Big(\frac{\varphi^{(j)}(z)}{j!}\Big)^{k_j},$$

where $k = k_1 + k_2 + \cdots + k_n$, and the sum is over all non-negative integers k_1, k_2, ..., k_n satisfying $k_1 + 2k_2 + \cdots + nk_n = n$. For the information related to this formula see [7].

Theorem 2.4. *Suppose that $p \geq 1$ and φ is an analytic self-map of $\mathbb{H}$, then the operator $C_\varphi : H^p(\mathbb{H}) \to \mathcal{W}_\mu^{(n)}(\mathbb{H})$ is bounded if and only if for each $k \in \{1, 2, ..., n\}$ it follows that*

$$I_k := \sup_{z \in \mathbb{H}} \frac{\mu(z)\Big|\sum \frac{n!}{k_1! \cdots k_n!} \prod_{j=1}^{n} \Big(\frac{\varphi^{(j)}(z)}{j!}\Big)^{k_j}\Big|}{(\mathrm{Im}\varphi(z))^{k+\frac{1}{p}}} < \infty, \tag{4}$$

where the sum is over all non-negative integers k_1, k_2, ..., k_n satisfying $k_1 + 2k_2 + \cdots + nk_n = n$.

Moreover, if the operator $C_\varphi : H^p(\mathbb{H}) \to \mathcal{W}_\mu^{(n)}(\mathbb{H})/\mathbb{P}_{n-1}$ is bounded, then

$$\|C_\varphi\|_{H^p(\mathbb{H}) \to \mathcal{W}_\mu^{(n)}(\mathbb{H})/\mathbb{P}_{n-1}} \asymp \sum_{k=1}^{n} I_k. \tag{5}$$

Proof. First assume that the operator $C_\varphi : H^p(\mathbb{H}) \to \mathcal{W}_\mu^{(n)}(\mathbb{H})$ is bounded. For a fixed $w \in \mathbb{H}$ and constants c_1, c_2, ..., c_n, set the function

$$f_w(z) = \sum_{j=1}^{n} \frac{c_j}{n-2+j+\frac{2}{p}} \frac{(2i\mathrm{Im}w)^{n-2+j+\frac{1}{p}}}{(z-\overline{w})^{n-2+j+\frac{2}{p}}}.$$

Then by Lemma 2.1 we know that $f_w \in H^p(\mathbb{H})$ for every $w \in \mathbb{H}$, and

$$\sup_{w \in \mathbb{H}} \|f_w\|_{H^p(\mathbb{H})} \leq C. \tag{6}$$

Now we prove that for each $k \in \{1, ..., n\}$, there are constants c_1, c_2, ..., c_n such that

$$f_w^{(k)}(w) = \frac{1}{(2i\mathrm{Im}w)^{k+\frac{1}{p}}}, \quad f_w^{(l)}(w) = 0, \quad l \in \{1, ..., n\} \setminus \{k\}. \tag{7}$$

In fact, by differentiating function f_w for each $k \in \{1, ..., n\}$, the system in (7) becomes

$$c_1 + c_2 + \cdots + c_n = 0$$

$$\Big(n + \frac{2}{p}\Big)c_1 + \Big(n + 1 + \frac{2}{p}\Big)c_2 + \cdots + \Big(2n - 1 + \frac{2}{p}\Big)c_n = 0$$

$$\cdots$$

$$\prod_{j=0}^{k-2} \Big(n + j + \frac{2}{p}\Big)c_1 + \prod_{j=0}^{k-2} \Big(n + j + 1 + \frac{2}{p}\Big)c_2 + \cdots + \prod_{j=0}^{k-2} \Big(2n - 1 + \frac{2}{p}\Big)c_n = 1$$

$$\cdots$$

$$\prod_{j=0}^{n-2} \Big(n + j + \frac{2}{p}\Big)c_1 + \prod_{j=0}^{n-2} \Big(n + j + 1 + \frac{2}{p}\Big)c_2 + \cdots + \prod_{j=0}^{n-2} \Big(2n - 1 + \frac{2}{p}\Big)c_n = 0. \quad (8)$$

Applying Lemma 2.3 with $a = n + 2/p > 0$, we see that the determinant of system (8) is different from zero, from which the claim holds.

For each $k \in \{1, ..., n\}$, we choose the corresponding function which satisfy (7), and write it by $f_{w,k}$. For each $k \in \{1, ..., n\}$, the boundedness of the operator $C_\varphi : H^p(\mathbb{H}) \to \mathcal{W}_\mu^{(n)}(\mathbb{H})$, Faàdi Bruno's formula and (6) imply that

$$\begin{aligned} \frac{\mu(z)\Big|\sum \frac{n!}{k_1!\cdots k_n!} \prod_{j=1}^n \Big(\frac{\varphi^{(j)}(z)}{j!}\Big)^{k_j}\Big|}{(\mathrm{Im}\varphi(z))^{k+\frac{1}{p}}} &\le \sup_{w\in\mathbb{H}} \|C_\varphi f_{\varphi(w),k}\|_{\mathcal{W}_\mu^{(n)}(\mathbb{H})} \\ &\le C\|C_\varphi\|_{H^p(\mathbb{H})\to\mathcal{W}_\mu^{(n)}(\mathbb{H})}, \end{aligned} \quad (9)$$

where the sum is over all non-negative integers k_1, k_2, ..., k_n satisfying $k_1 + 2k_2 + \cdots + nk_n = n$.

Now assume that the condition in (4) holds. By Faàdi Bruno's formula and Lemma 2.2, we have

$$\begin{aligned} \|C_\varphi f\|_{\mathcal{W}_\mu^{(n)}(\mathbb{H})} &= \sum_{j=0}^{n-1} |f \circ \varphi(0)| + \sup_{z\in\mathbb{H}} \mu(z)|(C_\varphi f)^{(n)}(z)| \\ &= \sum_{j=0}^{n-1} \Big|\sum \frac{j!}{l_1!\cdots l_j!} f^{(l)}(\varphi(0)) \prod_{s=1}^{j} \Big(\frac{\varphi^{(s)}(0)}{s!}\Big)^{l_s}\Big| \\ &\quad + \sup_{z\in\mathbb{H}} \mu(z)\Big|\sum \frac{n!}{k_1!\cdots k_n!} f^{(k)}(\varphi(z)) \prod_{j=1}^{n} \Big(\frac{\varphi^{(j)}(z)}{j!}\Big)^{k_j}\Big| \\ &\le \sum_{j=0}^{n-1} \sum_{l=0}^{j} |f^{(l)}(\varphi(0))| \Big|\sum \frac{j!}{l_1!\cdots l_j!} \prod_{s=1}^{j} \Big(\frac{\varphi^{(s)}(0)}{s!}\Big)^{l_s}\Big| \\ &\quad + C\|f\|_{H^p(\mathbb{H})} \sum_{k=1}^{n} \sup_{z\in\mathbb{H}} \frac{\mu(z)\Big|\sum \frac{n!}{k_1!\cdots k_n!} \prod_{j=1}^n \Big(\frac{\varphi^{(j)}(z)}{j!}\Big)^{k_j}\Big|}{(\mathrm{Im}\varphi(z))^{k+\frac{1}{p}}}. \end{aligned} \quad (10)$$

From this, Lemma 2.2 with $z = \varphi(0)$ and the condition in (4), we prove that $C_\varphi : H^p(\mathbb{H}) \to \mathcal{W}_\mu^{(n)}(\mathbb{H})$ is bounded. Moreover, if we consider the bounded operator

$C_\varphi : H^p(\mathbb{H}) \to \mathcal{W}_\mu^{(n)}(\mathbb{H})/\mathbb{P}_{n-1}$, then

$$\|C_\varphi\|_{H^p(\mathbb{H})\to\mathcal{W}_\mu^{(n)}(\mathbb{H})/\mathbb{P}_{n-1}} \le C\sum_{k=1}^{n}\sup_{z\in\mathbb{H}}\frac{\mu(z)\Big|\sum\frac{n!}{k_1!\cdots k_n!}\prod_{j=1}^{n}\Big(\frac{\varphi^{(j)}(z)}{j!}\Big)^{k_j}\Big|}{(\mathrm{Im}\varphi(z))^{k+\frac{1}{p}}}. \quad (11)$$

Combining (9) and (11), we obtain the desired asymptotic relation in (5).

Acknowledgement. This work is supported by the Science Foundation of Sichuan Province (Grant No.11ZA120).

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J. COMPUTATIONAL ANALYSIS AND APPLICATIONS, VOL. 15, NO.7, 1194-1201, 2013, COPYRIGHT 2013 EUDOXUS PRESS, LLC

Notes on Generalized Gamma, Beta and Hypergeometric Functions

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Abstract

Recently, some generalizations of the generalized Gamma, Beta, Gauss hypergeometric and Confluent hypergeometric functions has been introduced in [11]. In this paper we obtain some integral representations of the above mentioned functions and Mellin transform representation of the generalized Gamma function. Furthermore, some recurrence relations of these functions are given.

Key words : Gamma Function, Beta Function, Hypergeometric Function, Confluent Hypergeometric Function, Mellin transform.

2000 Mathematics Subject Classification. 33C45, 33C50.

1 Introduction

In recent years, some extensions of the well known special functions have been considered by several authors [1], [2], [4], [5], [6], [9]. In 1994, Chaudhry and Zubair [1] have introduced the following extension of gamma function

$$\Gamma_p(x) := \int_0^\infty t^{x-1} \exp\left(-t - pt^{-1}\right) dt, \tag{1}$$
$$\mathrm{Re}\,(p) > 0.$$

In 1997, Chaudhry *et al.* [2] has presented the following extension of Euler's beta function

$$B_p\,(x,y) := \int_0^1 t^{x-1}\,(1-t)^{y-1} \exp\left[-\frac{p}{t(1-t)}\right] dt, \tag{2}$$
$$(\mathrm{Re}(p) > 0, \mathrm{Re}(x) > 0, \mathrm{Re}(y) > 0)$$

and they proved that this extension has connection with the Macdonald, error and Whittakers function. It is clearly seen that $\Gamma_0(x) = \Gamma\,(x)$ and $B_0\,(x,y) = B\,(x,y)$.

Afterwards, Chaudhry *et al.* [3] used $B_p\,(x,y)$ to extend the hypergeometric functions (and confluent hypergeometric functions) as follows:

$$F_p\,(a,b;c;z) = \sum_{n=0}^{\infty} \frac{B_p\,(b+n,c-b)}{B\,(b,c-b)}\,(a)_n\,\frac{z^n}{n!}$$
$$p \geq 0\,;\ \mathrm{Re}\,(c) > \mathrm{Re}\,(b) > 0,$$

$$\phi_p(b;c;z) = \sum_{n=0}^{\infty} \frac{B_p(b+n, c-b)}{B(b, c-b)} \frac{z^n}{n!}$$
$$p \geq 0\,;\ \mathrm{Re}(c) > \mathrm{Re}(b) > 0,$$

where $(\lambda)_\nu$ denotes the Pochhammer symbol defined by

$$(\lambda)_0 \equiv 1 \text{ and } (\lambda)_\nu := \frac{\Gamma(\lambda+\nu)}{\Gamma(\lambda)}$$

and gave the Euler type integral representation

$$F_p(a,b;c;z) = \frac{1}{B(b,c-b)} \int_0^1 t^{b-1}(1-t)^{c-b-1}(1-zt)^{-a} \exp\left[-\frac{p}{t(1-t)}\right] dt$$
$$p > 0\,;\ p = 0 \text{ and } |\arg(1-z)| < \pi < p;\ \mathrm{Re}(c) > \mathrm{Re}(b) > 0.$$

They called these functions as extended Gauss hypergeometric function (EGHF) and extended confluent hypergeometric function (ECHF), respectively. They have discussed the differentiation properties and Mellin transforms of $F_p(a,b;c;z)$ and obtained transformation formulas, recurrence relations, summation and asymptotic formulas for this function. Note that $F_0(a,b;c;z) = {}_2F_1(a,b;c;z)$.

Note that, very recently, the second auıthor obtained some representations of these extended functions in terms of a finite number of well known higher transcendental functions, specially, as an infinite series containing hypergeometric, confluent hypergeometric, Whittaker's, Lagrange functions, Laguerre polynomials, and products of them [10].

We consider the following generalizations of gamma and Euler's beta functions

$$\Gamma_p^{(\alpha,\beta)}(x) := \int_0^\infty t^{x-1}\, {}_1F_1\left(\alpha;\beta;-t-\frac{p}{t}\right) dt \tag{3}$$
$$\mathrm{Re}(\alpha) > 0, \mathrm{Re}(\beta) > 0, \mathrm{Re}(p) > 0, \mathrm{Re}(x) > 0,$$

$$B_p^{(\alpha,\beta)}(x,y) := \int_0^1 t^{x-1}(1-t)^{y-1}\ {}_1F_1\left(\alpha;\beta;\frac{-p}{t(1-t)}\right) dt, \tag{4}$$
$$(\mathrm{Re}(p) > 0, \mathrm{Re}(x) > 0, \mathrm{Re}(y) > 0, \mathrm{Re}(\alpha) > 0, \mathrm{Re}(\beta) > 0).$$

respectively. It is obvious by (1), (3) and (2), (4) that, $\Gamma_p^{(\alpha,\alpha)}(x) = \Gamma_p(x)$, $\Gamma_0^{(\alpha,\alpha)}(x) = \Gamma(x)$, $B_p^{(\alpha,\alpha)}(x,y) = B_p(x,y)$ and $B_0^{(\alpha,\beta)}(x,y) = B(x,y)$. We call the functions $\Gamma_p^{(\alpha,\beta)}(x)$ and $B_p^{(\alpha,\beta)}(x,y)$ as generalized Euler's gamma function (GEGF) and generalized Euler's beta function (GEBF), respectively.

On the other hand using the new generalization (4) of beta function the generalized Gauss hypergeometric (GGHF) and generalized confluent hypergeometric functions (GCHF) is defined by

$$F_p^{(\alpha,\beta)}(a,b;c;z) := \sum_{n=0}^{\infty} (a)_n \frac{B_p^{(\alpha,\beta)}(b+n, c-b)}{B(b,c-b)} \frac{z^n}{n!}$$

and

$${}_1F_1^{(\alpha,\beta;p)}(b;c;z) := \sum_{n=0}^{\infty} \frac{B_p^{(\alpha,\beta)}(b+n,c-b)}{B(b,c-b)} \frac{z^n}{n!},$$

respectively (see [11]). The following integral representations were obtained in [11]:

$$F_p^{(\alpha,\beta)}(a,b;c;z) := \frac{1}{B(b,c-b)} \int_0^1 t^{b-1}(1-t)^{c-b-1}\ {}_1F_1\left(\alpha;\beta;\frac{-p}{t(1-t)}\right)(1-zt)^{-a}\, dt,$$
$$\mathrm{Re}(p) > 0;\ p = 0 \text{ and } |arg(1-z)| < \pi; \mathrm{Re}(c) > \mathrm{Re}(b) > 0;$$

and

$$ {}_1F_1^{(\alpha,\beta;p)}(b;c;z) := \frac{1}{B(b,c-b)} \int_0^1 t^{b-1}(1-t)^{c-b-1} e^{zt} {}_1F_1\left(\alpha;\beta;\frac{-p}{t(1-t)}\right) dt, \tag{6} $$
$$ p \geq 0; \text{ and } \operatorname{Re}(c) > \operatorname{Re}(b) > 0. $$

Observe that [3],

$$ F_p^{(\alpha,\alpha)}(a,b;c;z) = F_p(a,b;c;z), \; F_0^{(\alpha,\beta)}(a,b;c;z) = {}_2F_1(a,b;c;z), $$

and

$$ {}_1F_1^{(\alpha,\alpha;p)}(b;c;z) = {}_1F_1^{(p)}(b;c;z) = \phi_p(b;c;z), \; {}_1F_1^{(\alpha,\beta;0)}(b;c;z) = {}_1F_1(b;c;z). $$

In section 2, we obtain some integral representations of generalized beta, Gauss hypergeometric and Confluent hypergeometric functions. Mellin transform representation of the generalized Gamma function is also be given. Furthermore, some recurrence relations of the above mentioned functions are presented.

2 New integral representations of GEBF, GGHF and GCHF

It is important and useful to obtain different integral representations of the new generalized beta function, for later use. Also it is useful to discuss the relationships between classical gamma functions and new generalizations. We start with the following integral representation for $B_p^{(\alpha,\beta)}(x)$ by means of Chaudhry's extended beta function.

Theorem 1 *For the new generalized beta function, we have*

$$ B_p^{(\alpha,\beta)}(x,y) = \frac{\Gamma(\beta)}{\Gamma(\alpha)\Gamma(\beta-\alpha)} \int_0^1 B_{pt}(x,y)\, t^{\alpha-1}(1-t)^{\beta-\alpha-1} dt. $$

Proof. Using the integral representation of confluent hypergeometric function, we have

$$ B_p^{(\alpha,\beta)}(x,y) = \frac{\Gamma(\beta)}{\Gamma(\alpha)\Gamma(\beta-\alpha)} \int_0^1 \int_0^1 u^{x-1}(1-u)^{y-1} \exp\left[-\frac{pt}{u(1-u)}\right] t^{\alpha-1}(1-t)^{\beta-\alpha-1} dt du. $$

From the uniform convergence of the integrals, the order of integration can be interchanged to yield that

$$ B_p^{(\alpha,\beta)}(x,y) = \frac{\Gamma(\beta)}{\Gamma(\alpha)\Gamma(\beta-\alpha)} \int_0^1 \left\{ \int_0^1 u^{x-1}(1-u)^{y-1} \exp\left[-\frac{pt}{u(1-u)}\right] du \right\} t^{\alpha-1}(1-t)^{\beta-\alpha-1} dt. $$

In view of (2), we get

$$ B_p^{(\alpha,\beta)}(x,y) = \frac{\Gamma(\beta)}{\Gamma(\alpha)\Gamma(\beta-\alpha)} \int_0^1 B_{pt}(x,y)\, t^{\alpha-1}(1-t)^{\beta-\alpha-1} dt. $$

Whence the result. ■

Theorem 2 *For the following representation holds true for the GGHF:*

$$ F_p^{(\alpha,\beta)}(a,b;c;z) = \frac{\Gamma(\beta)}{\Gamma(\alpha)\Gamma(\beta-\alpha)} \int_0^1 F_{pt}(a,b;c;z)\, t^{\alpha-1}(1-t)^{\beta-\alpha-1} dt. $$

Proof. Since

$$F_p^{(\alpha,\beta)}(a,b;c;z)=\sum_{n=0}^{\infty}(a)_n\frac{B_p^{(\alpha,\beta)}(b+n,c-b)}{B(b,c-b)}\frac{z^n}{n!}$$

we have from the above theorem that

$$F_p^{(\alpha,\beta)}(a,b;c;z)=\frac{\Gamma(\beta)}{\Gamma(\alpha)\Gamma(\beta-\alpha)}\sum_{n=0}^{\infty}\frac{(a)_n}{B(b,c-b)}\int_0^1 B_{pt}(b+n,c-b)\,t^{\alpha-1}(1-t)^{\beta-\alpha-1}dt\frac{z^n}{n!}.$$

From the uniform convergence of the series involved and the absolute convergence of the integral, interchanging the order of series and the integral, we get

$$F_p^{(\alpha,\beta)}(a,b;c;z)=\frac{\Gamma(\beta)}{\Gamma(\alpha)\Gamma(\beta-\alpha)}\int_0^1\left\{\sum_{n=0}^{\infty}(a)_n\frac{B_{pt}(b+n,c-b)}{B(b,c-b)}\frac{z^n}{n!}\right\}t^{\alpha-1}(1-t)^{\beta-\alpha-1}dt.$$

Whence the result. ■

In a similar manner, we are led fairly easily to the theorem below:

Theorem 3 *For the GGCF, we have the foolowing integral representation:*

$${}_1F_1^{(\alpha,\beta;p)}(b;c;z):=\frac{\Gamma(\beta)}{\Gamma(\alpha)\Gamma(\beta-\alpha)}\int_0^1\phi_{pt}(b;c;z)\,t^{\alpha-1}(1-t)^{\beta-\alpha-1}dt.$$

3 Mellin Transform Representation of the GEGF

In this section, we obtain the Mellin transform representations of the GEGF.

Theorem 4 *For the GEGF, we have the following Mellin transform representation:*

$$\mathfrak{M}\left\{\Gamma_p^{(\alpha,\beta)}(y):s\right\}:=\frac{\Gamma(\beta)\Gamma(s)\Gamma(s+y)B(\alpha-2s-y,\beta-\alpha)}{\Gamma(\alpha)\Gamma(\beta-\alpha)}.$$

Proof. Using the integral representation of confluent hypergeometric function, we have

$$\Gamma_p^{(\alpha,\beta)}(s)=\frac{\Gamma(\beta)}{\Gamma(\alpha)\Gamma(\beta-\alpha)}\int_0^{\infty}\int_0^1 u^{s-1}e^{-ut-\frac{pt}{u}}t^{\alpha-1}(1-t)^{\beta-\alpha-1}dtdu. \tag{7}$$

Multiplying both sides of (7) by p^{s-1} and integrate with respect to p over the interval $[0,\infty)$, we get

$$\mathfrak{M}\left\{\Gamma_p^{(\alpha,\beta)}(s):s\right\}:=\int_0^{\infty}p^{s-1}\Gamma_p^{(\alpha,\beta)}(s)dp=\frac{\Gamma(\beta)}{\Gamma(\alpha)\Gamma(\beta-\alpha)}\int_0^{\infty}p^{s-1}\int_0^{\infty}\int_0^1 u^{y-1}e^{-ut-\frac{pt}{u}}t^{\alpha-1}(1-t)^{\beta-\alpha-1}dtdudp.$$

Since the integrals involved are absolutely convergent, we get by interchanging the order of integrals that

$$\mathfrak{M}\left\{\Gamma_p^{(\alpha,\beta)}(s):s\right\}=\frac{\Gamma(\beta)}{\Gamma(\alpha)\Gamma(\beta-\alpha)}\int\limits_0^\infty\int\limits_0^1\left\{\int\limits_0^\infty p^{s-1}\,e^{-\frac{pt}{u}}dp\right\}u^{y-1}t^{\alpha-1}(1-t)^{\beta-\alpha-1}e^{-ut}\,dtdu$$

$$=\frac{\Gamma(\beta)}{\Gamma(\alpha)\Gamma(\beta-\alpha)}\int\limits_0^\infty\int\limits_0^1 u^st^{-s}\Gamma(s)u^{y-1}e^{-ut}t^{\alpha-1}(1-t)^{\beta-\alpha-1}dtdu$$

$$=\frac{\Gamma(\beta)\Gamma(s)}{\Gamma(\alpha)\Gamma(\beta-\alpha)}\int\limits_0^1 t^{\alpha-s-1}(1-t)^{\beta-\alpha-1}\int\limits_0^\infty u^{s+y-1}e^{-ut}dudt$$

$$=\frac{\Gamma(\beta)\Gamma(s)}{\Gamma(\alpha)\Gamma(\beta-\alpha)}\int\limits_0^1 t^{\alpha-2s-y-1}(1-t)^{\beta-\alpha-1}\int\limits_0^\infty v^{s+y-1}e^{-v}dvdt$$

$$=\frac{\Gamma(\beta)\Gamma(s)\Gamma(s+y)}{\Gamma(\alpha)\Gamma(\beta-\alpha)}\int\limits_0^1 t^{\alpha-2s-y-1}(1-t)^{\beta-\alpha-1}dt=\frac{\Gamma(\beta)\Gamma(s)\Gamma(s+y)}{\Gamma(\alpha)\Gamma(\beta-\alpha)}B(\alpha-2s-y,\beta-\alpha).$$

■

Corollary 5 *By Mellin inversion formula, we have the following complex integral representation for* $\Gamma_p^{(\alpha,\beta)}$ *:*

$$\Gamma_p^{(\alpha,\beta)}(y)=\frac{1}{2\pi i}\int_{-i\infty}^{i\infty}\frac{\Gamma(\beta)\Gamma(s)\Gamma(s+y)}{\Gamma(\alpha)\Gamma(\beta-\alpha)}B(\alpha-2s-y,\beta-\alpha)\,p^{-s}ds.$$

Corollary 6 *Taking* $s=1$ *in the above theorem, we get*

$$\int_0^\infty\Gamma_p^{(\alpha,\beta)}(s)dp=\frac{\Gamma(\beta)\Gamma(y+1)\Gamma(\alpha-2-y)}{\Gamma(\alpha)\Gamma(\beta-2-y)}.$$

4 Recurrence Relations for GEBF, GGHF and GCHF

In this section we obtain new recurrence relations for GEBF, GEGF, GGHF and GCHF by using their Mellin transform representation. We start with the following theorem.

Theorem 7 *We have the following difference formula for* $B_p^{(\alpha,\beta)}(x,y)$ *:*

$$xB_p^{(\alpha,\beta)}(x,y+1)-yB_p^{(\alpha,\beta)}(x+1,y)=-\frac{\alpha p}{\beta}B_p^{(\alpha+1,\beta+1)}(x-1,y+1)+\frac{\alpha p}{\beta}B_p^{(\alpha+1,\beta+1)}(x+1,y-1).$$

$$(\mathrm{Re}(p)>0)$$

Proof. *Recalling that the Mellin transform operator is defined by*

$$\mathfrak{M}\{f(t):s\}:=\int\limits_0^\infty t^{s-1}f(t)dt,$$

we observe that $B_p^{(\alpha,\beta)}(x,y)$ *is the Mellin transform of the function*

$$f(t:y;\alpha,\beta;p)=H(1-t)\,(1-t)^{y-1}\,{}_1F_1\left(\alpha;\beta;\frac{-p}{t(1-t)}\right)dt,$$

where

$$H(t)=\begin{cases}1\;if\;t>0\\0\;if\;t<0\end{cases}$$

is the Heaviside unit function. Hence $B_p^{(\alpha,\beta)}(x,y)$ *has the Mellin transform representation*

$$B_p^{(\alpha,\beta)}(x,y) = \mathfrak{M}\left\{f(t:y;\alpha,\beta;p):x\right\}.$$

Taking derivative of $f(t:y;\alpha,\beta;p)$*, we get*

$$\frac{\partial}{\partial t}\left(f(t:y;\alpha,\beta;p)\right) = -\left[\delta(1-t)\,(1-t)^{y-1} + (y-1)H(1-t)\,(1-t)^{y-2}\right]\ {}_1F_1\left(\alpha;\beta;\frac{-p}{t(1-t)}\right)dt$$
$$-\frac{\alpha p}{\beta}H(1-t)\,(1-t)^{y-1}\left[\frac{-1}{t^2}+\frac{1}{(1-t)^2}\right]\ {}_1F_1\left(\alpha+1;\beta+1;\frac{-p}{t(1-t)}\right),$$

where

$$\delta(t-t_0) = \begin{cases}\infty \text{ if } t=t_0\\ 0 \text{ if } t\neq t_0\end{cases}$$

is the Dirac delta function. Since

$$\mathfrak{M}\left\{f'(t):x\right\} = (1-x)\mathfrak{M}\left\{f(t):x-1\right\}$$

we have

$$(x-1)B_p^{(\alpha,\beta)}(x-1,y) = \mathfrak{M}\left\{\left[\delta(1-t)\,(1-t)^{y-1} + (y-1)\,(1-t)^{y-2}\right]\ {}_1F_1\left(\alpha;\beta;\frac{-p}{t(1-t)}\right):x\right\}$$
$$+\mathfrak{M}\left\{\frac{\alpha p}{\beta}H(1-t)\,(1-t)^{y-1}\left[\frac{-1}{t^2}+\frac{1}{(1-t)^2}\right]\ {}_1F_1\left(\alpha+1;\beta+1;\frac{-p}{t(1-t)}\right):x\right\}$$
$$= (y-1)B_p^{(\alpha,\beta)}(x,y-1) - \frac{\alpha p}{\beta}B_p^{(\alpha+1,\beta+1)}(x-2,y) + \frac{\alpha p}{\beta}B_p^{(\alpha+1,\beta+1)}(x,y-2).$$

This completes the proof. ■

Remark 8 *For* $\alpha=\beta$*, we get the recurrence obtained in [[12], pp.222, Eq(5.65)]*

$$xB_p(x,y+1) - yB_p(x+1,y) = p\left[B_p(x+1,y-1) - B_p(x-1,y+1)\right].$$
$$(\mathrm{Re}(p)>0)$$

Theorem 9 *We have the following difference formula for* $\Gamma_p^{(\alpha,\beta)}(s)$:

$$(s-1)\Gamma_p^{(\alpha,\beta)}(s-1) = \frac{\alpha}{\beta}\Gamma_p^{(\alpha+1,\beta+1)}(s) - \frac{p\alpha}{\beta}\Gamma_p^{(\alpha+1,\beta+1)}(s-2).$$

Proof. By (3), $\Gamma_p^{(\alpha,\beta)}(s)$ is the Mellin transform of the function

$$f(t:\alpha,\beta;p) = {}_1F_1(\alpha;\beta;-t-pt^{-1}).$$

Hence

$$\Gamma_p^{(\alpha,\beta)}(s) = \mathfrak{M}\left\{f(t:\alpha,\beta;p):s\right\}.$$

Taking derivative of $f(t:\alpha,\beta;p)$, we get

$$\frac{\partial}{\partial t}\left(f(t:\alpha,\beta;p)\right) = \left(-1+pt^{-2}\right)\frac{\alpha}{\beta}\ {}_1F_1(\alpha+1;\beta+1;-t-\frac{p}{t})$$

Since

$$\mathfrak{M}\left\{f'(t):s\right\} = (1-s)\mathfrak{M}\left\{f(t):s-1\right\}$$

we get

$$(1-s)\Gamma_p^{(\alpha,\beta)}(s-1) = -\frac{\alpha}{\beta}\Gamma_p^{(\alpha+1,\beta+1)}(s) + \frac{p\alpha}{\beta}\Gamma_p^{(\alpha+1,\beta+1)}(s-2).$$

This completes the proof. ■

Remark 10 *When $p = 0$ and $\alpha = \beta$, we have the well known identity*

$$\Gamma(s) = (s-1)\Gamma(s-1).$$

Theorem 11 *We have the following difference formula for $F_p^{(\alpha,\alpha)}(a, b; c; z)$:*

$$\begin{aligned}&(b-1)B(b-1, c-b-1)F_p^{(\alpha,\beta)}(a, b-1; c; z)\\&= (c-b-1)B(b, c-b-1)F_p^{(\alpha,\beta)}(a, b; c-1; x) + azB(b, c-b)F_p^{(\alpha,\beta)}(a+1, b; c; x)\\&-\frac{\alpha p}{\beta}B(b-2, c-b)F_p^{(\alpha+1,\beta+1)}(a, b-2; c-2; z) + \frac{\alpha p}{\beta}B(b, c-b-2)F_p^{(\alpha+1,\beta+1)}(a, b; c-2; z).\end{aligned}$$

Proof. *Observe from (5) that $B(b, c-b)F_p^{(\alpha,\beta)}(a, b; c; z)$ is the Mellin transform of the function*

$$f_{a,b,c}(t : z; \alpha, \beta; p) = H(1-t)(1-t)^{c-b-1}\ {}_1F_1\left(\alpha; \beta; \frac{-p}{t(1-t)}\right)(1-zt)^{-a}.$$

Hence $B(b, c-b)F_p^{(\alpha,\beta)}(a, b; c; z)$ has the Mellin transform representation

$$B(b, c-b)F_p^{(\alpha,\beta)}(a, b; c; z) = \mathfrak{M}\{f_{a,b,c}(t : z; \alpha, \beta; p) : b\}.$$

Taking derivative of $f_{a,b}(t : y; \alpha, \beta; p)$, we get

$$\begin{aligned}&\frac{\partial}{\partial t}(f_{a,b}(t : z; \alpha, \beta; p)) = -\Big[\delta(1-t)(1-t)^{c-b-1}(1-zt)^{-a} + (c-b-1)H(1-t)(1-t)^{c-b-2}(1-zt)^{-a}\\&+azH(1-t)(1-t)^{c-b-1}(1-zt)^{-a-1}\Big]\ {}_1F_1\left(\alpha; \beta; \frac{-p}{t(1-t)}\right)dt\\&-\frac{\alpha p}{\beta}H(1-t)(1-t)^{c-b-1}(1-zt)^{-a}\left[\frac{-1}{t^2} + \frac{1}{(1-t)^2}\right]\ {}_1F_1\left(\alpha+1; \beta+1; \frac{-p}{t(1-t)}\right).\end{aligned}$$

Since

$$\mathfrak{M}\{f'(t) : b\} = (1-b)\mathfrak{M}\{f(t) : b-1\}$$

we get

$$\begin{aligned}&(b-1)B(b-1, c-b-1)F_p^{(\alpha,\beta)}(a, b-1; c; z) = (c-b-1)B(b, c-b-1)F_p^{(\alpha,\beta)}(a, b; c-1; x)\\&+ azB(b, c-b)F_p^{(\alpha,\beta)}(a+1, b; c; x) - \frac{\alpha p}{\beta}B(b-2, c-b)F_p^{(\alpha+1,\beta+1)}(a, b-2; c-2; z)\\&+ \frac{\alpha p}{\beta}B(b, c-b-2)F_p^{(\alpha+1,\beta+1)}(a, b; c-2; z).\end{aligned}$$

This completes the proof. ■

Similarly, using (6), we get:

Theorem 12 *We have the following difference formula for ${}_1F_1^{(\alpha,\beta;p)}(b; c; z)$:*

$$\begin{aligned}&(b-1)B(b-1, c-b-1)_1F_1^{(\alpha,\beta;p)}(b-1; c; z)\\&= (c-b-1)B(b, c-b-1)_1F_1^{(\alpha,\beta;p)}(b; c-1; z) + zB(b, c-b)_1F_1^{(\alpha,\beta;p)}(b; c; z)\\&-\frac{\alpha p}{\beta}B(b-2, c-b)_1F_1^{(\alpha+1,\beta+1;p)}(b-2; c-2; z) + \frac{\alpha p}{\beta}B(b, c-b-2)_1F_1^{(\alpha+1,\beta+1;p)}(b; c-2; z).\end{aligned}$$

Proof. *Observe from (6) that $B(b, c-b)_1F_1^{(\alpha,\beta;p)}(b; c; z)$ is the Mellin transform of the function*

$$f_{a,b,c}(t : z; \alpha, \beta; p) = H(1-t)(1-t)^{c-b-1}e^{zt}\ {}_1F_1\left(\alpha; \beta; \frac{-p}{t(1-t)}\right).$$

Hence $B(b,c-b)_1F_1^{(\alpha,\beta;p)}(b;c;z)$ *has the Mellin transform representation*

$$B(b,c-b)_1F_1^{(\alpha,\beta;p)}(b;c;z)=\mathfrak{M}\left\{f_{a,b,c}(t:z;\alpha,\beta;p):b\right\}.$$

Taking derivative of $f_{a,b}(t:y;\alpha,\beta;p)$, *we get*

$$\begin{aligned}&\frac{\partial}{\partial t}\left(f_{a,b}(t:z;\alpha,\beta;p)\right)=-\Big[\delta(1-t)\left(1-t\right)^{c-b-1}e^{zt}+(c-b-1)H(1-t)\left(1-t\right)^{c-b-2}e^{zt}\\&+zH(1-t)\left(1-t\right)^{c-b-1}e^{zt}\Big]\ {}_1F_1\left(\alpha;\beta;\frac{-p}{t(1-t)}\right)dt\\&-\frac{\alpha p}{\beta}H(1-t)\left(1-t\right)^{c-b-1}e^{zt}\left[\frac{-1}{t^2}+\frac{1}{(1-t)^2}\right]\ {}_1F_1\left(\alpha+1;\beta+1;\frac{-p}{t(1-t)}\right).\end{aligned}$$

Since

$$\mathfrak{M}\left\{f'(t):b\right\}=(1-b)\mathfrak{M}\left\{f(t):b-1\right\}$$

we get

$$\begin{aligned}&(b-1)B(b-1,c-b-1)_1F_1^{(\alpha,\beta;p)}(b-1;c;z)=(c-b-1)B(b,c-b-1)_1F_1^{(\alpha,\beta;p)}(b;c-1;z)\\&+zB(b,c-b)_1F_1^{(\alpha,\beta;p)}(b;c;z)-\frac{\alpha p}{\beta}B(b-2,c-b)_1F_1^{(\alpha+1,\beta+1;p)}(b-2;c-2;z)\\&+\frac{\alpha p}{\beta}B(b,c-b-2)_1F_1^{(\alpha+1,\beta+1;p)}(b;c-2;z).\end{aligned}$$

This completes the proof. ■

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J. COMPUTATIONAL ANALYSIS AND APPLICATIONS, VOL. 15, NO.7, 1202-1210, 2013, COPYRIGHT 2013 EUDOXUS PRESS, LLC

A modified AOR iterative method for new preconditioned linear systems for $L-$matrices*

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Abstract

In this paper, a preconditioned AOR iterative method is presented with a new preconditioner, and the corresponding convergence and comparison results are given. The optimum parameters and spectral radius for strictly diagonally dominant L-matrices are found. Numerical examples are given to illustrate the efficiency of our method.

Keywords: $AOR-$ iteration method; $L-$ matrix; Spectral radius; Optimum parameters; Preconditioner.

AMS subject classification: 65F10

1 Introduction

Consider the following linear system

$$Ax = b, \tag{1.1}$$

where $A \in R^{n\times n}$, b, $x \in R^n$. For any splitting, $A = M - N$ with $\det(M) \neq 0$, the basic iterative scheme for solving (1.1) is as follows

$$x_{k+1} = M^{-1}Nx_k + M^{-1}b, k = 0, 1, \cdots.$$

*The work is supported by the National Natural Science Foundation of China (11226331). G. Zeng was supported by the Doctor Fund of East China Institute of Technology (No. DHBK201201) and L. Lei was suppoted the Principal fund of East China Institute of Technology (No. DHXK1217). Partial results of this paper were represented at the 4th International Conference on Bioinformatics and Biomedical Engineering,Chengdu, China. June 18-20, 2010.

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For simplicity, without loss of generality, we assume throughout this paper that

$$A = I - L - U,$$

where I is the identity matrix, L and U are strictly lower and upper triangular matrices obtained from A, respectively. Thereby the iterative matrix of the classical AOR iterative method in [1] is defined as

$$L_{r\omega} = (I - rL)^{-1}[(1-\omega)I + (\omega - r)L + \omega U], \tag{1.2}$$

where ω and r are real parameters with $\omega \neq 0$.

The spectral radius of the iterative matrix determines the convergence and stability of the method, and the smaller it is, the faster the method converges when the spectral radius is smaller than 1. In order to accelerate the convergence of the iterative method solving (1.1), preconditioned methods are often utilized, which is, which is,

$$PAx = Pb, \tag{1.3}$$

where P is the nonsingular preconditioner.

Construct $P = (I + \hat{S})$ with

$$\hat{S} = \begin{pmatrix} 0 & 0 & \cdots & 0 & 0 \\ -\alpha_2 a_{2,1} & 0 & \cdots & 0 & 0 \\ -\alpha_3 a_{3,1} & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ -\alpha_n a_{n,1} & 0 & \cdots & 0 & 0 \end{pmatrix}, \text{ and } \alpha_i \in R,\ i = 2,\ \cdots,\ n.$$

The Equation (1.3) transform to

$$\hat{A}x = \hat{b}, \tag{1.4}$$

where $\hat{A} = (I + \hat{S})A$ and $\hat{b} = (I + \hat{S})b$. The coefficient matrix of (1.4) is splited as

$$\hat{A} = \hat{D} - \hat{L} - \hat{U}, \tag{1.5}$$

where $\hat{D} = diag(\hat{A})$, $\hat{L}$ and $\hat{U}$ are strictly lower and upper triangular matrices obtained from $\hat{A}$, respectively. Through some trivial calculation, we obtain that

$$\hat{D} = diag(1,\ 1 - \alpha_2 a_{2,1} a_{1,2},\ \cdots,\ 1 - \alpha_n a_{n,1} a_{1,n}),$$

and

$$\hat{L} = -\begin{pmatrix} 0 & 0 & \cdots & 0 & 0 \\ a_{2,1} - \alpha_2 a_{2,1} & 0 & \cdots & 0 & 0 \\ a_{3,1} - \alpha_3 a_{3,1} & a_{3,2} - \alpha_3 a_{3,1} a_{1,2} & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ a_{n,1} - \alpha_n a_{n,1} & a_{n,2} - \alpha_n a_{n,1} a_{1,2} & a_{n,3} - \alpha_n a_{n,1} a_{1,3} & \ldots & 0 \end{pmatrix},$$

and

$$\hat{U} = -\begin{pmatrix} 0 & a_{1,2} & a_{1,3} & \cdots & a_{1,n} \\ 0 & 0 & a_{2,3} - \alpha_2 a_{2,1} a_{1,3} & \cdots & a_{2,n} - \alpha_2 a_{2,1} a_{1,n} \\ 0 & 0 & 0 & \cdots & a_{3,n} - \alpha_3 a_{3,1} a_{1,n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 0 \end{pmatrix}.$$

Applying the *AOR* method to the preconditioned linear system (1.4), the corresponding preconditioned *AOR* iterative method is obtained with iterative matrix

$$\hat{L}_{r\omega} = (\hat{D} - r\hat{L})^{-1}[(1-\omega)\hat{D} + (\omega - r)\hat{L} + \omega\hat{U}], \tag{1.6}$$

where ω and r are real parameters with $\omega \neq 0$.

The rest of the article is organized as follows. In Section 2, we briefly explain some notation and some Lemma which are used to state and to prove our results. In Section 3, we sate our result with its proof. Examples are given to illustrate our main theorem in Section 4.

2 Preliminaries

Some notation and Lemmas as follows are needed in this article.

A matrix A is nonnegative(positive) if each entry of A is nonnegative(positive), respectively, which is denoted by $A \geq 0,\ (A > 0)$. Let $\rho(A)$ be the spectral radius of A. In addition, A matrix A is irreducible if the directed graph associated to A is strongly connected. Lastly, A matrix A is an $L-$matrix if $a_{i,i} > 0,\ i = 1,\ 2,\ \cdots,\ n$ and $a_{i,j} \leq 0$, for all $i,\ j = 1,\ 2,\ \cdots,\ n$ such that $i \neq j$.

The following Lemma will be useful to prove the main results.

Lemma 2.1a([5]). *Let $A \in R^{m\times n}$, $A = M - N$ is a splitting of A. Then*
(a). If $M^{-1} \geq 0$ and $N \geq 0$, then $A = M - N$ is a regular splitting;
(b). If $M^{-1} \geq 0$ and $M^{-1}N \geq 0$, then $A = M - N$ is a weak regular splitting.

Lemma 2.1b([5]). *Let $A = M_1 - N_1 = M_2 - N_2$ are two regular splitting for matrix A and suppose that A^{-1} and $N_2 \geq N_1 \geq 0$. Then*

$$0 \leq \rho(M_1^{-1}N_1) \leq \rho(M_2^{-1}N_2) < 1.$$

Lemma 2.2a([6]). *Let $A \in C^{n\times n}$ be a non-negative and irreducible $n \times n$ matrix. Then*
(a). A has a positive real eigenvalue equal to its spectral radius $\rho(A)$;
(b). There exists an eigenvector $x > 0$ corresponding to $\rho(A)$,
(c). $\rho(A)$ is a simple eigenvalue of A;

(d). $\rho(A)$ increases when any entry of A increases.

Lemma 2.2b([6]). *Let A be a non-negative matrix. Then*
(a). If $\alpha x \leq Ax$ for some non-negative vector x, $x \neq 0$, then $\alpha \leq \rho(A)$.
(b). If $Ax \leq \beta x$ for some positive vector x, then $\rho(A) \leq \beta$. Moreover, if A is irreducible and if $0 \neq \alpha x \leq Ax \leq \beta x$, $\alpha x \neq Ax$, $Ax \neq \beta x$ for non-negative vector x, then $\alpha < \rho(A) < \beta$ and x is a positive vector.

3 Main results

Our main goal in this section is to establish the following results with proof.

Lemma 3.1. *Let A and $\hat{A}$ be the coefficient matrices of the linear systems (1.1) and (1.4), where A is an $L-$matrix for which there exists i such that $a_{i,1} \neq 0$, $i = 2, \cdots, n$, with $a_{i+1,i}a_{i,i+1} \neq 0$, $i = 1, \cdots, n-1$. If $0 \leq r \leq \omega \leq 1$ $(\omega \neq 0,\ r \neq 1)$ and one of the following conditions is also satisfied simultaneously*
(a). $0 < \alpha_i \leq 1$ if $0 < a_{1,i}a_{i,1} < 1$, or $a_{i,1} \neq 0$ and $a_{1,i} = 0$;
(b). $0 < \alpha_i \leq 1$ if $a_{1,i}a_{i,1} = 1$;
(c). $0 < \alpha_i < \frac{1}{a_{1,i}a_{i,1}}$ if $a_{1,i}a_{i,1} > 1$;
(d). $\alpha_i > 0$ if $a_{i,1} = 0$, $i = 1, 2, \cdots, n$.
Then the iterative matrices $L_{r,\omega}$ and $\hat{L}_{r,\omega}$ are nonnegative and irreducible.

Proof. From (1.2) we have

$$L_{r,\omega} = (1-\omega)I + \omega(1-r)L + \omega U + T, \tag{3.1}$$

where

$$T = rL[\omega - r)L + \omega U] + (r^2L^2 + \cdots + r^{n-1}L^{n-1})[(1-\omega)I + (\omega - r)L + \omega U] \geq 0. \tag{3.2}$$

Then $L_{r,\omega}$ and $\hat{L}_{r,\omega}$ are nonnegative and irreducible according to lemma 1 of [4]. □

Theorem 3.2. *Under the assumptions of Lemma 3.1, and let $L_{r,\omega}$ and $\hat{L}_{r,\omega}$ be the iterative matrices of the AOR method obtained from (1.1) and (1.4), respectively. Then we have*
(a) $\rho(\hat{L}_{r,\omega}) < \rho(L_{r,\omega}) < 1$, if $\rho(L_{r,\omega}) < 1$;
(b) $\rho(\hat{L}_{r,\omega}) = \rho(L_{r,\omega}) = 1$, if $\rho(L_{r,\omega}) = 1$;
(c) $\rho(\hat{L}_{r,\omega}) > \rho(L_{r,\omega}) > 1$, if $\rho(L_{r,\omega}) > 1$.

Proof. From Lemma 3.1 it is obvious that $L_{r,\omega}$ and $\hat{L}_{r,\omega}$ are nonnegative and irreducible. Therefore, according to Lemma 2.2a there is a positive vector x, such that

$$L_{r,\omega}x = \lambda x, \tag{3.3}$$

where $\lambda = \rho(L_{r,\omega})$, (3.3) can equivalently to

$$[(1-\omega)I + (\omega - r)L + \omega U]x = \lambda(I - rL)x. \tag{3.4}$$

Now consider

$$\begin{aligned}
\hat{L}_{r\omega}x - \lambda x &= (\hat{D} - r\hat{L})^{-1}[(1-\omega)\hat{D} + (\omega - r)\hat{L} + \omega\hat{U}]x - \lambda x \\
&= (\hat{D} - r\hat{L})^{-1}[(1-\omega)\hat{D} + (\omega - r)\hat{L} + \omega\hat{U} - \lambda(\hat{D} - r\hat{L})]x \\
&= (\hat{D} - r\hat{L})^{-1}[(1-\omega-\lambda)D_1 + \omega U_1 - (\omega - r + \lambda r)L_1]x \\
&= (\lambda - 1)(\hat{D} - r\hat{L})^{-1}(0,\ \eta_2,\ \eta_3,\ \cdots,\ ,\ \eta_n)^T,
\end{aligned} \tag{3.5}$$

where $\eta_i = \alpha_i a_{i,1}[r\sum_{1<j<i} a_{1,j}x_j + a_{1,i}x_i + (r-1)x_1] \geq 0,\ \ i = 2,\ \cdots,\ n.$

In the following, we give the comparison results based on the three cases of λ.

(a) If $0 < \lambda < 1$, then $\hat{L}_{r\omega}x - \lambda x \leq 0$ without the equality holding constantly. Therefore, $\hat{L}_{r\omega}x \leq \lambda x$. Furthermore, we get $\rho(\hat{L}_{r\omega}) < \lambda = \rho(L_{r\omega})$ by Lemma 2.2b.

(b) If $\lambda = 1$, then $\hat{L}_{r\omega}x - \lambda x = 0$, we get $\rho(\hat{L}_{r\omega}) = \lambda = \rho(L_{r\omega})$ still by Lemma 2.2b.

(c) If $\lambda > 1$, then $\hat{L}_{r\omega}x - \lambda x \geq 0$ without the equality holding constantly. Therefore, $\hat{L}_{r\omega}x \geq \lambda x$. Furthermore, we get $\rho(\hat{L}_{r\omega}) > \lambda = \rho(L_{r\omega})$ by Lemma 2.2b again. The proof of Theorem 3.2 is completed. □

According to our main result, we have the following corollary.

Corollary 3.3. *Let $L_{r,\omega}$ and $\bar{L}_{r,\omega}$ be the iterative matrices of the AOR method obtained from (1.1) and (1.4), respectively. Under the same conditions in Theorem 3.2 except for the ones for α_i, $i = 2, \cdots, n$, we have*

(a) $\rho(\bar{L}_{r,\omega}) < \rho(L_{r,\omega}) < 1$, if $\rho(L_{r,\omega}) < 1$;

(b) $\rho(\bar{L}_{r,\omega}) = \rho(L_{r,\omega}) = 1$, if $\rho(L_{r,\omega}) = 1$;

(c) $\rho(\bar{L}_{r,\omega}) > \rho(L_{r,\omega}) > 1$, if $\rho(L_{r,\omega}) > 1$.

Now we show how the Modified AOR optimum parameters and spectral radius are found. For convenience, let $\tilde{A} = \hat{S}A = I_{\hat{S}} - L_{\hat{S}} - U_{\hat{S}}$. We redefine (1.5), $\hat{D} = I + I_{\hat{S}}$, $\hat{L} = L + L_{\hat{S}}$, $\hat{U} = U + U_{\hat{S}}$.

Lemma 3.4 *Under the assumptions of Lemma 3.1, and A is a strictly diagonally dominant $L-$matrix. Then $\hat{A}$ is a strictly diagonally dominant $L-$matrix.*

Proof. We first prove $\hat{A}$ is an $L-$matrix.

$$\hat{A} = (I + \hat{S})A = \hat{D} - \hat{L} - \hat{U}$$

$$= \begin{pmatrix} 1 & a_{1,2} & a_{1,3} & \cdots & a_{1,n} \\ a_{2,1}-\alpha_2 a_{2,1} & 1-\alpha_2 a_{1,2}a_{2,1} & a_{2,3}-\alpha_2 a_{2,1}a_{1,3} & \cdots & a_{2,n}-\alpha_2 a_{2,1}a_{1,n} \\ a_{3,1}-\alpha_3 a_{3,1} & a_{3,2}-\alpha_3 a_{3,1}a_{1,2} & 1-\alpha_3 a_{1,3}a_{3,1} & \cdots & a_{3,n}-\alpha_3 a_{3,1}a_{1,n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ a_{n,1}-\alpha_n a_{n,1} & a_{n,2}-\alpha_n a_{n,1}a_{1,2} & a_{n,3}-\alpha_n a_{n,1}a_{1,3} & \vdots & 1-\alpha_n a_{1,n}a_{n,1} \end{pmatrix}.$$

Since A is a strictly diagonally dominant $L-$matrix, the non-diagonal elements of the first line of $\hat{A}$ are non-positive. For all the lines from the second line for $\hat{A}$, we have

$$\hat{a}_{i,j} = \begin{cases} a_{i,j} - \alpha_i a_{i,1}a_{1,j} \leq 0, if\ i \neq j \\ 1 - \alpha_i a_{i,1}a_{1,j} > 0, if\ i = j \end{cases}$$

Thus $\hat{A}$ is an L-matrix.

Below we prove $\hat{A}$ is a strictly diagonally dominant matrix.
For the first line of $\hat{A}$, $|a_{1,2} + a_{1,3} + \cdots + a_{1,n}| < 1|$ holds, and for the $i-$th line

$$\begin{aligned} &|(a_{i,1} - \alpha_i a_{i,1}) + (a_{i,i-1} - \alpha_i a_{i,1}a_{1,i-1}) + (a_{i,i+1} - \alpha_i a_{i,1}a_{1,i+1}) + \cdots + (a_{i,n} - \alpha_i a_{i,1}a_{1,n})| \\ &= -(a_{i,1} + \cdots + a_{i,i-1} + a_{i,i+1} + \cdots + a_{i,n}) + \alpha_i a_{i,1}(1 + \cdots + a_{1,i-1} + a_{1,i+1} + \cdots + a_{1,n}) \\ &\leq 1 - \alpha_i a_{i,1}a_{1,i} \end{aligned}$$

holds too. Then, $\hat{A}$ is a strictly diagonally dominant $L-$matrix. □

Theorem 3.5. *Under the assumptions of Lemma 3.4, $\rho(L_{r,\omega}) < 1$, $0 \leq r \leq \omega \leq 1$ and $\omega > 0$. Then*

$$\rho(\hat{L}_{1,1}) \leq \rho(\hat{L}_{r,\omega}) < 1. \tag{3.6}$$

If $r = 1,\ \omega = 1$ and $\alpha = [1, 1, \cdots, 1]$, then equality holds in (3.6).
Proof. From Equation (1.6), we get

$$\begin{aligned} \hat{L}_{r,\omega} =& [(I + I_{\hat{S}}) - r(L + L_{\hat{S}})]^{-1} \\ & \times [(1-\omega)(I + I_{\hat{S}}) + (\omega - r)(L + L_{\hat{S}}) + \omega(U + U_{\hat{S}})], \end{aligned}$$

let′s denote

$$M_2 = [(I + I_{\hat{S}}) - r(L + L_{\hat{S}})],$$
$$N_2 = [(1-\omega)(I + I_{\hat{S}}) + (\omega - r)(L + L_{\hat{S}}) + \omega(U + U_{\hat{S}})],$$

according to Lemma 3.4, we known that $\hat{A} = (I + \hat{S})A$ is a strictly diagonally dominant $L-$matrix. Hence,

$$(I + I_{\hat{S}})^{-1} \geq 0 \ \ and \ \ \rho[(I + I_{\hat{S}})^{-1}(L + L_{\hat{S}})] < 1.$$

Moreover,

$$M_2^{-1} = \{I + [r(I + I_{\hat{S}})^{-1}(L + L_{\hat{S}})] + [r(I + I_{\hat{S}})^{-1}(L + L_{\hat{S}})]^2 + \cdots\} \times (I + I_{\hat{S}})^{-1} \geq 0, \tag{3.7}$$

$$N_2 = [(1-\omega)(I + I_{\hat{S}}) + (\omega - r)(L + L_{\hat{S}}) + \omega(U + U_{\hat{S}})] \geq 0, \tag{3.8}$$

and

$$\begin{aligned} M_2 - N_2 =& [(I + I_{\hat{S}}) - r(L + L_{\hat{S}})] - \\ & [(1-\omega)(I + I_{\hat{S}}) + (\omega - r)(L + L_{\hat{S}}) + \omega(U + U_{\hat{S}})] \\ =& \omega[(I + I_{\hat{S}}) - (L + L_{\hat{S}}) - (U + U_{\hat{S}})] \\ =& \omega\hat{A}. \end{aligned} \tag{3.9}$$

Therefore, $\omega\hat{A} = M_2 - N_2$ is a regular splitting.

On the other hand,

$$\begin{aligned} \hat{L}_{1,1} =& [(I + I_{\hat{S}}) - (L + L_{\hat{S}})]^{-1}(U + U_{\hat{S}}) \\ =& [\omega(I + I_{\hat{S}}) - \omega(L + L_{\hat{S}})]^{-1}\omega(U + U_{\hat{S}}). \end{aligned}$$

Let

$$M_1 = \omega(I + I_{\hat{S}}) - \omega(L + L_{\hat{S}}), \quad N_1 = \omega(U + U_{\hat{S}}),$$

since $(I + I_{\hat{S}})^{-1} \geq 0$ and $\rho[(I + I_{\hat{S}})^{-1}(L + L_{\hat{S}})] < 1$, we have

$$\begin{aligned} M_1^{-1} =& \frac{1}{\omega}\{I + [(I + I_{\hat{S}})^{-1}(L + L_{\hat{S}})] \\ & + [(I + I_{\hat{S}})^{-1}(L + L_{\hat{S}})]^2 + \cdots\} \times (I + I_{\hat{S}})^{-1} \geq 0, \end{aligned} \tag{3.10}$$

$$N_1 = \omega(U + U_{\hat{S}}) \geq 0, \tag{3.11}$$

and

$$\begin{aligned} M_1 - N_1 =& \omega(I + I_{\hat{S}}) - \omega(L + L_{\hat{S}}) - \omega(U + U_{\hat{S}}) \\ =& \omega[(I + I_{\hat{S}}) - (L + L_{\hat{S}}) - (U + U_{\hat{S}})] \\ =& \omega\hat{A}. \end{aligned} \tag{3.12}$$

According to (3.7-12), $\omega\hat{A} = M_2 - N_2 = M_1 - N_1$ are two different regular splitting of $\omega\hat{A}$, and $N_2 = (1-\omega)(I+I_{\hat{S}})+(\omega-r)(L+L_{\hat{S}})+\omega(U+U_{\hat{S}}) \geq \omega(U+U_{\hat{S}}) = N_1 \geq 0$, we can obtain $\rho(M_1^{-1}N_1) \leq \rho(M_2^{-1}N_2) < 1$ by Lemma 2.1b. Hence,

$$\rho(\hat{L}_{1,1}) \leq \rho(\hat{L}_{r,\omega}) < 1.$$

In particular, if $r = 1, \ \omega = 1$ and $\alpha = [1, 1, \cdots, 1]$, then $\rho(\hat{L}_{r,\omega}) = \rho(\hat{L}_{1,1})$ hold. □

4 Numerical experiments

In this section we give some numerical examples to illustrate the results obtained in Section 3.

Example 4.1. Consider the matrix A of (1.1), and given by

$$A = \begin{pmatrix} 1 & -0.01 & -0.5 & -0.3 & -0.05 & -0.3 \\ -0.2 & 1 & -0.1 & -0.15 & -0.12 & -0.14 \\ -0.1 & -0.14 & 1 & -0.05 & -0.4 & -0.2 \\ -0.2 & -0.05 & -0.11 & 1 & -0.2 & -0.1 \\ -0.4 & -0.03 & -0.05 & -0.15 & 1 & -0.2 \\ -0.08 & -0.3 & -0.1 & -0.1 & -0.3 & 1 \end{pmatrix}.$$

For the modified AOR iterative method, we have the following results. The digital of following table is formed by Matlab R2010a program.

Table 1. Numerical illustration of our main results

$(\alpha_2, \alpha_3, \cdots, \alpha_6)$	ω	r	$\rho(L_{r,\omega})$	$\rho(\hat{L}_{r,\omega})$
$(0.5, 0.5, 0.5, 0.5, 0.5)$	0.8	0.2	0.8890	0.8745
$(0.6, 0.6, 0.6, 0.6, 0.6)$	0.75	0.65	0.8674	0.8462
$(0.8, 0.8, 0.8, 0.8, 0.8)$	0.8	0.6	0.8629	0.8317
$(0.9, 0.9, 0.9, 0.9, 0.9)$	0.95	0.9	0.7999	0.7471
$(1, 1, 1, 1, 1)$	1	1	0.7710	0.7009

Remark 4.1. From the above table, we know $\rho(\hat{L}_{r,\omega}) < \rho(L_{r,\omega})$ when $\rho(L_{r,\omega}) < 1$. In particular, if $r = 1,\ \omega = 1$ and $\alpha = [1, 1, \cdots, 1]$, then $\rho(\hat{L}_{r,\omega}) = \rho(\hat{L}_{1,1})$ hold. So the results are in concord with our main results.

Example 4.2. Consider the matrix A of (1.1), and given by

$$A = \begin{pmatrix} 1 & -0.01 & -0.5 & -0.3 & -0.05 & -0.3 \\ -0.4 & 1 & -0.2 & -0.15 & -0.12 & -0.3 \\ -0.5 & -0.14 & 1 & -0.5 & -0.6 & -0.2 \\ -0.2 & -0.05 & -0.11 & 1 & -0.2 & -0.1 \\ -0.6 & -0.05 & -0.06 & -0.15 & 1 & -0.2 \\ -0.7 & -0.3 & -0.1 & -0.1 & -0.3 & 1 \end{pmatrix}.$$

For the modified AOR iterative method, we have the following results.

Table 2. Numerical illustration of Theorem 3.2

$(\alpha_2, \alpha_3, \cdots, \alpha_6)$	ω	r	$\rho(L_{r,\omega})$	$\rho(\hat{L}_{r,\omega})$
$(0.9, 0.1, 0.5, 0.2, 0.8)$	0.05	0.05	1.0121	1.0151
$(0.4, 0.7, 0.8, 0.3, 0.6)$	0.7	0.3	1.1970	1.2686
$(0.7, 0.2, 0.8, 0.3, 0.3)$	0.75	0.65	1.2706	1.3252
$(0.2, 0.4, 0.6, 0.7, 0.3)$	0.8	0.6	1.2778	1.3568
$(0.2, 0.4, 0.6, 0.7, 0.3)$	0.95	0.9	1.4209	1.5515

Remark 4.2. From the above table, it is easy to know that $\rho(\hat{L}_{r,\omega}) > \rho(L_{r,\omega})$ when $\rho(L_{r,\omega}) > 1$. The results are also in concord with Theorem 3.2 and Corollary 3.3.

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J. COMPUTATIONAL ANALYSIS AND APPLICATIONS, VOL. 15, NO.7, 1211-1222, 2013, COPYRIGHT 2013 EUDOXUS PRESS, LLC

Modern Algorithms of Simulation for Getting Some Random Numbers

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Abstract In order to carry out the simulation, we need a source of random numbers distributed according to the desired probability distribution. In this paper we have constructed algorithms for generating both continuous and discrete random variables. One simulates a discrete random variable having a geometric distribution, which is used in reliability.We also create some algorithms for generating: a normal continuous variable, other continuous variables having exponential distribution, Weibull distribution, gamma distribution. The aim of this paper is to see that if we have random numbers generated according to some distribution, we may perform a transformation to generate the desired distribution.

Key words generalized test likelihood ratio – parametric classification criterion – maximum likelihood estimates – likelihood function

1 Inverse Transform Method

We shall describe a method of simulating a discrete random variable that take a finite number of values, called inverse transform method. According to this method, we can simulate any random variable X if we know its distribution function F and we can calculate the inverse function F^{-1}.

Using this method, we build a Matlab program to simulate the discrete variable X, whose distribution is

$$X : \begin{pmatrix} a_1 & \cdots & a_k & \cdots & a_m \\ p_1 & \cdots & p_k & \cdots & p_m \end{pmatrix},$$

where

$$\sum_{k=1}^{m} p_k = 1.$$

Its distribution function is:

$$F_X(x) = \begin{cases} 0, & x \le a_1 \\ p_1, & x \in (a_1, a_2] \\ p_1 + p_2, & x \in (a_2, a_3] \\ \vdots & \\ p_1 + p_2 + \cdots + p_k, & x \in (a_k, a_{k+1}] \\ \vdots & \\ 1 & x > a_m \end{cases} \tag{1}$$

and the inverse function will be:

$$F_X^{-1}(u) = a_k, \quad x \in (F_X(a_{k-1}),\ F_X(a_k)], \quad (\forall)\ \ k = \overline{1, m},$$

where

$$a_0 = -\infty,\ F_X(a_0) = 0.$$

The algorithm for simulating the random variable X consists of:

- generating a value u uniformly distributed in $[0, 1]$;
- finding the index k for which

$$F_X(a_{k-1}) < u \le F_X(a_k). \tag{2}$$

The relation (2) results from the fact that the relation:

$$a_{k-1} < X \le a_k$$

involves

$$F_X(a_{k-1}) < F_X(x) \le F_X(a_k)$$

and using that

$$F_X(x) = u.$$

We shall construct the corresponding Matlab program:

```
function x=simdiscrv(F,a,m)
u=rand;
k=1;
while (u>F(k))
k=k+1;
end
x=a(k);
end
```

We shall apply the previous Matlab function to generate a discrete random variable that gives the number of points obtained in the experience when we roll a die and the possible outcomes are 1, 2, 3, 4, 5, 6 corresponding to the side that turns up.

Thereby

$$X : \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ \frac{1}{6} & \frac{1}{6} & \frac{1}{6} & \frac{1}{6} & \frac{1}{6} & \frac{1}{6} \end{pmatrix},$$

where

$$\sum_{k=1}^{6} p_k = 1$$

and

$$F_X(x) = \begin{cases} 0, & x \leq 1 \\ 1/6, & x \in (1,2] \\ 2/6, & x \in (2,3] \\ 3/6, & x \in (3,4] \\ 4/6, & x \in (4,5] \\ 5/6, & x \in (5,6] \\ 1, & x > 6. \end{cases}$$

In the command line of Matlab we shall write:

```
>> a = 1 : 7;
>> F = 0 : 1=6 : 1;
>> x = simdiscrv(F; a; 7)
```

It will display:

```
u =
0.6557
x =
5
```

2 Simulation of a random variable having a geometric distribution

Let X be a random variable signifying the number of failures until a certain success in a number of independent Bernoulli samples. So, X has the distribution:

$$X : \begin{pmatrix} 0 & 1 & 2 & \cdots & k & \cdots & n \\ p & pq & pq^2 & \cdots & pq^k & \cdots & pq^n \end{pmatrix}$$

and with the mean and respectively the variance:

$$\begin{cases} M(X) = \frac{q}{p} \\ Var(X) = \frac{q}{p^2}, \end{cases}$$

where p is the probability the probability of having a success, i.e the probability that a random event observable A to occur in a random experience and $q = 1 - p$ is the probability to achieve a failure, i.e the probability that the event contrary $\overline{A}$ to occur.

The distribution function of X is:

$$F(x) = P(X < x) = \sum_{k=0}^{x} pq^k = 1 - q^{x+1},\ x = 0, 1, 2, \cdots, n,$$

namely it is a discrete distribution function.

The name of geometric distribution comes from the fact that

$$P(X = x) = pq^x$$

is thew term of a geometric progression.

The simulation the random variable X, which has a geometric distribution can be also achieved by means of the inverse transform method, using the formula:

$$X = \left[\frac{\log(U)}{\log(q)} \right], \tag{3}$$

where:

- $[a]$ is the integer part of a,
- U is a random variable, uniformly distributed in $[0, 1]$.

3 Simulation of a random variable with a exponential and a Weibull distribution

A exponential variable $X \sim Exp(\lambda)$ has the probability density function:

$$f(x) = \begin{cases} \lambda e^{-\lambda x}, & x > 0 \\ 0, & x \leq 0 \end{cases}$$

$(\forall)\ \lambda \in \mathbb{R}$, the distribution function:

$$F_X(x) = \int_{-\infty}^{x} f(t)\,dt = 1 - e^{-\lambda x},\ x > 0$$

and

$$\begin{cases} M(X) = \frac{1}{\lambda} \\ Var(X) = \frac{1}{\lambda^2}. \end{cases}$$

To simulate a random variable X, which has an exponential distribution we shall use the inverse transform method, hence the algorithm for simulating the random variable X consists in:

- generating a value u uniformly distributed in $[0, 1]$,
- finding of

$$X = F^{-1}(u) = -\frac{1}{\lambda}\ln(1-u).$$

A Weibull variable (denoted $W(\alpha, \lambda, \gamma)$) is a random variable, closely related to the exponential random variable and which has the probability density function:

$$f(x) = \begin{cases} \gamma\lambda(x-\alpha)^{\gamma-1} e^{-\lambda(x-\alpha)\gamma}, & x > \alpha \\ 0, & x \leq \alpha \end{cases}$$

$(\forall)\ \alpha \in \mathbb{R},\ \gamma,\ \lambda > 0.$

If $X \sim Exp(1)$ then the Weibull variable is generated using the formula

$$W = \alpha + \left(\frac{X}{\lambda}\right)^{\frac{1}{\gamma}}. \tag{4}$$

Indeed, we have:

$$P(W < w) = P(X < \lambda(w-\alpha)^{\gamma}) = \int_{-\infty}^{\lambda(w-\alpha)^{\gamma}} e^{-t}dt$$

and further, using the change of variable

$$u = \alpha + \left(\frac{t}{\lambda}\right)^{\frac{1}{\gamma}}$$

it will result:

$$P\left(W < w\right) = \int_{-\infty}^{w} \gamma\lambda \left(u-\alpha\right)^{\gamma-1} e^{-\lambda(u-\alpha)\gamma} du.$$

The Weibull variable is used in reliability, it representing the service life without failures of a equipment or a industrial product.

4 Simulation of a random variable with a χ^2 distribution

Let Z_i, $1 \leq i \leq \gamma$ independent normal variables $N(0,1)$. A random variable χ^2 with γ degrees of freedom is a variable of the form

$$X_{\chi^2} = \sum_{i=1}^{\gamma} Z_i^2, \ \gamma \in \mathbb{N}^*. \tag{5}$$

A random variable χ^2 is continuous and admits the probability density function:

$$f\left(x\right) = \frac{1}{2^{\frac{\gamma}{2}} \Gamma\left(\frac{\gamma}{2}\right)} \cdot x^{\frac{\gamma}{2}-1} \cdot e^{-\frac{x}{2}}, \ x > 0,$$

where

$$\Gamma\left(\gamma\right) = \int_{0}^{\infty} x^{\gamma-1} e^{-x} dx \tag{6}$$

signifies the Euler Gamma function, $\Gamma : (0,1) \to \mathbb{R}$, which has the properties:

$$\begin{cases} \Gamma\left(\frac{1}{2}\right) = \sqrt{\pi} \\ \Gamma\left(1\right) = 1 \\ \Gamma\left(a+1\right) = a\Gamma\left(a\right), \ (\forall) \ a > 0 \\ \Gamma\left(n+1\right) = n!, \quad (\forall) \ n \in \mathbb{N} \end{cases}$$

and

$$\begin{cases} M\left(\chi^2\right) = \gamma \\ Var\left(\chi^2\right) = 2\gamma. \end{cases}$$

For the simulation in Matlab of a random variable χ^2 we shall use the formula (5):

```
function x=hip(n)
z=randn(n,1);
x=sum(z.^2);
end
```

5 Simulation of a random variable with a Gamma distribution

A random variable X has has the distribution $G(\alpha, \lambda, \gamma)$ if it has the probability density function:

$$f(x) = \begin{cases} \frac{\lambda^\gamma}{\Gamma(\gamma)} (x-\alpha)^{\gamma-1} e^{-\lambda(x-\alpha)} & x > \alpha \\ 0, & x \leq \alpha \end{cases}$$

where $(\forall)\ \alpha \in \mathbb{R},\ \gamma,\ \lambda > 0$ are respectively the parameters of location, scale and form of the variable.

We can notice that an exponential variable is a gamma variable $G(0, \lambda, 1)$ and χ^2 is a gamma variable $G(0, \frac{1}{2}, \frac{\gamma}{2})$.

If $Y \sim G(\alpha, \lambda, \frac{\gamma}{2})$ and $Z \sim G(0, \frac{1}{2}, \frac{\gamma}{2})$ then we have:

$$Y = \alpha + \frac{Z}{2\lambda}. \tag{7}$$

The relation (7) can be justified as follows:

$$\begin{aligned} F_Z(z) &= P(Z < z) = P(2\lambda(Y-\alpha) < z) = P\left(Y < \alpha + \frac{z}{2\lambda}\right) \\ &= F_Y\left(\alpha + \frac{z}{2\lambda}\right) = \int_{-\infty}^{\alpha + \frac{z}{2\lambda}} \frac{\lambda^{\frac{\gamma}{2}}}{\Gamma\left(\frac{\gamma}{2}\right)} (t-\alpha)^{\frac{\gamma}{2}-1} e^{-\lambda(t-\alpha)} dt \end{aligned}$$

and further, using the change of variable

$$w = 2\lambda(t-\alpha)$$

we shall achieve:

$$F_Z(z) = \int_{-\infty}^{z} \frac{\lambda^{\frac{\gamma}{2}}}{\Gamma\left(\frac{\gamma}{2}\right)} \left(\frac{w}{2\lambda}\right)^{\frac{\gamma}{2}-1} e^{-\lambda \cdot \frac{w}{2\lambda}} \frac{dw}{2\lambda} = \int_{-\infty}^{z} \frac{\lambda^{\frac{\gamma}{2}}}{\Gamma\left(\frac{\gamma}{2}\right)} \cdot w^{\frac{\gamma}{2}-1} e^{-\frac{w}{2}} dw.$$

For the simulation in Matlab of a random variable Y , whose distribution is $G\left(\alpha, \lambda, \frac{\gamma}{2}\right)$ we proceed as follows:

- one generates $Z = \chi^2$;
- one determines Y using (7).

Hence, we have:

```
function y =gam(al,la,n)
z=hip(n);
y=al+z/(2*la);
end
```

6 Validation of the Generators

The validation of the generators one refers both to the formal correctness of the programs and to the checking of the statistical hypothesis of concordance

$$H: \ X \sim F(x) \tag{8}$$

with regard to distribution function $F(x)$ of the random variable X, over which the simulated selection $X_1, X_2, \cdots, X_n$, of volume n big enough has been made.

The validation of the generators involves the following two steps:

A) Building the graphical histogram and comparing it with the probability density of X.
B) Application of the concordance test χ^2 to verify the hypothesis (8).

The histogram construction is done using the following algorithm:

Step 1. We simulate a number $n1 << n$ of selection values $X_1, X_2, \cdots, X_{n_1}$ and we store them.

Step 2. We choose a number k, which means the number of the histogram intervals: $I_1, I_2, \cdots, I_k$.

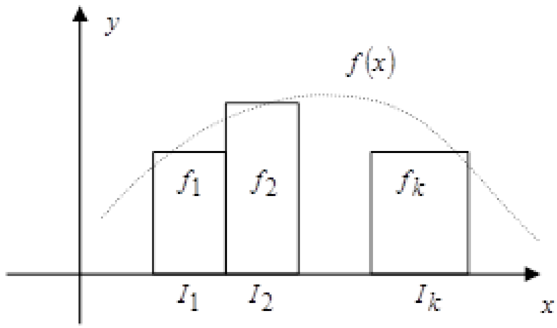

Fig. 1. Histogram

The dashed line suggests the probability density form of the variable X.

Step 3. We determine on the basis of the selection, the following limits of the histogram intervals:

$$\begin{cases} a_2 = \min\{X_1, X_2, \cdots, X_{n_1}\} \\ a_k = \max\{X_1, X_2, \cdots, X_{n_1}\}. \end{cases}$$

Then we form the intervals $I_i = (a_i, a_{i+1}]$, $(\forall)\ i = \overline{2, k-1}$, where

$$a_i = a_2 + (i-2)h, \ h = \frac{a_k - a_2}{k-2}, (\forall)\ i = \overline{3, k-1}.$$

Step 4. We compute the relative frequencies $f_i = \frac{n_i}{n}$, $(\forall)\ i = \overline{2, k-1}$, where n_i represents the absolute frequencies, namely the number of selection values that belong to the interval I_i. One makes the initializations:

$$\begin{cases} f_1 = f_k = 0 \\ a_1 = a_2 \\ a_{k+1} = a_k. \end{cases}$$

Step 5. We simulate every one of the other $n - n_1$ selection values and for each X such simulated we shall achieve the following operations:

a) if $X \leq a2$ then then we set: $a_1 = \min\{a_1, X\}$ and $f_1 = f_1 + 1$;
b) if $X > a_k$ then then we set: $a_{k+1} = \max\{a_{k+1}, X\}$ and $f_k = f_k + 1$;
c) if $a2 < X \leq a_k$ then we set: $p = \left[\frac{X-a_2}{h}\right] + 2$ and $f_{p+1} = f_{p+1} + 1$.

Step 6. We represent graphically the selection histogram $X_1, X_2, \cdots, X_n$, as follows: we take on the abscissa the intervals I_i, then we build some rectangles having these intervals as their bases and the relative frequencies f_i as their heights.

Remark 1 For a discrete random variable X, which takes the values $a_1, a_2, \cdots, a_m$ with the probabilities $p_1, p_2, \cdots, p_m$, the probability density function $f(x)$ is defined by:

$$f(x) = \begin{cases} p_i, \text{ if } x = x_i, \ i = \overline{1, m} \\ 0, \text{ otherwise} \end{cases} \tag{9}$$

and the and distribution function is given in (1).

With the built histogram, we can apply the test χ^2 to verify the hypothesis (8). Therefore, we have to compute the statistics

$$\chi^2 = \sum_{i=1}^{k} \frac{(n_i - np_i)^2}{np_i},$$

which has a distribution χ^2, with $k-1$ degrees of freedom (see Karl Pearson's theorem), where:

- k is the number of intervals in the histogram,
- n_i $(\forall)$ $i = \overline{1, k}$ represent the absolute frequencies,
- p_i $(\forall)$ $i = \overline{1, k}$ are the probabilities that an observation to belong to the interval I_i and they are expressed by:

$$\begin{cases} p_1 = P(a_1 < X \leq a_2) = F(a_2), \\ p_i = P(a_i < X \leq a_{i+1}) = F(a_{i+1}) - F(a_i), (\forall) \ i = \overline{2, k-1}, \\ p_k = P(a_k < X \leq a_{k+1}) = 1 - F(a_k). \end{cases} \tag{10}$$

Hypothesis H is accepted if

$$\chi^2 \leq \chi^2_{k-s-1,\ \alpha}$$

and is reject otherwise, α being the probability of type I error(it is also called level of significance or risk or probability of transgression) and s meaning the number of estimated parameters.

The next figures show the graphic representation of the histogram and respectively of the probability density function, in the case of validation of the algorithm for the simulation of a random variable, which has a normal distribution (see Fig. 2) or an exponential distribution (see Fig. 3) and respectively a geometric distribution (see Fig. 4).

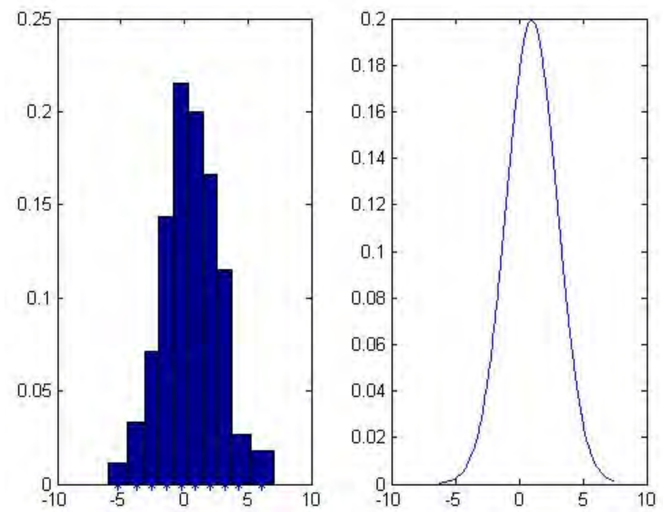

Fig. 2. A normal distribution

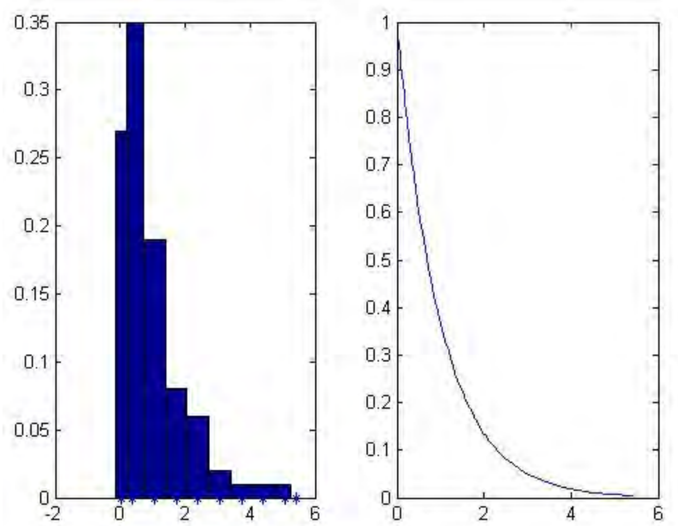

Fig. 3. An exponential distribution

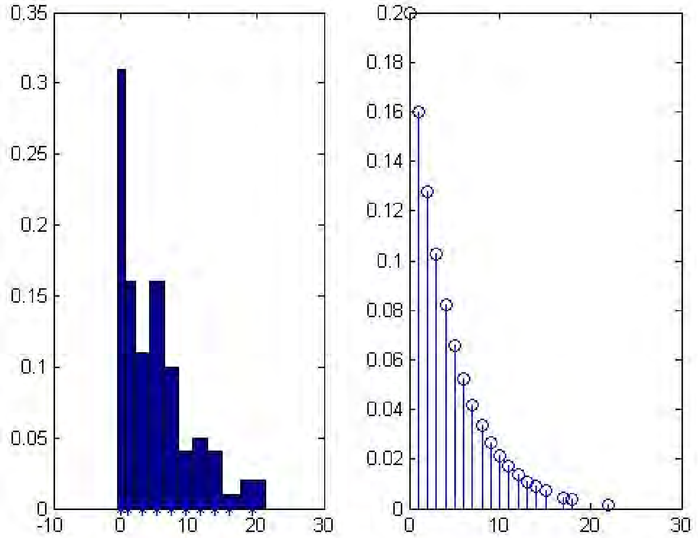

Fig. 4. A geometric distribution

7 Conclusion

We have built some algorithms for generating both continuous and discrete random variables.

We performed the the implementation of the Inverse Transform Method, according to which we can simulate any random variable X if we know its distribution function F and we can calculate the inverse function F^{-1}.

One simulates a discrete random variable having a geometric distribution, which is used in reliability. We also create some algorithms for generating: a normal continuous variable, other continuous variables having exponential distribution, Weibull distribution, gamma distribution.

Our goal is to see that if we have random numbers generated according to some distribution, we may perform a transformation to generate the desired distribution.

A cknowledgements

This work was supported by the strategic grant POSDRU/89/1.5/S/58852, Project "Postdoctoral programme for training scientific researchers" cofinanced by the European Social Fund within the Sectorial Operational Program Human Resources Development 2007-2013.

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J. COMPUTATIONAL ANALYSIS AND APPLICATIONS, VOL. 15, NO.7, 1223-1230, 2013, COPYRIGHT 2013 EUDOXUS PRESS, LLC

Second order Mond-Weir type duality for multiobjective programming involving Second order (C, α, ρ, d)-convexity *

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Abstract

In this paper, we introduce a class of second order (C, α, ρ, d)-convexity. Under the (C, α, ρ, d)-convexity assumptions on the functions involved, weak, strong and strict converse duality theorems are established for a second order Mond-Weir type multiobjective dual. Our results generalize these existing dual results which were discussed by Ahmad et al [Second-order (F, α, ρ, d)-convexity and duality in multiobjective programming, Information Science, 176(2006)3094-3103].

Keywords. Multiobjective programming; Second order duality; Efficient; (C, α, ρ, d)-convexity

MR(2000)Subject Classification: 49N15,90C30

1. Introduction

It is well known that the convex functions are very important in optimization theory. But for many mathematical models in desision sciences, economics, management sciences, stochastics, applied mathematics and engineering, the notion of convexity does no longer suffice. So it is necessary to generalize the notion of convexity and to extend the corresponding results to larger classes of optimization problems. In the last decades, various generalization of convex functions have been introduced in the literature. Preda [16] introduced the concept of (F, ρ)-convexity, which is an extension of F-convexity defined by Hanson and Mond [8] and ρ-convexity given by Vial [17]. Gulati and Islam [7] and Ahmad [2] established optimality conditions and duality results for multiobjective programming involving F-convexity and (F, ρ)-convexity assumptions, respectively.

Mangasarian [13] introduced the notation of second-order duality for nonlinear programs. He has indicated a possible computational advantage of the second-order dual over the first order dual. Mond[14] reproved second order duality theorems under simpler assumptions than those previously

*Project supported by the National Social Science Foundation of China (12BJY055), the Social Science Foundation of Hunan province (No.09ZDB080) in China, Humanity and Social Science foundation of Ministry of Education of China(12YJAZH142) and the Construct Program of the Key Discipline in Hunan Province.

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given by [13]. Yang et al. [18] proposed several second order duals for nonlinear programming problem and discussed duality results under generalized F-convexity.

In [20], Zhang and Mond extended the class of (F,ρ)-convex functions to second order (F,ρ)-convex functions and obtained duality results for three types of multiobjective dual problems. Aghezzaf [1] formulated a mixed type dual for multiobjective programming problem and discussed various duality results by defining new classes of generalized second order (F,ρ)-convexity. Liang et al. [10, 11] introduced (F,α,ρ,d)-convexity and obtained some optimality conditions and duality results for the single objective fractional problems and multiobjective problems. Ahmad and Husian [5] introduced a class of second order (F,α,ρ,d)-convex functions, and established some duality theorems for a second order Mond-Weir type multiobjective dual by using the assumptions on the functions involved (F,α,ρ,d)-convexity. Recently, Yuan et al.[19] introduced a class of functions, which called (C,α,ρ,d)-convex functions. They obtained sufficient optimality conditions for nondifferentiable minimax fractional problems. Chinchuluun et ai. [6] studied nonsmooth multiobjective fractional programming problems in the framework of (C,α,ρ,d)-convexity. Long [12] derive some sufficient optimality conditions and duality results for weakly efficient solutions of nondifferentiable multiobjective fractional programming problems under the assumptions of (C,α,ρ,d)-convexity.

In this paper, we introduce a class of second order (C,α,ρ,d)-convexity. Under the (C,α,ρ,d)-convexity assumptions on the functions involved, weak, strong and strict converse duality theorems are established for a second order Mond-Weir type multiobjective dual. Our results generalize these existing dual results which were discussed by Ahmad et al. in [5].

2. Preliminaries

Throughout the paper, the following convention for vectors in R^n will be necessary: $x \leqq y$ if and only if $x_i \leqq y_i,\ i=1,2,\cdots,n$, $x \leq y$ if and only if $x \leqq y$ and $x \neq y$, $x > y$ if and only if $x_i > y_i,\ i=1,2,\cdots,n$.

In this paper, we consider the following multiobjective programming problem:

$$\begin{array}{lll}(P) & Minimize & f(x)\\ & \text{s.t.} & g(x) \leqq 0,\quad x \in X,\end{array}$$

where $f=(f_1,f_2,\cdots,f_k): X \to R^k$, $g=(g_1,g_2,\cdots,g_m): X \to R^m$ are assumed to be twice differentiable functions over X, an open subset of R^n.

Definition 2.1 *A feasible point $\overline{x}$ is said to be an efficient solution of the vector minimum problem (P) if there exists no other feasible point x such that $f(x) \leq f(\overline{x})$.*

Assume that $\alpha : X \times X \to R_+ \setminus \{0\}, \rho \in R$ and $d : X \times X \to R_+$ satisfies $d(x,x_0)=0 \Leftrightarrow x=x_0$. Let $C : X \times X \times R^n \to R$ be a function which satisfies $C_{(x,x_0)}(0)=0$ for any $(x,x_0) \in X \times X$.

Definition 2.2 [19]A function $C : X \times X \times R^n \to R$ is said to be convex on R^n iff for any fixed $(x,x_0) \in X \times X$ and for any $y_1,y_2 \in R^n$, one has

$$C_{(x,x_0)}(\lambda y_1+(1-\lambda)y_2) \leq \lambda C_{(x,x_0)}(y_1)+(1-\lambda)C_{(x,x_0)}(y_2),\quad \forall \lambda \in (0,1).$$

Definition 2.3 [19]A differentiable function $h : X \to R$ is said to be (C, α, ρ, d)-convex at x_0 iff for any $x \in X$

$$\frac{h(x) - h(x_0)}{\alpha(x, x_0)} \geq C_{(x,x_0)}(\nabla h(x_0)) + \rho \frac{d(x, x_0)}{\alpha(x, x_0)}.$$

The function h is said to be (C, α, ρ, d)-convex on X iff h is (C, α, ρ, d)-convex at every point in X.

In the sequel, we introduce a class of second order (C, α, ρ, d)-convexity.

Definition 2.4 *A twice differentiable function f_i over X is said to be (strict) second order (C, α, ρ, d)-convex at x_0 if for all $x \in X$ and for all $p \in R^n$,*

$$\frac{f_i(x) - f_i(x_0) + \frac{1}{2}p^T \nabla^2 f_i(x_0)p}{\alpha(x, x_0)} (>) \geq C_{(x,x_0)}(\nabla f_i(x_0) + \nabla^2 f_i(x_0)p) + \rho \frac{d(x, x_0)}{\alpha(x, x_0)}.$$

A twice differentiable vector function $f : X \to R^k$ is said to be second order (C, α, ρ, d)-convex at x_0 if each of its components f_i is second order (C, α, ρ, d)-convex at x_0.

Remark 2.1 *From the above definition, second order (F, α, ρ, d)-convexity[5] is a special case of (C, α, ρ, d)-convexity, since any linear function is also a convex function.*

The following convention will be followed. If f is an k-dimensional vector function, then $f(u) - \nabla f(u)r - \frac{1}{2}p^T \nabla^2 f(u)p$ denotes the vector of components $f_1(u) - \nabla f_1(u)r - \frac{1}{2}p^T \nabla^2 f_1(u)p, \cdots, f_k(u) - \nabla f_k(u)r - \frac{1}{2}p^T \nabla^2 f_k(u)p$.

In order to prove the strong duality theorem, we need the following Kuhn-Tucker type necessary conditions [9].

Theorem 2.1 *(Kuhn-Tucker type necessary conditions)Assume that x^* is an efficient solution for (P) at which Kuhn-Tucker constraint qualification is satisfied. Then there exist $\lambda^* \in R^k$ and $y^* \in R^m$ such that*

$$\lambda^{*T} \nabla f(x^*) + y^{*T} \nabla g(x^*) = 0,$$

$$y^{*T} g(x^*) = 0,$$

$$y^* \geqq 0, \ \lambda^* \geq 0.$$

3. Second order Mond-Weir type duality

In this section, we consider the following Mond-Weir type second order dual associated with multiobjective problem (P) and establish weak, strong and strict converse duality theorems under second order (C, α, ρ, d)-convexity.

$$
\begin{array}{lll}
(MD) & Maximize & f(u) - \nabla f(u)^T r - \frac{1}{2} p^T \nabla^2 f(u) p, \\
\text{s.t.} & & \sum\limits_{i=1}^{k} \lambda_i \nabla f_i(u) + \sum\limits_{i=1}^{k} \lambda_i \nabla^2 f_i(u) p + \sum\limits_{i=1}^{m} y_i \nabla g_i(u) + \sum\limits_{i=1}^{m} y_i \nabla^2 g_i(u) p = 0, \\
& & \sum\limits_{i=1}^{m} y_i g_i(u) - \sum\limits_{i=1}^{m} y_i \nabla g_i(u)^T r - \sum\limits_{i=1}^{m} y_i \frac{1}{2} p^T \nabla^2 g_i(u) p \geq 0, \\
& & \sum\limits_{i=1}^{k} \lambda_i \nabla f_i(u)^T r \geq 0, \\
& & \sum\limits_{i=1}^{m} y_i \nabla g_i(u)^T r \geq 0, \\
& & y \geqq 0, \ \ \lambda \geqq 0, \\
& & r \in R^n, \quad y \in R^m, \quad \lambda \in R^k.
\end{array}
$$

Remark 3.1 *If $r = 0$, then (MD) becomes the dual considered in* [5].

Theorem 3.1 *(Weak duality)Suppose that for all feasible x in (P) and all feasible (u, y, λ, r, p) in (MD). If $g_i(\cdot)(i = 1, 2, \cdots, m)$ is second order $(C, \alpha_1, \rho_1, d_1)$-convex and $f_i(\cdot)(i = 1, 2, \cdots, k)$ is second order $(C, \alpha_2, \rho_2, d_2)$-convex, and*

$$
\rho_1 \tfrac{d_1}{\alpha_1} \sum_{i=1}^{m} y_i + \rho_2 \tfrac{d_2}{\alpha_2} \sum_{i=1}^{k} \lambda_i \geq 0, \tag{3.1}
$$

then the following cannot hold:

$$
f(x) \leq f(u) - \nabla f(u)^T r - \frac{1}{2} p^T \nabla^2 f(u) p.
$$

Proof. Suppose the conclusion is not true, i.e.,

$$
f(x) \leq f(u) - \nabla f(u)^T r - \frac{1}{2} p^T \nabla^2 f(u) p.
$$

In view of $(C, \alpha_2, \rho_2, d_2)$-convexity of $f_i(\cdot)$ at u, we obtain

$$
\begin{array}{rcl}
-\sum\limits_{i=1}^{k} \frac{\lambda_i}{\alpha_2} \nabla f_i(u)^T r & > & \sum\limits_{i=1}^{k} \lambda_i \frac{f_i(x) - f_i(u) + \frac{1}{2} p^T \nabla^2 f_i(u) p}{\alpha_2} \\
& \geq & \sum\limits_{i=1}^{k} \lambda_i C_{(x,u)}(\nabla f_i(u) + \nabla^2 f_i(u) p) + \rho_2 \frac{d_2}{\alpha_2} \sum\limits_{i=1}^{k} \lambda_i.
\end{array} \tag{3.2}
$$

Let x be any feasible solution in (P) and (u, y, λ, r, p) be any feasible solution in (MD). Then we have

$$
\sum_{i=1}^{m} y_i g_i(x) \leq 0 \leq \sum_{i=1}^{m} y_i g_i(u) - \frac{1}{2} \sum_{i=1}^{m} y_i p^T \nabla^2 g_i(u) p - \sum_{i=1}^{m} y_i \nabla g_i(u)^T r.
$$

Using second order $(C, \alpha_1, \rho_1, d_1)$-convexity of $g_i(\cdot)$ at u and the above inequality, we get

$$
\begin{array}{rcl}
-\sum\limits_{i=1}^{m} \frac{y_i}{\alpha_1} \nabla g_i(u)^T r & \geq & \sum\limits_{i=1}^{m} y_i \frac{g_i(x) - g_i(u) + \frac{1}{2} p^T \nabla^2 g_i(u) p}{\alpha_1} \\
& \geq & \sum\limits_{i=1}^{m} y_i C_{(x,u)}(\nabla g_i(u) + \nabla^2 g_i(u) p) + \rho_1 \frac{d_1}{\alpha_1} \sum\limits_{i=1}^{m} y_i.
\end{array} \tag{3.3}
$$

Taking into account convexity of $C_{(x,u)}(\cdot)$, (3.2) and (3.3), one gets

$$\begin{aligned}
-\sum_{i=1}^{k}\tfrac{\lambda_i}{\alpha_2}\nabla f_i(u)^T r &- \sum_{i=1}^{m}\tfrac{y_i}{\alpha_1}\nabla g_i(u)^T r\\
&> (\sum_{i=1}^{k}\lambda_i+\sum_{i=1}^{m}y_i)C_{(x,u)}\{\frac{\sum_{i=1}^{k}\lambda_i}{\sum_{i=1}^{k}\lambda_i+\sum_{i=1}^{m}y_i}(\nabla f_i(u)+\nabla^2 f_i(u)p)\\
&+ \frac{\sum_{i=1}^{m}y_i}{\sum_{i=1}^{k}\lambda_i+\sum_{i=1}^{m}y_i}(\nabla g_i(u)+\nabla^2 g_i(u)p)\}\\
&+ \rho_2\tfrac{d_2}{\alpha_2}\sum_{i=1}^{k}\lambda_i+\rho_1\tfrac{d_1}{\alpha_1}\sum_{i=1}^{m}y_i.
\end{aligned} \tag{3.4}$$

From the first, third, fourth dual constraint in (MD) and $C_{(x,u)}(0)=0$, we obtain

$$0>\rho_2\frac{d_2}{\alpha_2}\sum_{i=1}^{k}\lambda_i+\rho_1\frac{d_1}{\alpha_1}\sum_{i=1}^{m}y_i,$$

which contradicts the condition (3.1). Hence the following cannot hold:

$$f(x)\leq f(u)-\nabla f(u)^T r-\frac{1}{2}p^T\nabla^2 f(u)p.$$

Theorem 3.2 *(Strong duality) Let $\overline{x}$ be an efficient solution of (P) at which the Kuhn-Tucker constraint qualification is satisfied. Then there exist $\overline{y}\in R^m$ and $\overline{\lambda}\in R^k$, such that $(\overline{x},\overline{y},\overline{\lambda},\overline{r}=0,\overline{p}=0)$ is feasible for (MD) and the objective values of (P) and (D) are equal. Furthermore, if the assumptions of Weak duality hold for all feasible solutions of (P) and (MD), then $(\overline{x},\overline{y},\overline{\lambda},\overline{r}=0,\overline{p}=0)$ is an efficient solution of (MD).*

Proof. Since $\overline{x}$ is an efficient solution of (P) at which the Kuhn-Tucker constraint qualification is satisfied, then by Theorem 2.1, there exist $\overline{y}\in R^m$ and $\overline{\lambda}\in R^k$ such that

$$\begin{gathered}
\overline{\lambda}^T\nabla f(\overline{x})+\overline{y}^T\nabla g(\overline{x})=0,\\
\overline{y}^T g(\overline{x})=0,\\
\overline{y}\geqq 0,\ \overline{\lambda}\geq 0.
\end{gathered}$$

Therefore $(\overline{x},\overline{y},\overline{\lambda},\overline{r}=0,\overline{p}=0)$ is feasible for (MD) and the objective values of (P) and (MD) are equal. The efficiency of this feasible solution for (MD) follows from the weak duality theorem.

Theorem 3.3 *(Strict Converse duality) Let $\overline{x}$ and $(\overline{u},\overline{y},\overline{\lambda},\overline{r},\overline{p})$ be the efficient solution of (P) and (MD), respectively, such that*

$$f(\overline{x})=f(\overline{u})-\nabla f(\overline{x})^T r-\tfrac{1}{2}\overline{p}^T\nabla^2\overline{f}(\overline{u})\overline{p}. \tag{3.5}$$

If $g_i(\cdot)(i=1,2,\cdots,m)$ is strict second order (C,α_1,ρ_1,d_1)-convex and $f_i(\cdot)(i=1,2,\cdots,k)$ is second order (C,α_2,ρ_2,d_2)-convex, and

$$\rho_1\tfrac{d_1}{\alpha_1}\sum_{i=1}^{m}\overline{y}_i+\rho_2\tfrac{d_2}{\alpha_2}\sum_{i=1}^{k}\overline{\lambda}_i\geq 0. \tag{3.6}$$

Then $\overline{x}=\overline{u}$; that is, $\overline{u}$ is an efficient solution of (P).

Proof. Suppose the conclusion is not true, i.e.,$\overline{x} \neq \overline{u}$. In view of $(C, \alpha_2, \rho_2, d_2)$-convexity of $f_i(\cdot)$ at $\overline{u}$ and (3.5), we obtain

$$\begin{aligned} -\sum_{i=1}^{k} \frac{\overline{\lambda}_i}{\alpha_2} \nabla f_i(\overline{u})^T \overline{r} &= \sum_{i=1}^{k} \overline{\lambda}_i \frac{f_i(\overline{x}) - f_i(\overline{u}) + \frac{1}{2}\overline{p}^T \nabla^2 f_i(\overline{u})\overline{p}}{\alpha_2} \\ &\geq \sum_{i=1}^{k} \overline{\lambda}_i C_{(\overline{x},\overline{u})}(\nabla f_i(\overline{u}) + \nabla^2 f_i(\overline{u})\overline{p}) + \rho_2 \frac{d_2}{\alpha_2} \sum_{i=1}^{k} \overline{\lambda}_i. \end{aligned} \tag{3.7}$$

Let $\overline{x}$ be any feasible solution in (P) and $(\overline{u}, \overline{y}, \overline{\lambda}, \overline{r}, \overline{p})$ be any feasible solution in (MD). Then we have

$$\sum_{i=1}^{m} \overline{y}_i g_i(\overline{x}) \leq 0 \leq \sum_{i=1}^{m} \overline{y}_i g_i(\overline{u}) - \frac{1}{2} \sum_{i=1}^{m} \overline{y}_i p^T \nabla^2 g_i(\overline{u}) \overline{p} - \sum_{i=1}^{m} \overline{y}_i \nabla g_i(\overline{u})^T \overline{r}.$$

Using strict second order $(C, \alpha_1, \rho_1, d_1)$-convexity of $g_i(\cdot)$ at $\overline{u}$ and the above inequality, we get

$$\begin{aligned} -\sum_{i=1}^{m} \frac{\overline{y}_i}{\alpha_1} \nabla g_i(\overline{u})^T \overline{r} &\geq \sum_{i=1}^{m} \overline{y}_i \frac{g_i(\overline{x}) - g_i(\overline{u}) + \frac{1}{2}\overline{p}^T \nabla^2 g_i(\overline{u})\overline{p}}{\alpha_1} \\ &> \sum_{i=1}^{m} \overline{y}_i C_{(\overline{x},\overline{u})}(\nabla g_i(\overline{u}) + \nabla^2 g_i(\overline{u})\overline{p}) + \rho_1 \frac{d_1}{\alpha_1} \sum_{i=1}^{m} \overline{y}_i. \end{aligned} \tag{3.8}$$

Taking into account convexity of $C_{(\overline{x},\overline{u})(\cdot)}$, (3.7) and (3.8), one gets

$$\begin{aligned} -\sum_{i=1}^{k} \frac{\overline{\lambda}_i}{\alpha_2} \nabla f_i(\overline{u})^T \overline{r} &- \sum_{i=1}^{m} \frac{\overline{y}_i}{\alpha_1} \nabla g_i(\overline{u})^T \overline{r} \\ &> (\sum_{i=1}^{k} \overline{\lambda}_i + \sum_{i=1}^{m} \overline{y}_i) C_{(\overline{x},\overline{u})} \{ \frac{\sum_{i=1}^{k} \overline{\lambda}_i}{\sum_{i=1}^{k} \overline{\lambda}_i + \sum_{i=1}^{m} \overline{y}_i} (\nabla f_i(\overline{u}) + \nabla^2 f_i(\overline{u})\overline{p}) \\ &+ \frac{\sum_{i=1}^{m} \overline{y}_i}{\sum_{i=1}^{k} \overline{\lambda}_i + \sum_{i=1}^{m} \overline{y}_i} (\nabla g_i(\overline{u}) + \nabla^2 g_i(\overline{u})\overline{p}) \} \\ &+ \rho_2 \frac{d_2}{\alpha_2} \sum_{i=1}^{k} \overline{\lambda}_i + \rho_1 \frac{d_1}{\alpha_1} \sum_{i=1}^{m} \overline{y}_i. \end{aligned} \tag{3.9}$$

From the first, third, fourth dual constraint in (MD) and $C_{(\overline{x},\overline{u})}(0) = 0$, we obtain

$$0 > \rho_2 \frac{d_2}{\alpha_2} \sum_{i=1}^{k} \overline{\lambda}_i + \rho_1 \frac{d_1}{\alpha_1} \sum_{i=1}^{m} \overline{y}_i,$$

which contradicts the condition (3.6). Hence $\overline{x} = \overline{u}$.

4. Conclusions

In this paper, we introduce a class of second order (C, α, ρ, d)-convexity, which includes many other generalized convexity concepts in mathematical programming as special cases. Using the (C, α, ρ, d)-convexity assumptions on the functions involved, weak, strong and strict converse duality theorems are established for a second order Mond-Weir type multiobjective dual. Our results generalize these existing dual results which were discussed by Ahmad et al. in [5], These results can be further generalized to a class of nondifferentiable multiobjective programming.

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J. COMPUTATIONAL ANALYSIS AND APPLICATIONS, VOL. 15, NO.7, 1231-1239, 2013, COPYRIGHT 2013 EUDOXUS PRESS, LLC

Fractional Voronovskaya type asymptotic expansions for bell and squashing type neural network operators

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Abstract

Here we introduce the normalized bell and squashing type neural network operators of one hidden layer. Based on fractional calculus theory we derive fractional Voronovskaya type asymptotic expansions for the error of approximation of these operators to the unit operator.

2010 AMS Mathematics Subject Classification: 26A33, 41A25, 41A36, 41A60.

Keywords and Phrases: Neural Network Fractional Approximation, Voronovskaya Asymptotic Expansion, fractional derivative.

1 Background

We need

Definition 1 *Let $f : \mathbb{R} \to \mathbb{R}$, $\nu > 0$, $n = \lceil \nu \rceil$ ($\lceil \cdot \rceil$ is the ceiling of the number), such that $f \in AC^n([a,b])$ (space of functions f with $f^{(n-1)} \in AC([a,b])$, absolutely continuous functions), $\forall$ $[a,b] \subset \mathbb{R}$. We call left Caputo fractional derivative (see [8], pp. 49-52) the function*

$$D_{*a}^{\nu} f(x) = \frac{1}{\Gamma(n-\nu)} \int_a^x (x-t)^{n-\nu-1} f^{(n)}(t)\,dt, \tag{1}$$

$\forall$ $x \geq a$, where Γ is the gamma function $\Gamma(\nu) = \int_0^{\infty} e^{-t} t^{\nu-1} dt$, $\nu > 0$. Notice $D_{*a}^{\nu} f \in L_1([a,b])$ and $D_{*a}^{\nu} f$ exists a.e.on $[a,b]$, $\forall$ $b > a$.

We set $D_{*a}^{0} f(x) = f(x)$, $\forall$ $x \in [a, +\infty)$.

We also need

Definition 2 *(see also [2], [9], [10]). Let $f : \mathbb{R} \to \mathbb{R}$, such that $f \in AC^m([a,b])$, $\forall\ [a,b] \subset \mathbb{R}$, $m = \lceil \alpha \rceil$, $\alpha > 0$. The right Caputo fractional derivative of order $\alpha > 0$ is given by*

$$D_{b-}^{\alpha} f(x) = \frac{(-1)^m}{\Gamma(m-\alpha)} \int_x^b (J-x)^{m-\alpha-1} f^{(m)}(J)\, dJ, \tag{2}$$

$\forall\ x \leq b$. We set $D_{b-}^0 f(x) = f(x)$, $\forall\ x \in (-\infty, b]$. Notice that $D_{b-}^{\alpha} f \in L_1([a,b])$ and $D_{b-}^{\alpha} f$ exists a.e.on $[a,b]$, $\forall\ a < b$.

We mention the left Caputo fractional Taylor formula with integral remainder.

Theorem 3 *([8], p. 54) Let $f \in AC^m([a,b])$, $\forall\ [a,b] \subset \mathbb{R}$, $m = \lceil \alpha \rceil$, $\alpha > 0$. Then*

$$f(x) = \sum_{k=0}^{m-1} \frac{f^{(k)}(x_0)}{k!} (x-x_0)^k + \frac{1}{\Gamma(\alpha)} \int_{x_0}^{x} (x-J)^{\alpha-1} D_{*x_0}^{\alpha} f(J)\, dJ, \tag{3}$$

$\forall\ x \geq x_0$.

Also we mention the right Caputo fractional Taylor formula.

Theorem 4 *([2]) Let $f \in AC^m([a,b])$, $\forall\ [a,b] \subset \mathbb{R}$, $m = \lceil \alpha \rceil$, $\alpha > 0$. Then*

$$f(x) = \sum_{k=0}^{m-1} \frac{f^{(k)}(x_0)}{k!} (x-x_0)^k + \frac{1}{\Gamma(\alpha)} \int_{x}^{x_0} (J-x)^{\alpha-1} D_{x_0-}^{\alpha} f(J)\, dJ, \tag{4}$$

$\forall\ x \leq x_0$.

Convention 5 *We assume that*

$$D_{*x_0}^{\alpha} f(x) = 0, \ \ \text{for } x < x_0,$$

and

$$D_{x_0-}^{\alpha} f(x) = 0, \text{ for } x > x_0,$$

for all $x, x_0 \in \mathbb{R}$.

We mention

Proposition 6 *(by [3]) i) Let $f \in C^n(\mathbb{R})$, where $n = \lceil \nu \rceil$, $\nu > 0$. Then $D_{*a}^{\nu} f(x)$ is continuous in $x \in [a, \infty)$.*

ii) Let $f \in C^m(\mathbb{R})$, $m = \lceil \alpha \rceil$, $\alpha > 0$. Then $D_{b-}^{\alpha} f(x)$ is continuous in $x \in (-\infty, b]$.

We also mention

Theorem 7 *([5]) Let $f \in C^m(\mathbb{R})$, $f^{(m)} \in L_\infty(\mathbb{R})$, $m = \lceil \alpha \rceil$, $\alpha > 0$, $\alpha \notin \mathbb{N}$, $x, x_0 \in \mathbb{R}$. Then $D^\alpha_{*x_0} f(x)$, $D^\alpha_{x_0-} f(x)$ are jointly continuous in (x, x_0) from $\mathbb{R}^2$ into $\mathbb{R}$.*

For more see [4], [6].

We need the following (see [7]).

Definition 8 *A function $b : \mathbb{R} \to \mathbb{R}$ is said to be bell-shaped if b belongs to L^1 and its integral is nonzero, if it is nondecreasing on $(-\infty, a)$ and nonincreasing on $[a, +\infty)$, where a belongs to $\mathbb{R}$. In particular $b(x)$ is a nonnegative number and at a b takes a global maximum; it is the center of the bell-shaped function. A bell-shaped function is said to be centered if its center is zero. The function $b(x)$ may have jump discontinuities. In this work we consider only centered bell-shaped functions of compact support $[-T, T]$, $T > 0$.*

Example 9 (1) *$b(x)$ can be the characteristic function over $[-1, 1]$.*

(2) *$b(x)$ can be the hat function over $[-1, 1]$, i.e.,*

$$b(x) = \begin{cases} 1 + x, & -1 \leq x \leq 0, \\ 1 - x, & 0 < x \leq 1 \\ 0, & \textit{elsewhere.} \end{cases}$$

Here we consider functions $f \in C(\mathbb{R})$.

We study the following "normalized bell type neural network operators" (see also related [1], [7])

$$(H_n(f))(x) := \frac{\sum_{k=-n^2}^{n^2} f\left(\frac{k}{n}\right) b\left(n^{1-\alpha}\left(x - \frac{k}{n}\right)\right)}{\sum_{k=-n^2}^{n^2} b\left(n^{1-\alpha}\left(x - \frac{k}{n}\right)\right)}, \tag{5}$$

where $0 < \alpha < 1$ and $x \in \mathbb{R}$, $n \in \mathbb{N}$.

We find a fractional Voronovskaya type asymptotic expansion for $H_n(f)(x)$.

The terms in $H_n(f)(x)$ are nonzero iff

$$\left| n^{1-\alpha}\left(x - \frac{k}{n}\right) \right| \leq T, \text{ i.e. } \left| x - \frac{k}{n} \right| \leq \frac{T}{n^{1-\alpha}}$$

iff

$$nx - Tn^\alpha \leq k \leq nx + Tn^\alpha. \tag{6}$$

In order to have the desired order of numbers

$$-n^2 \leq nx - Tn^\alpha \leq nx + Tn^\alpha \leq n^2, \tag{7}$$

it is sufficient enough to assume that

$$n \geq T + |x|. \tag{8}$$

When $x \in [-T, T]$ it is enough to assume $n \geq 2T$ which implies (7).

Proposition 10 *(see [1]) Let $a \leq b$, $a, b \in \mathbb{R}$. Let card(k) (≥ 0) be the maximum number of integers contained in $[a, b]$. Then*

$$\max(0, (b-a)-1) \leq card(k) \leq (b-a)+1. \tag{9}$$

Remark 11 *We would like to establish a lower bound on card(k) over the interval $[nx - Tn^{\alpha}, nx + Tn^{\alpha}]$. From Proposition 10 we get that*

$$card(k) \geq \max(2Tn^{\alpha} - 1, 0).$$

We obtain card$(k) \geq 1$, if

$$2Tn^{\alpha} - 1 \geq 1 \quad \textit{iff } n \geq T^{-\frac{1}{\alpha}}.$$

So to have the desired order (7) and card$(k) \geq 1$ over $[nx - Tn^{\alpha}, nx + Tn^{\alpha}]$, we need to consider

$$n \geq \max\left(T + |x|, T^{-\frac{1}{\alpha}}\right). \tag{10}$$

Also notice that card$(k) \to +\infty$, as $n \to +\infty$.

Denote by $[\cdot]$ the integral part of a number.

Remark 12 *Clearly we have that*

$$nx - Tn^{\alpha} \leq nx \leq nx + Tn^{\alpha}. \tag{11}$$

We prove in general that

$$nx - Tn^{\alpha} \leq [nx] \leq nx \leq \lceil nx \rceil \leq nx + Tn^{\alpha}. \tag{12}$$

Indeed we have that, if $[nx] < nx - Tn^{\alpha}$, then $[nx] + Tn^{\alpha} < nx$, and $[nx] + [Tn^{\alpha}] \leq [nx]$, resulting into $[Tn^{\alpha}] = 0$, which for large enough n is not true. Therefore $nx - Tn^{\alpha} \leq [nx]$. Similarly, if $\lceil nx \rceil > nx + Tn^{\alpha}$, then $nx + Tn^{\alpha} \geq nx + [Tn^{\alpha}]$, and $\lceil nx \rceil - [Tn^{\alpha}] > nx$, thus $\lceil nx \rceil - [Tn^{\alpha}] \geq \lceil nx \rceil$, resulting into $[Tn^{\alpha}] = 0$, which again for large enough n is not true.

Therefore without loss of generality we may assume that

$$nx - Tn^{\alpha} \leq [nx] \leq nx \leq \lceil nx \rceil \leq nx + Tn^{\alpha}. \tag{13}$$

Hence $\lceil nx - Tn^{\alpha} \rceil \leq [nx]$ and $\lceil nx \rceil \leq [nx + Tn^{\alpha}]$. Also if $[nx] \neq \lceil nx \rceil$, then $\lceil nx \rceil = [nx] + 1$. If $[nx] = \lceil nx \rceil$, then $nx \in \mathbb{Z}$; and by assuming $n \geq T^{-\frac{1}{\alpha}}$, we get $Tn^{\alpha} \geq 1$ and $nx + Tn^{\alpha} \geq nx + 1$, so that $[nx + Tn^{\alpha}] \geq nx + 1 = [nx] + 1$.

We need also

Definition 13 *Let the nonnegative function $S : \mathbb{R} \to \mathbb{R}$, S has compact support $[-T, T]$, $T > 0$, and is nondecreasing there and it can be continuous only on either $(-\infty, T]$ or $[-T, T]$, S can have jump discontinuites. We call S the "squashing function", see [1], [7].*

Let $f \in C(\mathbb{R})$. For $x \in \mathbb{R}$ we define the following "normalized squashing type neural network operators" (see also related [1])

$$(K_n(f))(x) := \frac{\sum_{k=-n^2}^{n^2} f\left(\frac{k}{n}\right) S\left(n^{1-\alpha}\left(x - \frac{k}{n}\right)\right)}{\sum_{k=-n^2}^{n^2} S\left(n^{1-\alpha}\left(x - \frac{k}{n}\right)\right)}, \tag{14}$$

$0 < \alpha < 1$ and $n \in \mathbb{N} : n \geq \max\left(T + |x|, T^{-\frac{1}{\alpha}}\right)$.

It is clear that

$$(K_n(f))(x) := \frac{\sum_{k=\lceil nx-Tn^\alpha\rceil}^{[nx+Tn^\alpha]} f\left(\frac{k}{n}\right) S\left(n^{1-\alpha}\left(x - \frac{k}{n}\right)\right)}{\sum_{k=\lceil nx-Tn^\alpha\rceil}^{[nx+Tn^\alpha]} S\left(n^{1-\alpha}\left(x - \frac{k}{n}\right)\right)}. \tag{15}$$

We find a fractional Voronovskaya type asymptotic expansion for $(K_n(f))(x)$.

2 Main Results

We present our first main result.

Theorem 14 *Let $\beta > 0$, $N \in \mathbb{N}$, $N = \lceil \beta \rceil$, $f \in AC^N([a,b])$, $\forall$ $[a.b] \subset \mathbb{R}$, with $\left\| D_{x_0-}^{\beta} f \right\|_{\infty}, \left\| D_{*x_0}^{\beta} f \right\|_{\infty} \leq M$, $M > 0$, $x_0 \in \mathbb{R}$. Let $T > 0$, $n \in \mathbb{N} : n \geq \max\left(T + |x_0|, T^{-\frac{1}{\alpha}}\right)$ Then*

$$(H_n(f))(x_0) - f(x_0) = \sum_{j=1}^{N-1} \frac{f^{(j)}(x_0)}{j!} H_n\left((\cdot - x_0)^j\right)(x_0) + o\left(\frac{1}{n^{(1-\alpha)(\beta-\varepsilon)}}\right), \tag{16}$$

where $0 < \varepsilon \leq \beta$.

If $N = 1$, the sum in (16) disappears.

The last (16) implies that

$$n^{(1-\alpha)(\beta-\varepsilon)} \left[(H_n(f))(x_0) - f(x_0) - \sum_{j=1}^{N-1} \frac{f^{(j)}(x_0)}{j!} H_n\left((\cdot - x_0)^j\right)(x_0) \right] \to 0, \tag{17}$$

as $n \to \infty$, $0 < \varepsilon \leq \beta$.

When $N = 1$, or $f^{(j)}(x_0) = 0$, $j = 1, ..., N-1$, then we derive

$$n^{(1-\alpha)(\beta-\varepsilon)} \left[(H_n(f))(x_0) - f(x_0) \right] \to 0$$

as $n \to \infty$, $0 < \varepsilon \leq \beta$. Of great interest is the case of $\beta = \frac{1}{2}$.

Proof. From [8], p. 54; (3), we get by the left Caputo fractional Taylor formula that

$$f\left(\frac{k}{n}\right) = \sum_{j=0}^{N-1} \frac{f^{(j)}(x_0)}{j!} \left(\frac{k}{n} - x_0\right)^j + \frac{1}{\Gamma(\beta)} \int_{x_0}^{\frac{k}{n}} \left(\frac{k}{n} - J\right)^{\beta-1} D_{*x_0}^{\beta} f(J)\, dJ, \tag{18}$$

for all $x_0 \leq \frac{k}{n} \leq x_0 + Tn^{\alpha-1}$, iff $\lceil nx_0 \rceil \leq k \leq [nx_0 + Tn^{\alpha}]$, where $k \in \mathbb{Z}$.

Also from [2]; (4), using the right Caputo fractional Taylor formula we get

$$f\left(\frac{k}{n}\right) = \sum_{j=0}^{N-1} \frac{f^{(j)}(x_0)}{j!}\left(\frac{k}{n} - x_0\right)^j + \frac{1}{\Gamma(\beta)} \int_{\frac{k}{n}}^{x_0} \left(J - \frac{k}{n}\right)^{\beta-1} D_{x_0-}^{\beta} f(J)\, dJ, \tag{19}$$

for all $x_0 - Tn^{\alpha-1} \leq \frac{k}{n} \leq x_0$, iff $\lceil nx_0 - Tn^{\alpha} \rceil \leq k \leq [nx_0]$, where $k \in \mathbb{Z}$. Notice that $\lceil nx_0 \rceil \leq [nx_0] + 1$.

Call

$$V(x_0) := \sum_{k=\lceil nx_0 - Tn^{\alpha} \rceil}^{[nx_0 + Tn^{\alpha}]} b\left(n^{1-\alpha}\left(x_0 - \frac{k}{n}\right)\right).$$

Hence we have

$$\frac{f\left(\frac{k}{n}\right) b\left(n^{1-\alpha}\left(x_0 - \frac{k}{n}\right)\right)}{V(x_0)} = \sum_{j=0}^{N-1} \frac{f^{(j)}(x_0)}{j!}\left(\frac{k}{n} - x_0\right)^j \frac{b\left(n^{1-\alpha}\left(x_0 - \frac{k}{n}\right)\right)}{V(x_0)} + \tag{20}$$
$$\frac{b\left(n^{1-\alpha}\left(x_0 - \frac{k}{n}\right)\right)}{V(x_0)\,\Gamma(\beta)} \int_{x_0}^{\frac{k}{n}} \left(\frac{k}{n} - J\right)^{\beta-1} D_{*x_0}^{\beta} f(J)\, dJ,$$

and

$$\frac{f\left(\frac{k}{n}\right) b\left(n^{1-\alpha}\left(x_0 - \frac{k}{n}\right)\right)}{V(x_0)} = \sum_{j=0}^{N-1} \frac{f^{(j)}(x_0)}{j!}\left(\frac{k}{n} - x_0\right)^j \frac{b\left(n^{1-\alpha}\left(x_0 - \frac{k}{n}\right)\right)}{V(x_0)} + \tag{21}$$
$$\frac{b\left(n^{1-\alpha}\left(x_0 - \frac{k}{n}\right)\right)}{V(x_0)\,\Gamma(\beta)} \int_{\frac{k}{n}}^{x_0} \left(J - \frac{k}{n}\right)^{\beta-1} D_{x_0-}^{\beta} f(J)\, dJ,$$

Therefore we obtain

$$\frac{\sum_{k=[nx_0]+1}^{[nx_0+Tn^{\alpha}]} f\left(\frac{k}{n}\right) b\left(n^{1-\alpha}\left(x_0 - \frac{k}{n}\right)\right)}{V(x_0)} =$$
$$\sum_{j=0}^{N-1} \frac{f^{(j)}(x_0)}{j!}\left(\frac{\sum_{k=[nx_0]+1}^{[nx_0+Tn^{\alpha}]} \left(\frac{k}{n} - x_0\right)^j b\left(n^{1-\alpha}\left(x_0 - \frac{k}{n}\right)\right)}{V(x_0)}\right) + \tag{22}$$
$$\sum_{k=[nx_0]+1}^{[nx_0+Tn^{\alpha}]} \frac{b\left(n^{1-\alpha}\left(x_0 - \frac{k}{n}\right)\right)}{V(x_0)\,\Gamma(\beta)} \int_{x_0}^{\frac{k}{n}} \left(\frac{k}{n} - J\right)^{\beta-1} D_{*x_0}^{\beta} f(J)\, dJ,$$

and

$$\frac{\sum_{k=\lceil nx_0 - Tn^{\alpha} \rceil}^{[nx_0]} f\left(\frac{k}{n}\right) b\left(n^{1-\alpha}\left(x_0 - \frac{k}{n}\right)\right)}{V(x_0)} =$$

$$\sum_{j=0}^{N-1} \frac{f^{(j)}(x_0)}{j!} \frac{\sum_{k=\lceil nx_0 - Tn^\alpha \rceil}^{[nx_0]} \left(\frac{k}{n} - x_0\right)^j b\left(n^{1-\alpha}\left(x_0 - \frac{k}{n}\right)\right)}{V(x_0)} + \tag{23}$$

$$\frac{\sum_{k=\lceil nx_0 - Tn^\alpha \rceil}^{[nx_0]} b\left(n^{1-\alpha}\left(x_0 - \frac{k}{n}\right)\right)}{V(x_0)\Gamma(\beta)} \int_{\frac{k}{n}}^{x_0} \left(J - \frac{k}{n}\right)^{\beta-1} D_{x_0-}^{\beta} f(J)\, dJ.$$

We notice here that

$$(H_n(f))(x) := \frac{\sum_{k=-n^2}^{n^2} f\left(\frac{k}{n}\right) b\left(n^{1-\alpha}\left(x - \frac{k}{n}\right)\right)}{\sum_{k=-n^2}^{n^2} b\left(n^{1-\alpha}\left(x - \frac{k}{n}\right)\right)} \tag{24}$$

$$= \frac{\sum_{k=\lceil nx - Tn^\alpha \rceil}^{[nx+Tn^\alpha]} f\left(\frac{k}{n}\right) b\left(n^{1-\alpha}\left(x - \frac{k}{n}\right)\right)}{\sum_{k=\lceil nx - Tn^\alpha \rceil}^{[nx+Tn^\alpha]} b\left(n^{1-\alpha}\left(x - \frac{k}{n}\right)\right)}, \quad \forall\, x \in \mathbb{R}.$$

Adding the two equalities (22), (23) and rewriting it, we obtain

$$T(x_0) := (H_n(f))(x_0) - f(x_0) - \sum_{j=1}^{N-1} \frac{f^{(j)}(x_0)}{j!} H_n\left((\cdot - x_0)^j\right)(x_0) = \theta_n^*(x_0), \tag{25}$$

where

$$\theta_n^*(x_0) := \frac{\sum_{k=\lceil nx_0 - Tn^\alpha \rceil}^{[nx_0]} b\left(n^{1-\alpha}\left(x_0 - \frac{k}{n}\right)\right)}{V(x_0)\Gamma(\beta)} \int_{\frac{k}{n}}^{x_0} \left(J - \frac{k}{n}\right)^{\beta-1} D_{x_0-}^{\beta} f(J)\, dJ$$

$$+ \sum_{k=[nx_0]+1}^{[nx_0+Tn^\alpha]} \frac{b\left(n^{1-\alpha}\left(x_0 - \frac{k}{n}\right)\right)}{V(x_0)\Gamma(\beta)} \int_{x_0}^{\frac{k}{n}} \left(\frac{k}{n} - J\right)^{\beta-1} D_{*x_0}^{\beta} f(J)\, dJ. \tag{26}$$

We observe that

$$|\theta_n^*(x_0)| \le \frac{1}{V(x_0)\Gamma(\beta)} \cdot$$

$$\left\{ \sum_{k=\lceil nx_0 - Tn^\alpha \rceil}^{[nx_0]} b\left(n^{1-\alpha}\left(x_0 - \frac{k}{n}\right)\right) \int_{\frac{k}{n}}^{x_0} \left(J - \frac{k}{n}\right)^{\beta-1} \left|D_{x_0-}^{\beta} f(J)\right| dJ \right. \tag{27}$$

$$\left. + \sum_{k=[nx_0]+1}^{[nx_0+Tn^\alpha]} b\left(n^{1-\alpha}\left(x_0 - \frac{k}{n}\right)\right) \int_{x_0}^{\frac{k}{n}} \left(\frac{k}{n} - J\right)^{\beta-1} \left|D_{*x_0}^{\beta} f(J)\right| dJ \right\} \le$$

$$\frac{M}{V(x_0)\Gamma(\beta)} \left\{ \sum_{k=\lceil nx_0 - Tn^\alpha \rceil}^{[nx_0]} b\left(n^{1-\alpha}\left(x_0 - \frac{k}{n}\right)\right) \frac{\left(x_0 - \frac{k}{n}\right)^\beta}{\beta} + \right.$$

$$\left. \sum_{k=[nx_0]+1}^{[nx_0+Tn^\alpha]} b\left(n^{1-\alpha}\left(x_0 - \frac{k}{n}\right)\right) \frac{\left(\frac{k}{n} - x_0\right)^\beta}{\beta} \right\} \le$$

$$\frac{M}{V\left(x_{0}\right)\Gamma\left(\beta+1\right)}\left\{\left(\sum_{k=\lceil nx_{0}-Tn^{\alpha}\rceil}^{[nx_{0}]}b\left(n^{1-\alpha}\left(x_{0}-\frac{k}{n}\right)\right)\right)\left(\frac{T}{n^{1-\alpha}}\right)^{\beta}+\right.$$

$$\left.\left(\sum_{k=[nx_{0}]+1}^{[nx_{0}+Tn^{\alpha}]}b\left(n^{1-\alpha}\left(x_{0}-\frac{k}{n}\right)\right)\right)\left(\frac{T}{n^{1-\alpha}}\right)^{\beta}\right\}=\frac{M}{\Gamma\left(\beta+1\right)}\frac{T^{\beta}}{n^{(1-\alpha)\beta}}. \tag{28}$$

So we have proved that

$$\left|T\left(x_{0}\right)\right|=\left|\theta_{n}^{*}\left(x_{0}\right)\right|\leq\left(\frac{MT^{\beta}}{\Gamma\left(\beta+1\right)}\right)\left(\frac{1}{n^{(1-\alpha)\beta}}\right), \tag{29}$$

resulting to

$$\left|T\left(x_{0}\right)\right|=O\left(\frac{1}{n^{(1-\alpha)\beta}}\right), \tag{30}$$

and

$$\left|T\left(x_{0}\right)\right|=o\left(1\right). \tag{31}$$

And, letting $0<\varepsilon\leq\beta$, we derive

$$\frac{\left|T\left(x_{0}\right)\right|}{\left(\frac{1}{n^{(1-\alpha)(\beta-\varepsilon)}}\right)}\leq\frac{MT^{\beta}}{\Gamma\left(\beta+1\right)}\left(\frac{1}{n^{(1-\alpha)\varepsilon}}\right)\rightarrow0, \tag{32}$$

as $n\rightarrow\infty$.

I.e.

$$\left|T\left(x_{0}\right)\right|=o\left(\frac{1}{n^{(1-\alpha)(\beta-\varepsilon)}}\right), \tag{33}$$

proving the claim. ∎

Our second main result follows

Theorem 15 *Same assumptions as in Theorem 14. Then*

$$\left(K_{n}\left(f\right)\right)\left(x_{0}\right)-f\left(x_{0}\right)=\sum_{j=1}^{N-1}\frac{f^{(j)}\left(x_{0}\right)}{j!}K_{n}\left(\left(\cdot-x_{0}\right)^{j}\right)\left(x_{0}\right)+o\left(\frac{1}{n^{(1-\alpha)(\beta-\varepsilon)}}\right), \tag{34}$$

where $0<\varepsilon\leq\beta$.

If $N=1$, *the sum in (34) disappears.*

The last (34) implies that

$$n^{(1-\alpha)(\beta-\varepsilon)}\left[\left(K_{n}\left(f\right)\right)\left(x_{0}\right)-f\left(x_{0}\right)-\sum_{j=1}^{N-1}\frac{f^{(j)}\left(x_{0}\right)}{j!}K_{n}\left(\left(\cdot-x_{0}\right)^{j}\right)\left(x_{0}\right)\right]\rightarrow0, \tag{35}$$

as $n\rightarrow\infty$, $0<\varepsilon\leq\beta$.

When $N=1$, *or* $f^{(j)}\left(x_{0}\right)=0$, $j=1,...,N-1$, *then we derive*

$$n^{(1-\alpha)(\beta-\varepsilon)}\left[\left(K_{n}\left(f\right)\right)\left(x_{0}\right)-f\left(x_{0}\right)\right]\rightarrow0 \tag{36}$$

as $n\rightarrow\infty$, $0<\varepsilon\leq\beta$. *Of great interest is the case of* $\beta=\frac{1}{2}$.

Proof. As in Theorem 14. ∎

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ITERATES OF MULTIVARIATE CHENEY-SHARMA OPERATORS

TEODORA CĂTINAŞ AND DIANA OTROCOL

Abstract. Using the weakly Picard operators technique, we study the convergence of the iterates of some bivariate and trivariate Cheney-Sharma operators. Also, we generalize the procedure for the multivariate case.

Keywords: Cheney-Sharma operators, contraction principle, weakly Picard operators.

2000 Mathematics Subject Classification: 41A36, 41A05, 41A25, 39B12, 47H10.

1. Preliminaries

We recall some results regarding weakly Picard operators that will be used in the sequel (see, e.g., [17], [20]).

Let (X, d) be a metric space and $A : X \to X$ an operator. We denote by

$F_A := \{x \in X \mid A(x) = x\}$-the fixed point set of A;

$I(A) := \{Y \subset X \mid A(Y) \subset Y,\ Y \neq \emptyset\}$-the family of the nonempty invariant subset of A

$A^0 := 1_X,\ A^1 := A,\ ...,\ A^{n+1} := A \circ A^n, \quad n \in \mathbb{N}.$

Definition 1.1. *The operator $A : X \to X$ is a Picard operator if there exists $x^* \in X$ such that:*
(i) $F_A = \{x^*\}$;
(ii) *the sequence $(A^n(x_0))_{n\in\mathbb{N}}$ converges to x^* for all $x_0 \in X$.*

Definition 1.2. *The operator A is a weakly Picard operator if the sequence $(A^n(x))_{n\in\mathbb{N}}$ converges, for all $x \in X$, and the limit (which may depend on x) is a fixed point of A.*

Definition 1.3. *We define the operator A^∞, $A^\infty : X \to X$, by*

$$A^\infty(x) := \lim_{n\to\infty} A^n(x).$$

Theorem 1.4. [17] *An operator A is a weakly Picard operator if and only if there exists a partition of X, $X = \bigcup_{\lambda\in\Lambda} X_\lambda$, such that*

(a) $X_\lambda \in I(A)$, $\forall \lambda \in \Lambda$;
(b) $A|_{X_\lambda} : X_\lambda \to X_\lambda$ *is a Picard operator,* $\forall \lambda \in \Lambda$.

2. Cheney-Sharma operator

In [21] there was given an extension to two variables of the second univariate operator of Cheney-Sharma introduced in [5].

Let f be a real-valued function defined on $D = [0,1] \times [0,1]$. The bivariate Cheney-Sharma operator is defined by

$$(S_{m,n}f)(x,y;\beta,b) = \sum_{i=0}^{m} \sum_{j=0}^{n} p_{m,i}(x;\beta)\, q_{n,j}(y;b) f\left(\tfrac{i}{m}, \tfrac{j}{n}\right), \qquad (1)$$

with

$$p_{m,i}(x;\beta) = \frac{\binom{m}{i} x(x+i\beta)^{i-1}(1-x)\left[1-x+(m-i)\beta\right]^{m-i-1}}{(1+m\beta)^{m-1}},$$

and

$$q_{n,j}(y;b) = \frac{\binom{n}{j} y(y+jb)^{j-1}(1-y)\left[1-y+(n-j)b\right]^{n-j-1}}{(1+nb)^{n-1}},$$

where β and b are nonnegative parameters.

For a function f defined on $D_1 = [0,1] \times [0,1] \times [0,1]$, the trivariate operator Cheney-Sharma is defined by [22]

$$(S_{m,n,l}f)(x,y,z;\beta,\gamma,\delta) = \sum_{i=0}^{m} \sum_{j=0}^{n} \sum_{k=0}^{l} p_{m,i}(x;\beta)\, q_{n,j}(y;\gamma) r_{l,k}(z;\delta) f\left(\tfrac{i}{m}, \tfrac{j}{n}, \tfrac{k}{r}\right), \qquad (2)$$

with

$$p_{m,i}(x;\beta) = \frac{\binom{m}{i} x(x+i\beta)^{i-1}(1-x)\left[1-x+(m-i)\beta\right]^{m-i-1}}{(1+m\beta)^{m-1}},$$

$$q_{n,j}(y;\gamma) = \frac{\binom{n}{j} y(y+j\gamma)^{j-1}(1-y)\left[1-y+(n-j)\gamma\right]^{n-j-1}}{(1+n\gamma)^{n-1}},$$

and

$$r_{l,k}(z;\delta) = \frac{\binom{l}{k} z(z+k\delta)^{k-1}(1-z)\left[1-z+(l-k)\delta\right]^{l-k-1}}{(1+l\delta)^{l-1}}$$

where β, γ and δ are nonnegative parameters. This operator represents an extension to three variables of the second univariate operator of Cheney-Sharma [5].

Theorem 2.1. [21] *If f is a real-valued function defined on D then we have*

$$(S_{m,n}e_{ij})(x,y)=x^iy^j,\quad i,j=0,1,$$

and therefore, $\operatorname{span}\{e_{00},e_{10},e_{01},e_{11}\}\subset F_{S_{m,n}}$, *where $F_{S_{m,n}}$ denotes the fixed points set of $S_{m,n}$.*

Theorem 2.2. [22] *If f is a real-valued function defined on D_1 then we have*

$$(S_{m,n,l}e_{ijk})(x,y,z)=x^iy^jz^k,\quad i,j,k\in\{0,1\},$$

and therefore, $\operatorname{span}\{e_{000},e_{100},e_{001},e_{001},e_{110},e_{011},e_{101},e_{111}\}\subset F_{S_{m,n,l}}$, *where $F_{S_{m,n,l}}$ denotes the fixed points set of $S_{m,n,l}$.*

3. Iterates of Cheney-Sharma operator

Using the weakly Picard operators technique and the contraction principle, we study the convergence of the iterates of the bivariate Cheney-Sharma operator given in (1).

A similar approach for the univariate case was given in [4]. Some other linear and positive operators lead to similar results in [1], [2], [7], [18] and [19]. The limit behavior for the iterates of some classes of positive linear operators were also studied, for example, in [3], [8]-[16].

Let f be a real-valued function defined on D.

Theorem 3.1. *The operator $S_{m,n}$ is a weakly Picard operator and*

$$\begin{aligned}\left(S^{\infty}_{m,n}f\right)(x,y;\beta,b)=&(1-x)(1-y)f(0,0)+(1-x)yf(1,0)\\&+x(1-y)f(0,1)+xyf(1,1).\end{aligned}\tag{3}$$

Proof. Taking into account the interpolation properties (Theorem 2.1), of $S_{m,n}$, consider

$$X_{\alpha_1,\alpha_2,\alpha_3,\alpha_4}=\{f\in C(D)\mid f(0,0)=\alpha_1,f(1,0)=\alpha_2,f(0,1)=\alpha_3,f(1,1)=\alpha_4\},\tag{4}$$

and denote by

$$f^*_{\alpha_1,\alpha_2,\alpha_3,\alpha_4}(x,y):=(1-x)(1-y)\alpha_1+(1-x)y\alpha_2+x(1-y)\alpha_3+xy\alpha_4,$$

with $\alpha_1,\alpha_2,\alpha_3,\alpha_4\in\mathbb{R}$.

We have the following properties:

(i) $X_{\alpha_1,\alpha_2,\alpha_3,\alpha_4}$ is closed subset of $C(D)$;
(ii) $X_{\alpha_1,\alpha_2,\alpha_3,\alpha_4}$ is an invariant subset of $S_{m,n}$, for $\alpha_1,\alpha_2,\alpha_3,\alpha_4\in\mathbb{R}$, $m,n\in\mathbb{N}_+$;
(iii) $C(D)=\underset{\alpha_1,\alpha_2,\alpha_3,\alpha_4\in\mathbb{R}}{\cup}X_{\alpha_1,\alpha_2,\alpha_3,\alpha_4}$ is a partition of $C(D)$;
(iv) $X_{\alpha_1,\alpha_2,\alpha_3,\alpha_4}\cap F_{S_{m,n}}=\{f^*_{\alpha_1,\alpha_2,\alpha_3,\alpha_4}\}$.

The statements (i) and (iii) are obvious.

(ii) By interpolation properties of $S_{m,n}$ we have that $X_{\alpha_1,\alpha_2,\alpha_3,\alpha_4}$ is an invariant subset of $S_{m,n}$, for any $\alpha_1,\alpha_2,\alpha_3,\alpha_4\in\mathbb{R},\ m,n\in\mathbb{N}_+$;

(iv) We prove that

$$S_{m,n}|_{X_{\alpha_1,\alpha_2,\alpha_3,\alpha_4}}:X_{\alpha_1,\alpha_2,\alpha_3,\alpha_4}\to X_{\alpha_1,\alpha_2,\alpha_3,\alpha_4}$$

is a contraction for $\alpha_1,\alpha_2,\alpha_3,\alpha_4\in\mathbb{R},\ m,n\in\mathbb{N}_+$.

Let $f,g\in X_{\alpha_1,\alpha_2,\alpha_3,\alpha_4}$. From (1) and (4) we obtain

$$\begin{aligned}
&|S_{m,n}(f)(x,y)-S_{m,n}(g)(x,y)|=\\
&=|S_{m,n}(f-g)(x,y)|\leq\\
&\leq |p_{m,0}\,(x;\beta)\,q_{n,0}(y;b)\,[f\,(0,0)-g(0,0)]|\\
&\quad+\left|\sum_{i=1}^{m}\sum_{j=1}^{n}p_{m,i}\,(x;\beta)\,q_{n,j}(y;b)\left[f\left(\tfrac{i}{m},\tfrac{j}{n}\right)-g\left(\tfrac{i}{m},\tfrac{j}{n}\right)\right]\right|\\
&=\sum_{i=1}^{m}\sum_{j=1}^{n}p_{m,i}\,(x;\beta)\,q_{n,j}(y;b)\left|f\left(\tfrac{i}{m},\tfrac{j}{n}\right)-g\left(\tfrac{i}{m},\tfrac{j}{n}\right)\right|\\
&\leq\sum_{i=1}^{m}p_{m,i}\,(x;\beta)\sum_{j=1}^{n}q_{n,j}(y;b)\,\|f-g\|_\infty\\
&=\left[\sum_{i=0}^{m}p_{m,i}\,(x;\beta)-p_{m,0}\,(x;\beta)\right]\left[\sum_{j=0}^{n}q_{n,j}(y;b)-q_{n,0}(y;b)\right]\|f-g\|_\infty\\
&=\left[1-\left(1-\tfrac{x}{1+m\beta}\right)^{m-1}\right]\left[1-\left(1-\tfrac{y}{1+nb}\right)^{n-1}\right]\|f-g\|_\infty\\
&\leq\left[1-\left(1-\tfrac{1}{1+m\beta}\right)^{m-1}\right]\left[1-\left(1-\tfrac{1}{1+nb}\right)^{n-1}\right]\|f-g\|_\infty.
\end{aligned}$$

where $\|\cdot\|_\infty$ denotes the Chebyshev norm.

From [2, Lemma 8] it follows that

$$\begin{aligned}
&|S_{m,n}(f)(x,y)-S_{m,n}(g)(x,y)|=\\
&\leq\left[1-\left(1-\tfrac{1}{1+m\beta}\right)^{m-1}\left(1-\tfrac{1}{1+nb}\right)^{n-1}\right]\|f-g\|_\infty.
\end{aligned}$$

So,

$$\begin{aligned}
&\|S_{m,n}(f)(x,y)-S_{m,n}(g)(x,y)\|_\infty\\
&\leq\left[1-\left(1-\tfrac{1}{1+m\beta}\right)^{m-1}\left(1-\tfrac{1}{1+nb}\right)^{n-1}\right]\|f-g\|_\infty,\ \forall f,g\in X_{\alpha_1,\alpha_2,\alpha_3,\alpha_4},
\end{aligned}$$

i.e., $S_{mn}|_{X_{\alpha_1,\alpha_2,\alpha_3,\alpha_4}}$ is a contraction for $\alpha_1,\alpha_2,\alpha_3,\alpha_4\in\mathbb{R}$.

On the other hand, we have that

$$f^*_{\alpha_1,\alpha_2,\alpha_3,\alpha_4}(x,y) := (1-x)(1-y)\alpha_1 + (1-x)y\alpha_2 + x(1-y)\alpha_3 + xy\alpha_4$$

and

$$\begin{aligned} &S_{m,n}\left((1-x)(1-y)\alpha_1 + (1-x)y\alpha_2 + x(1-y)\alpha_3 + xy\alpha_4\right) = \\ &\quad = (1-x)(1-y)\alpha_1 + (1-x)y\alpha_2 + x(1-y)\alpha_3 + xy\alpha_4. \end{aligned}$$

From the contraction principle we have that $f^*_{\alpha_1,\alpha_2,\alpha_3,\alpha_4}$ is the unique fixed point of $S_{m,n}$ in $X_{\alpha_1,\alpha_2,\alpha_3,\alpha_4}$ and $S_{m,n}|_{X_{\alpha_1,\alpha_2,\alpha_3,\alpha_4}}$ is a Picard operator, so (3) holds. Consequently, taking into account (ii), by Theorem 1.4 it follows that the operator $S_{m,n}$ is a weakly Picard operator. We remark that $F_{S_{m,n}} = \operatorname{span}\{e_{00}, e_{10}, e_{01}, e_{11}\}$. □

Next, we study the convergence of the iterates of the trivariate Cheney-Sharma operator given in (2).

Let f be a real-valued function defined on D_1.

Theorem 3.2. *The operator $S_{m,n,l}$ is a weakly Picard operator and*

$$\begin{aligned} &\left(S^\infty_{m,n,l} f\right)(x,y,z;\beta,\gamma,\delta) = \qquad (5)\\ &= (1-x)(1-y)(1-z)f(0,0,0) + x(1-y)(1-z)f(1,0,0)\\ &\quad + (1-x)y(1-z)f(0,1,0) + (1-x)(1-y)zf(0,0,1) + xy(1-z)f(1,1,0)\\ &\quad + x(1-y)zf(1,0,1) + (1-x)yzf(0,1,1) + xyzf(1,1,1). \end{aligned}$$

Proof. The proof follows the same steps as in Theorem 3.1. Using the following inequality

$$\begin{aligned} &|S_{m,n,l}(f)(x,y,z) - S_{m,n,l}(g)(x,y,z)| \le \\ &\le \left[1-\left(1-\tfrac{1}{1+m\beta}\right)^{m-1}\right]\left[1-\left(1-\tfrac{1}{1+n\gamma}\right)^{n-1}\right]\left[1-\left(1-\tfrac{1}{1+l\delta}\right)^{l-1}\right]\|f-g\|_\infty, \end{aligned}$$

and further [2, Lemma 8]

$$\begin{aligned} &\|S_{m,n,l}(f)(x,y,z) - S_{m,n,l}(g)(x,y,z)\|_\infty \le \\ &\le \left[1-\left(1-\tfrac{1}{1+m\beta}\right)^{m-1}\left(1-\tfrac{1}{1+n\gamma}\right)^{n-1}\left(1-\tfrac{1}{1+l\delta}\right)^{l-1}\right]\|f-g\|_\infty, \end{aligned}$$

$\forall f,g \in X_{\alpha_1,\alpha_2,\alpha_3,\alpha_4}$, we prove that $S_{m,n,l}$ is a contraction. □

We generalize these results to multivariate case.

Theorem 3.3. *Consider a function $f \in C(D_p)$, with $D_p = [0,1] \times \underbrace{\ldots}_{p\ times} \times [0,1]$. The p-variate Cheney-Sharma operator, denoted by $S_{i_1,\ldots,i_p}$,* *is a weakly Picard operator and*

$$\left(S^{\infty}_{i_1,\ldots,i_p}f\right)(x_1,\ldots,x_p)=\sum_{\alpha_i\in\{0,1\},i=\overline{1,p}} s^{\infty}_{i_1,\ldots,i_p}(x_1,\ldots,x_p)\,f(\alpha_1,\ldots,\alpha_p), \tag{6}$$

where $\alpha_i\in\{0,1\}$, $i=1,\ldots,p$ *and*

$$s^{\infty}_{i_1,\ldots,i_p}(x_1,\ldots,x_p)=x_1^{\alpha_1}\cdot\ldots\cdot x_p^{\alpha_p}(1-x_1)^{(1-\alpha_1)}\cdot\ldots\cdot(1-x_p)^{(1-\alpha_p)}.$$

Proof. The proof follows the same steps as in Theorem 3.1. □

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J. COMPUTATIONAL ANALYSIS AND APPLICATIONS, VOL. 15, NO.7, 1247-1256, 2013, COPYRIGHT 2013 EUDOXUS PRESS, LLC

Convergence Analysis of the Over-relaxed Proximal Point Algorithms with Errors for Generalized Nonlinear Random Operator Equations[1]

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Abstract. The purpose of this paper is to introduce and study the over-relaxed proximal point algorithms with errors for generalized nonlinear random operator equations with H-maximal monotonicity framework. Further, by using the generalized proximal operator technique associated with the H-maximal monotone operators, we discuss the approximation solvability of generalized nonlinear random operator equations in Hilbert spaces and the convergence analysis of iterative sequences generated by the over-relaxed proximal point algorithms with errors under some suit conditions, which generalize and improve the the over-relaxed proximal point algorithms due to Verma [R.U. Verma, The over-relaxed proximal point algorithm based on H-maximal monotonicity design and applications, Computers and Mathematics with Applications 55 (2008) 2673-2679].

Key words and phrases: H-maximal monotonicity, generalized proximal operator technique, over-relaxed proximal point algorithms with errors, generalized nonlinear random operator equation, convergence analysis.

2000 Mathematics Subject classification: 49J40, 47H05

1 Introduction

In 2008, Verma [1] developed the general framework for a generalized over-relaxed proximal point algorithm using the notion of H-maximal monotonicity (also referred to as H-monotonicity), and examined the convergence analysis for this algorithm in the context of solving the following general class of nonlinear inclusion problems along with some auxiliary results on the resolvent operators corresponding to H-maximal monotonicity:

$$0 \in M(x), \tag{1.1}$$

where $M : \mathcal{X} \to 2^{\mathcal{X}}$ is a multi-valued mapping on a real Hilbert space $\mathcal{X}$.

[1]This work was supported by the Opening Project of Shanghai Key Laboratory of Complex Prescription (Shanghai University of Traditional Chinese Medicine) (10DZ2270900), Sichuan Province Youth Fund project (2011JTD0031), the Scientific Research Fund of Sichuan Provincial Education Department (10ZA136) and the Cultivation Project of Sichuan University of Science and Engineering (2011PY01).

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However, in [2], Huang illustrated that the conditions and the main proof of two main theorems of [1] concerning the strong convergence of the over-relaxed proximal point algorithm for H-maximal monotone mappings in Hilbert spaces are incorrect. Furthermore, Huang [2] provided the following **open question**:

Does the strong convergence hold for the sequence $\{x_n\}$ generated by the over-relaxed proximal point algorithm for H-maximal monotone mappings in the setting of Hilbert spaces?

Very recently, Verma [3] also pointed out "the over-relaxed proximal point algorithm is of interest in the sense that it is quite application-oriented, but nontrivial in nature". Agarwal and Verma [4] explored the approximation solvability of a general class of variational inclusion problems (1.1) based on the relative maximal monotonicity frameworks, while generalizing most of the investigations on weak convergence using the proximal point algorithm in a real Hilbert space setting. Furthermore, it seems that the obtained results can be used to generalize the Yosida approximation, which, in turn, can be applied to first-order evolution inclusions, and the obtained results can further be applied to the Douglas-Rachford splitting method for finding the zero of the sum of two relatively monotone mappings as well.

On the other hand, it is well known that the random equations involving the random operators in view of their need in dealing with probabilistic models in applied sciences is very important. In recent years, many researchers introduced and studied the research works in these fascinating areas, the random variational inequality problems, random quasi-variational inequality problems, random variational inclusion problems and random quasi-complementarity problems, respectively. For more literature, we recommend to the reader [5-11] and the references therein.

Motivated and inspired by the above works, we shall introduce and study the over-relaxed proximal point algorithms with errors for the following generalized nonlinear random operator equations: find a solution $x : \Omega \to \mathcal{X}$ to

$$f_t(x) - J^{M_t}_{\rho(t),H_t}(H_t(x)) = 0, \tag{1.2}$$

where $(\Omega, \mathcal{A}, \mu)$ is a complete σ-finite measure spaces, $\mathcal{X}$ is a real Hilbert space, $f_t(x) = f(t, x(t))$ for $(t, x) \in \Omega \times \mathcal{X}$, $J^{M_t}_{\rho(t),H_t} = (H_t + \rho(t)M_t)^{-1}$, $M : \Omega \times \mathcal{X} \to 2^{\mathcal{X}}$ is a multi-valued mapping.

We remark that the determinate form of the problem (1.2) includes the problem (1.1) by using the generalized proximal operator technique associated with the H-maximal monotone operators. Indeed, based on the definition of the generalized resolvent operator associated with the H-maximal monotone operators, Eqn. (1.2) can be written as

$$0 \in H_t(f_t(x)) - H_t(x) + \rho(t)M_t(f_t(x)),$$

which is reduced to (1.1) when $f_t(x) \equiv x$ and $M_t(x) \equiv M(x)$ for all $(t, x) \in \Omega \times \mathcal{X}$.

Further, the problem (1.2) provide us a general and unified framework for studying a wide range of interesting and important problems arising in mathematics, physics, engineering sciences and economics finance, etc. For more details, see [1, 3-12] and the following determinate example.

Example 1.1. ([13]) Let $V : \mathbb{R}^n \to \mathbb{R}$ be a local Lipschitz continuous function, and let K be a closed convex set in $\mathbb{R}^n$. If $x^* \in \mathbb{R}^n$ is a solution to the following problem:

$$\min_{x \in K} V(x),$$

then

$$0 \in \partial V(x^*) + \mathcal{N}_K(x^*),$$

where $\partial V(x^*)$ denotes the subdifferential of V at x^*, and $\mathcal{N}_K(x^*)$ the normal cone of K at x^*.

Moreover, by using the generalized proximal operator technique associated with the H-maximal monotone operators, we will discuss the approximation solvability of generalized nonlinear random operator equations in Hilbert spaces and the convergence analysis of iterative sequences generated by the over-relaxed proximal point algorithms with errors under some suit conditions.

2 Preliminaries

Throughout this paper, we suppose that $(\Omega, \mathcal{A}, \mu)$ is a complete σ-finite measure space and $\mathcal{X}$ is a separable real Hilbert space endowed with the norm $\|\cdot\|$ and an inner product $\langle \cdot, \cdot \rangle$. We denote by $\mathcal{B}(\mathcal{X})$ the class of Borel σ-fields in $\mathcal{X}$. Let $2^{\mathcal{X}}$ denote the family of all the nonempty subsets of $\mathcal{X}$.

In this paper, we will use the following definitions and lemmas.

Definition 2.1. An operator $x : \Omega \to \mathcal{X}$ is said to be measurable if for any $\mathcal{X} \in \mathcal{B}(\mathcal{X})$, $\{t \in \Omega : x(t) \in \mathcal{X}\} \in \mathcal{A}$.

Definition 2.2. An operator $f : \Omega \times \mathcal{X} \to \mathcal{X}$ is called a random operator if for any $x \in \mathcal{X}$, $f(t,x) = h(t)$ is measurable. A random operator f is said to be continuous (resp. linear, bounded) if for any $t \in \Omega$, the operator $f(t,\cdot) : \mathcal{X} \to \mathcal{X}$ is continuous (resp. linear, bounded).

It is well known that a measurable operator is necessarily a random operator.

Definition 2.3. A multi-valued operator $G : \Omega \to 2^{\mathcal{X}}$ is said to be measurable if for any $\mathcal{X} \in \mathcal{B}(\mathcal{X})$, $G^{-1}(\mathcal{X}) = \{t \in \Omega : G(t) \cap \mathcal{X} \neq \emptyset\} \in \mathcal{A}$.

Definition 2.4. A operator $u : \Omega \to \mathcal{X}$ is called a measurable selection of a multi-valued measurable operator $\Gamma : \Omega \to 2^{\mathcal{X}}$ if u is measurable and for any $t \in \Omega$, $u(t) \in \Gamma(t)$.

Definition 2.5. Let $\mathcal{X}$ be a separable real Hilbert space. Then a random operator $g : \Omega \times \mathcal{X} \to \mathcal{X}$ is said to be

(i) s-cocoercive in the second argument, if there exists a real-valued random variable $s(t) > 0$ such that

$$\langle g_t(x) - g_t(y), x(t) - y(t)\rangle \geq s(t)\|g_t(x) - g_t(y)\|^2, \quad \forall x(t),\, y(t) \in \mathcal{X},\, t \in \Omega;$$

(ii) γ-relaxed cocoercive in the second argument, if there exists a positive real-valued random variable $\gamma(t)$ such that

$$\langle g_t(x) - g_t(y), x(t) - y(t)\rangle \geq -\gamma(t)\|g_t(x) - g_t(y)\|^2, \quad \forall x(t),\, y(t) \in \mathcal{X},\, t \in \Omega;$$

(iii) (β, ϵ)-relaxed cocoercive in the second argument, if there exist positive real-valued random variables $\alpha(t)$ and $\epsilon(t)$ such that

$$\langle g_t(x) - g_t(y), x(t) - y(t)\rangle \geq -\beta(t)\|g_t(x) - g_t(y)\|^2 + \epsilon(t)\|x(t) - y(t)\|^2,$$

for all $x(t)$, $y(t) \in \mathcal{X}$, $t \in \Omega$;

(iv) μ-Lipschitz continuous in the second argument if there exists a real-valued random variable $\mu(t) > 0$ such that

$$\|g_t(x) - g_t(y)\| \leq \mu(t)\|x(t) - y(t)\|, \ \forall x(t),\ y(t) \in \mathcal{X},\ t \in \Omega.$$

Definition 2.6. Let $H : \Omega \times \mathcal{X} \to \mathcal{X}$ be a nonlinear (in general) operators. A multi-valued operator $M : \Omega \times \mathcal{X} \to 2^{\mathcal{X}}$ is said to be

(i) monotone in the second argument if

$$\langle u(t) - v(t), x(t) - y(t)\rangle \geq 0,\ \forall (x(t), u(t)), (y(t), v(t)) \in Graph(M_t),$$

where $Graph(M_t) = \{(z(t), w(t)) \in \mathcal{X} \times \mathcal{X} :\ w(t) \in M(t, x(t)), t \in \Omega\}$;

(ii) r-strongly monotone in the second argument if there exists a measurable function $r : \Omega \to (0, +\infty)$ such that for any $t \in \Omega$,

$$\langle u(t) - v(t), x(t) - y(t)\rangle \geq r(t)\|x(t) - y(t)\|^2,\ \forall (x(t), u(t)), (y(t), v(t)) \in Graph(M_t);$$

(iii) m-relaxed monotone in the second argument if, there exists a real-valued random variable $m(t) > 0$ such that for any $t \in \Omega$,

$$\langle u(t) - v(t), x(t) - y(t)\rangle \geq -m(t)\|x(t) - y(t)\|^2,\ \forall (x(t), u(t)), (y(t), v(t)) \in Graph(M_t);$$

(iv) H-maximal monotone if M is monotone in the second argument and $R(H_t + \rho(t)M_t) = \mathcal{X}$ for every $t \in \Omega$ and $\rho(t) > 0$.

Lemma 2.1. ([1]) Let $\mathcal{X}$ be a separable real Hilbert space, $H : \Omega \times \mathcal{X} \to \mathcal{X}$ be r-strongly monotone in the second argument, and $M : \Omega \times \mathcal{X} \to 2^{\mathcal{X}}$ be H-maximal monotone. Then the generalized resolvent operator associated with M is defined by

$$J^{M_t}_{\rho(t),H_t}(x) = (H_t + \rho(t)M_t)^{-1}(x),\ \forall x \in \mathcal{X},\ t \in \Omega$$

and is $\frac{1}{r(t)}$-Lipschitz continuous for any $t \in \Omega$. Moreover,

$$\|J^{M_t}_{\rho(t),H_t}(H_t(x)) - J^{M_t}_{\rho(t),H_t}(H_t(y))\| \leq \frac{1}{r(t) - \rho(t)}\|H_t(x) - H_t(y)\|, \forall x, y \in \mathcal{X}, t \in \Omega,$$

where $r(t) - \rho(t) > 1$ for all $t \in \Omega$.

Lemma 2.2. Let H, f, M and $\mathcal{X}$ be the same as in the problem (1.2). If $I_t(x) = H_t(f_t(x)) - H_t(J^{M_t}_{\rho(t),H_t}(H_t(x)))$ for $x \in \mathcal{X}$, and for all $x_1(t), x_2(t) \in \mathcal{X}$, $\rho(t) > 0$ and $\gamma(t) > \frac{1}{2}$, $t \in \Omega$,

$$\begin{aligned}&\langle H_t(J^{M_t}_{\rho(t),H_t}(H_t(x_1))) - H_t(J^{M_t}_{\rho(t),H_t}(H_t(x_2))), H_t(f_t(x_1)) - H_t(f_t(x_2))\rangle\\ &\geq \gamma(t)\|H_t(J^{M_t}_{\rho(t),H_t}(H_t(x_1))) - H_t(J^{M_t}_{\rho(t),H_t}(H_t(x_2)))\|^2,\end{aligned}$$

then

$$\begin{aligned}&(2\gamma(t) - 1)\|H_t(J^{M_t}_{\rho(t),H_t}(H_t(x_1))) - H_t(J^{M_t}_{\rho(t),H_t}(H_t(x_2)))\|^2\\ &+\|I_t(x_1) - I_t(x_2)\|^2 \leq \|H_t(f_t(x_1)) - H_t(f_t(x_2))\|^2.\end{aligned}$$

Proof. By the assumption, now we know

$$
\begin{aligned}
&\|I_t(x_1) - I_t(x_2)\|^2 \\
&\le \|H_t(J^{M_t}_{\rho(t),H_t}(H_t(x_1))) - H_t(J^{M_t}_{\rho(t),H_t}(H_t(x_2)))\|^2 + \|H_t(f_t(x_1)) - H_t(f_t(x_2))\|^2 \\
&\quad -2\langle H_t(J^{M_t}_{\rho(t),H_t}(H_t(x_1))) - H_t(J^{M_t}_{\rho(t),H_t}(H_t(x_2))), H_t(f_t(x_1)) - H_t(f_t(x_2))\rangle \\
&\le -(2\gamma(t) - 1)\|H_t(J^{M_t}_{\rho(t),H_t}(H_t(x_1))) - H_t(J^{M_t}_{\rho(t),H_t}(H_t(x_2)))\|^2 \\
&\quad +\|H_t(f_t(x_1)) - H_t(f_t(x_2))\|^2.
\end{aligned}
$$

This completes the proof. □

3 Main Results

In this section, we shall introduce a new class of the over-relaxed proximal point algorithms with errors to approximate solvability of the generalized nonlinear random operator equation (1.2) with H-maximal monotonicity framework.

Definition 3.1. An operator M^{-1}, the inverse of $M : \mathcal{X} \to 2^{\mathcal{X}}$, is (s, c)-Lipschitz continuous at 0 if for any $c \ge 0$, there exist a constant $s \ge 0$ and a solution x^* of $0 \in M(x)$ (equivalently $x^* \in M^{-1}(0)$) such that

$$\|x - x^*\| \le s\|w - 0\|, \quad \forall x \in M^{-1}(w),$$

where $w \in B_t = \{w : \|w\| \le c, w \in \mathcal{X}, c > 0\}$.

Algorithm 3.1. *Step* 1. For all $t \in \Omega$, choose an arbitrary initial point $x_0(t) \in \mathcal{X}$.

Step 2. Choose sequences $\{\alpha_n\}$, $\{\delta_n(t)\}$ and $\{\rho_n(t)\}$ such that for $n \ge 0$ and $t \in \Omega$, sequence real-value $\{\alpha_n\} \subset [0, \infty)$ and real-value random sequences $\{\delta_n(t)\}$ and $\{\rho_n(t)\}$ are in $[0, \infty)$ satisfying

$$\sum_{n=0}^{\infty} \delta_n(t) < \infty, \quad \rho_n(t) \uparrow \rho(t), \quad \forall t \in \Omega.$$

Step 3. Let $\{x_n(t)\} \subset \mathcal{X}$ be generated by the following iterative procedure

$$H_t(f_t(x_{n+1})) = (1 - \alpha_n)H_t(f_t(x_n)) + \alpha_n y_n(t) + e_n(t), \quad \forall n \ge 0, \tag{3.1}$$

where $\{e_n(t)\}$ is a random error sequence in $\mathcal{X}$ to take into account a possible inexact computation of the operator point, which satisfies $\sum_{n=0}^{\infty} \|e_n(t)\| < \infty$, and $y_n(t)$ satisfies

$$\|y_n(t) - H_t(J^{M_t}_{\rho_n(t),H_t}(H_t(x_n)))\| \le \delta_n(t)\|y_n(t) - H_t(f_t(x_n))\|, \forall t \in \Omega.$$

Step 4. If $x_n(t)$ and $y_n(t)$ satisfy (3.1) to sufficient accuracy, stop; otherwise, set $n := n + 1$ and return to *Step* 2.

Algorithm 3.2. For any $t \in \Omega$ and an arbitrary initial point $x_0(t) \in \mathcal{X}$, sequence $\{x_n(t)\} \subset \mathcal{X}$ is generated by the following iterative procedure

$$H_t(x_{n+1}) = (1 - \alpha_n)H_t(x_n) + \alpha_n y_n(t) + e_n(t), \quad \forall n \ge 0,$$

where $\{e_n(t)\}$ is a random error sequence in $\mathcal{X}$ to take into account a possible inexact computation of the operator point, which satisfies $\sum_{n=0}^{\infty} \|e_n(t)\| < \infty$, and $y_n(t)$ satisfies

$$\|y_n(t) - H_t(J^{M_t}_{\rho_n(t),H_t}(H_t(x_n)))\| \leq \delta_n(t)\|y_n(t) - H_t(x_n)\|,$$

and $J^{M_t}_{\rho(t),H_t} = (H_t + \rho_n(t)M_t)^{-1}$, $\{\alpha_n\}$, $\{\delta_n(t)\}$ and $\{\rho_n(t)\}$ are three sequences in $[0, \infty)$ satisfying

$$\sum_{n=0}^{\infty} \delta_n < \infty, \quad \rho_n(t) \uparrow \rho(t), \forall t \in \Omega.$$

Remark 3.1 If $e_n(t) \equiv 0$ for all $t \in \Omega$, then the determinate form of Algorithm 3.2 is reduced to the generalized proximal point algorithm in Theorem 3.2 of [1].

Next, we apply the over-relaxed proximal point algorithm 3.1 to approximate the solution of the problems (1.1) and (1.2), and as a result, we end up showing linear convergence.

Theorem 3.1. Let $\mathcal{X}$ be a separable real Hilbert space, $H : \Omega \times \mathcal{X} \to \mathcal{X}$ be r-strongly monotone and κ-Lipschitz continuous in the second argument, $f : \Omega \times \mathcal{X} \to \mathcal{X}$ is σ-Lipschitz continuous and (β, ϵ)-relaxed cocoercive in the second argument with the inverse f^{-1} is μ-expanding and $M : \Omega \times \mathcal{X} \to 2^{\mathcal{X}}$ be H-maximal monotone. If, in addition,

(i) $(H_t \circ f_t - H_t + \rho(t)M_t)^{-1}$ is (s, c)-Lipschitz continuous in the second argument at 0, where $H_t \circ f_t$ is defined by $H_t \circ f_t(x) = H(t, f(t, x(t)))$ for $(t, x) \in \Omega \times \mathcal{X}$;

(ii) for any $t \in \Omega$ and $x_1(t), x_2(t) \in \mathcal{X}$, there exists a real-value random variable $\gamma(t) > \frac{1}{2}$ such that

$$\begin{aligned}&\langle H_t(J^{M_t}_{\rho(t),H_t}(H_t(x_1))) - H_t(J^{M_t}_{\rho(t),H_t}(H_t(x_2))), H_t(f_t(x_1)) - H_t(f_t(x_2))\rangle \\ &\geq \gamma(t)\|H_t(J^{M_t}_{\rho(t),H_t}(H_t(x_1))) - H_t(J^{M_t}_{\rho(t),H_t}(H_t(x_2)))\|^2;\end{aligned}$$

(iii) there exists a real-value random variable $\rho(t) > 0$ such that

$$\begin{cases} r(t)\sqrt{1 - 2\epsilon(t) + \beta(t)\sigma^2(t) + \sigma^2(t)} + \kappa(t) < r(t), \\ 2\beta(t)\kappa(t)\sigma(t)\vartheta(t) < r(t)(\sqrt{1 + 4\beta(t)\epsilon(t)} - 1), \\ \vartheta(t) = \sqrt{(1-\alpha)^2 + \kappa^2(t)\varepsilon^2(t)[\alpha^2 - 2\gamma(t)\alpha(\alpha - 1)]} < 1, \\ \varepsilon(t) = \frac{s(t)}{\sqrt{\mu^2(t)\rho^2(t) + s^2(t)r^2(t)(2\gamma(t)-1)}} < 1, \end{cases} \tag{3.2}$$

then (1) the generalized nonlinear random operator equation (1.2) has a unique solution $x^*(t)$ in $\mathcal{X}$.

(2) the sequence $\{x_n(t)\}$ generated by Algorithm 3.1 converges linearly to the solution $x^*(t)$ with convergence rate

$$\frac{2\beta(t)\kappa(t)\sigma(t)\vartheta(t)}{r(t)(\sqrt{1 + 4\beta(t)\epsilon(t)} - 1)} < 1,$$

where $\vartheta(t) = \sqrt{(1-\alpha)^2 + \kappa^2(t)\varepsilon^2(t)[\alpha^2 - 2\gamma(t)\alpha(\alpha - 1)]}$, $\alpha = \limsup_{n\to\infty} \alpha_n > 1$, $\varepsilon(t) = \frac{s(t)}{\sqrt{\mu^2(t)\rho^2(t) + s^2(t)r^2(t)(2\gamma(t)-1)}}$, $\rho_n(t) \uparrow \rho(t)$ for all $t \in \Omega$.

Proof. Firstly, for any given positive real-valued random variable $\rho(t)$, define $F : \Omega \times \mathcal{X} \to \mathcal{X}$ by

$$F_t(x) = x(t) - f_t(x) + J^{M_t}_{\rho(t),H_t}(H_t(x)),\ \forall x \in \mathcal{H}.$$

By the assumptions of the theorem and Lemma 2.1, for all $x(t), y(t) \in \mathcal{X}$ we have

$$\begin{aligned}
&\|F_t(x) - F_t(y)\| \\
&\le \|x(t) - y(t) - [f_t(x) - f_t(y)]\| + \|J^{M_t}_{\rho(t),H_t}(H_t(x)) - J^{M_t}_{\rho(t),H_t}(H_t(y))\| \\
&\le \theta(t)\|x(t) - y(t)\|,
\end{aligned}$$

where $\theta(t) = \sqrt{1 - 2\epsilon(t) + \beta(t)\sigma^2(t) + \sigma^2(t)} + \frac{\kappa(t)}{r(t)}$. It follows from condition (3.2) that $0 < \theta(t) < 1$ and so $F(t,\cdot)$ is a contractive mapping for any $t \in \Omega$, which shows that $F(t,\cdot)$ has a unique fixed point in $\mathcal{X}$.

Now, we prove the conclusion (2). Let $x^*(t)$ be a solution of Eqn. (1.2). Then for any given positive real-valued random variable $\rho_n(t)$ and $n \ge 0$, we have

$$H_t(f_t(x^*)) = (1 - \alpha_n)H_t(f_t(x^*)) + \alpha_n H_t(J^{M_t}_{\rho_n(t),H_t}(H_t(x^*))). \tag{3.3}$$

For $I_t = H_t \circ f_t - H_t(J^{M_t}_{\rho(t),H_t})$ and under the assumptions, it follows that $I_t(x_n) \to 0 (n \to \infty)$. Since $\rho_n^{-1}(t)I_t(x_n) \in (H_t \circ f_t - H_t + \rho_n(t)M_t)(f_t^{-1}(J^{M_t}_{\rho(t),H_t}(H_t(x_n))))$, this implies $f_t^{-1}(J^{M_t}_{\rho(t),H_t}(H_t(x_n))) \in (H_t \circ f_t - H_t + \rho_n(t)M_t)^{-1}(\rho_n^{-1}(t)I_t(x_n))$. Then, applying Lemma 2.2, the strong monotonicity of H, and the Lipschitz continuity of H (and hence, H being expanding), and the Lipschitz continuity at 0 of $(H_t \circ f_t - H_t + \rho_n(t)M_t)^{-1}$ by setting $w = \rho_n^{-1}(t)I_t(x_n)$ and $x(t) = H_t^{-1}(J^{M_t}_{\rho(t),H_t}(H_t(x_n)))$, we know

$$\begin{aligned}
&\mu^2\|J^{M_t}_{\rho_n(t),H_t}(H_t(x_n)) - J^{M_t}_{\rho_n(t),H_t}(H_t(x^*))\|^2 \\
&\le \|H_t^{-1}(J^{M_t}_{\rho_n(t),H_t}(H_t(x_n))) - H_t^{-1}(J^{M_t}_{\rho_n(t),H_t}(H_t(x^*)))\|^2 \\
&\le s^2(t)\|\rho_n^{-1}(t)I_t(x_n) - \rho_n^{-1}(t)I_t(x^*)\|^2 \\
&\le s^2(t)\rho_n^{-2}(t)\{\|H_t(f_t(x_n)) - H_t(f_t(x^*))\|^2 \\
&\qquad\qquad - r^2(t)(2\gamma(t) - 1)\|J^{M_t}_{\rho(t),H_t}(H_t(x_n)) - J^{M_t}_{\rho(t),H_t}(H_t(x^*))\|^2\},
\end{aligned}$$

which implies

$$\|J^{M_t}_{\rho(t),H_t}(H_t(x_n)) - J^{M_t}_{\rho(t),H_t}(H_t(x^*))\| \le \varepsilon_n(t)\|H_t(f_t(x_n)) - H_t(f_t(x^*))\|, \tag{3.4}$$

where $\varepsilon_n(t) = \frac{s(t)}{\sqrt{\mu^2(t)\rho_n^2(t) + s^2(t)r^2(t)(2\gamma(t)-1)}} < 1$.

For $n \ge 0$, let

$$H_t(f_t(z_{n+1})) = (1 - \alpha_n)H_t(f_t(x_n)) + \alpha_n H_t(J^{M_t}_{\rho_n(t),H_t}(H_t(x_n))).$$

Thus, by the assumptions of the theorem, (3.3) and and (3.4), now we find the estimate

$$\begin{aligned}
&\|H_t(f_t(z_{n+1})) - H_t(f_t(x^*))\|^2 \\
&= \|(1-\alpha_n)(H_t(f_t(x_n)) - H_t(f_t(x^*)))\|^2 \\
&\quad + \alpha_n^2 \|H_t(J^{M_t}_{\rho_n(t),H_t}(H_t(x_n))) - H_t(J^{M_t}_{\rho_n(t),H_t}(H_t(x^*)))\|^2 \\
&\quad + 2\langle \alpha_n [H_t(J^{M_t}_{\rho_n(t),H_t}(H_t(x_n))) - H_t(J^{M_t}_{\rho_n(t),H_t}(H_t(x^*)))], \\
&\qquad\qquad\qquad (1-\alpha_n)(H_t(f_t(x_n)) - H_t(f_t(x^*)))\rangle \\
&\le (1-\alpha_n)^2 \|H_t(f_t(x_n)) - H_t(f_t(x^*))\|^2 \\
&\quad + [\alpha_n^2 + 2\gamma(t)\alpha_n(1-\alpha_n)]\kappa^2(t)\|J^{M_t}_{\rho_n(t),H_t}(H_t(x_n)) - J^{M_t}_{\rho_n(t),H_t}(H_t(x^*))\|^2 \\
&\le \vartheta_n^2(t)\|H_t(f_t(x_n)) - H_t(f_t(x^*))\|^2, \qquad (3.5)
\end{aligned}$$

where $\vartheta_n(t) = \sqrt{(1-\alpha_n)^2 + \kappa^2(t)\varepsilon_n^2(t)[\alpha_n^2 - 2\gamma(t)\alpha_n(\alpha_n - 1)]}$.

Since

$$H_t(f_t(x_{n+1})) = (1-\alpha_n)H_t(f_t(x_n)) + \alpha_n y_n + e_n(t),$$

we have $H_t(f_t(x_{n+1})) - H_t(f_t(x_n)) = \alpha_n[y_n - H_t(f_t(x_n))] + e_n(t)$ and

$$\begin{aligned}
&\|H_t(f_t(x_{n+1})) - H_t(f_t(z_{n+1}))\| = \alpha_n\|y_n - H_t(J^{M_t}_{\rho_n(t),H_t}(H_t(x_n)))\| + \|e_n(t)\| \\
&\le \alpha_n\delta_n(t)\|y_n - H_t(f_t(x_n))\| + \|e_n(t)\| \\
&\le \delta_n(t)\|H_t(f_t(x_{n+1})) - H_t(f_t(x^*))\| \\
&\quad + \delta_n(t)\|H_t(f_t(x_n)) - H_t(f_t(x^*))\| + \|e_n(t)\|. \qquad (3.6)
\end{aligned}$$

In the sequel, we estimate using (3.5) and (3.6) that

$$\begin{aligned}
&\|H_t(f_t(x_{n+1})) - H_t(f_t(x^*))\| \\
&\le \|H_t(f_t(x_{n+1})) - H_t(f_t(z_{n+1}))\| + \|H_t(f_t(z_{n+1})) - H_t(f_t(x^*))\| \\
&\le \delta_n(t)\|H_t(f_t(x_{n+1})) - H_t(f_t(x^*))\| + \|e_n(t)\| \\
&\quad + (\delta_n(t) + \vartheta_n(t))\|H_t(f_t(x_n)) - H_t(f_t(x^*))\|,
\end{aligned}$$

which implies

$$\begin{aligned}
&\|H_t(f_t(x_{n+1})) - H_t(f_t(x^*))\| \\
&\le \frac{\vartheta_n(t) + \delta_n(t)}{1-\delta_n(t)}\|H_t(f_t(x_n)) - H_t(f_t(x^*))\| + \frac{1}{1-\delta_n(t)}\|e_n(t)\|. \qquad (3.7)
\end{aligned}$$

It follows from (3.7), the strong monotonicity and the Lipschitz continuity of H and f that for any $t \in \Omega$ and all $x(t), y(t) \in \mathcal{X}$,

$$\begin{aligned}
\frac{r(t)(\sqrt{1+4\beta(t)\epsilon(t)} - 1)}{2\beta(t)}\|x(t) - y(t)\| &\le \|H_t(f_t(x)) - H_t(f_t(y))\| \\
&\le \kappa(t)\sigma(t)\|x(t) - y(t)\|,
\end{aligned}$$

and

$$\begin{aligned}
\|x_{n+1} - x^*\| \quad &\le \frac{2\beta(t)\kappa(t)\sigma(t)}{r(t)(\sqrt{1+4\beta(t)\epsilon(t)} - 1)} \cdot \frac{\vartheta_n(t) + \delta_n(t)}{1-\delta_n(t)}\|x_n - x^*\| \\
&\quad + \frac{2\beta(t)}{r(t)(\sqrt{1+4\beta(t)\epsilon(t)} - 1)} \cdot \frac{1}{1-\delta_n(t)}\|e_n(t)\|. \qquad (3.8)
\end{aligned}$$

By (3.8), we know that the $\{x_n\}$ converges linearly to a solution x^* for

$$\frac{2\beta(t)\kappa(t)\sigma(t)\vartheta_n}{r(t)(\sqrt{1+4\beta(t)\epsilon(t)}-1)}.$$

Hence, we have

$$\limsup_{n\to\infty} \frac{2\beta(t)\kappa(t)\sigma(t)}{r(t)(\sqrt{1+4\beta(t)\epsilon(t)}-1)} \cdot \frac{\vartheta_n+\delta_n}{1-\delta_n} = \frac{2\beta(t)\kappa(t)\sigma(t)\vartheta(t)}{r(t)(\sqrt{1+4\beta(t)\epsilon(t)}-1)},$$

where $t \in \Omega$,

$$\vartheta(t) = \limsup_{n\to\infty} \vartheta_n(t) = \sqrt{(1-\alpha)^2 + \kappa^2(t)\varepsilon^2(t)[\alpha^2 - 2\gamma(t)\alpha(\alpha-1)]},$$

$\varepsilon(t) = \limsup_{n\to\infty} \varepsilon_n(t) = \frac{s(t)}{\sqrt{\mu^2(t)\rho^2(t)+s^2(t)r^2(t)(2\gamma(t)-1)}}$, $\rho_n(t) \uparrow \rho(t)$, $\alpha = \limsup_{n\to\infty} \alpha_n$. This completes the proof. □

Remark 3.2. The conditions (3.2) in Theorem 3.1 hold for some suitable value of constant or real-valued random variable, for example, $\alpha = 1.35$, and $r(t) = 1.25, \epsilon(t) = 0.4, \beta(t) = 0.15, \sigma(t) = 0.025, s(t) = 0.25, \kappa(t) = 0.98, \gamma(t) = 1.5262, \mu(t) = 0.6, \rho(t) = 0.7348$ and the convergence rate $\theta(t) = 0.0220 < 1$ for all $t \in \Omega$.

From Theorem 3.1, we have the following results as an application of Theorem 3.1.

Theorem 3.2. Let H, M and $\mathcal{X}$ be the same as in Theorem 3.1. If, in addition,

(i) M_t^{-1} is (s, c)-Lipschitz continuous in the second argument at 0;

(ii) for any $t \in \Omega$ and $x_1(t), x_2(t) \in \mathcal{X}$, there exists a real-value random variable $\gamma(t) > \frac{1}{2}$ such that

$$\begin{aligned}
&\langle H_t(J^{M_t}_{\rho(t),H_t}(H_t(x_1))) - H_t(J^{M_t}_{\rho(t),H_t}(H_t(x_2))), H_t(x_1) - H_t(x_2)\rangle \\
&\geq \gamma(t)\|H_t(J^{M_t}_{\rho(t),H_t}(H_t(x_1))) - H_t(J^{M_t}_{\rho(t),H_t}(H_t(x_2)))\|^2;
\end{aligned}$$

(iii) there exists a real-value random variable $\rho(t) > 0$ such that

$$\begin{cases}
\kappa(t)\vartheta(t) < r(t), \\
\vartheta(t) = \sqrt{(1-\alpha)^2 + \kappa^2(t)\varepsilon^2(t)[\alpha^2 - 2\gamma(t)\alpha(\alpha-1)]} < 1, \\
\varepsilon_n(t) = \frac{s(t)}{\sqrt{\rho_n^2(t)+s^2(t)r^2(t)(2\gamma(t)-1)}} < 1,
\end{cases}$$

then the sequence $\{x_n(t)\}$ generated by Algorithm 3.2 converges linearly to the solution $x^*(t)$ of the problem (1.1) with convergence rate

$$\frac{\kappa(t)}{r(t)}\sqrt{1-\alpha\{2(1-\gamma(t)\kappa^2(t)\varepsilon^2(t)) - \alpha[1-(2\gamma(t)-1)\kappa^2(t)\varepsilon^2(t)]\}} < 1,$$

where $\alpha = \limsup_{n\to\infty} \alpha_n > 1$, $\varepsilon(t) = \frac{s(t)}{\sqrt{\rho^2(t)+s^2(t)r^2(t)(2\gamma(t)-1)}}$, $\rho_n(t) \uparrow \rho(t)$ for all $t \in \Omega$.

Theorem 3.3. Let H, M and $\mathcal{X}$ be the same as in Theorem 3.1. If, in addition, condition (ii) of Theorem 3.2 holds and there exists a real-value random variable $\rho(t) \in (0, r(t)-1)$ such that

$$\kappa(t)\sqrt{(1-\alpha_n)^2 + \frac{\kappa^2(t)\alpha_n[\alpha_n - 2\gamma(t)(\alpha_n-1)]}{(r(t)-\rho(t))^2}} < r(t),$$

then the sequence $\{x_n(t)\}$ generated by Algorithm 3.2 converges linearly to the solution $x^*(t)$ of the problem (1.1) with convergence rate

$$\frac{\kappa(t)}{r(t)}\sqrt{1-\alpha\{2(1-\gamma(t)\kappa^2(t)\varepsilon^2(t))-\alpha[1-(2\gamma(t)-1)\kappa^2(t)\varepsilon^2(t)]\}}<1,$$

where $\alpha = \limsup_{n\to\infty}\alpha_n > 1$, $\varepsilon(t) = \frac{1}{r(t)-\rho(t)}$ with $r(t)-\rho(t) > 1$, $\rho_n(t) \uparrow \rho(t)$ for all $t \in \Omega$.

Remark 3.3. In Theorem 3.3, we apply Lemma 2.1, the Lipschitz continuity of the generalized resolvent operator associated with M instead, it seems that the conditions in Theorem 3.3 is less than that in Theorem 3.2. Further, if real-valued random variables $\gamma(t) = 1$ or $e_n(t) \equiv 1$ or $\kappa(t) = 1$ (that is, H is nonexpansive) for all $t \in \Omega$, then we can obtain corresponding results of Theorems 3.1-3.3. Therefore, the results presented in this paper improve, generalize and unify the corresponding results of recent works.

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J. COMPUTATIONAL ANALYSIS AND APPLICATIONS, VOL. 15, NO.7, 1257-1265, 2013, COPYRIGHT 2013 EUDOXUS PRESS, LLC

FIXED POINT THEOREM FOR CIRIC'S TYPE CONTRACTIONS IN GENERALIZED QUASI-METRIC SPACES

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Abstract. A fixed point theorem in generalized quasi-metric spaces is proved. The obtained result extends in generalized quasi-metric spaces the Ciric's fixed point theorem on quasi-contraction mapping. An example shows that the main theorem of this paper provides a larger class of mappings than the Ciric's fixed point theorem.

Keywords: Cauchy sequence, fixed point, generalized quasi-metric space, quasi-contraction.

Mathematics Subject Classification: 47H10, 54H25

1. Introduction and Preliminaries

The concept of metric space, as an ambient space in fixed point theory, has been generalized in several directions. Some of such generalizations are: the quasi-metric spaces, the generalized metric spaces and the generalized quasi-metric spaces.

The concept of quasi-metric space is treated differently by many authors. In [2], [8], [14], [15], [18], [19], etc the quasi-metric space is in line of metric space in which the triangular inequality $d(x,y) \le d(x,z) + d(z,y)$ is replaced by quasi- triangular inequality $d(x,y) \le k[d(x,z) + d(z,y)], k \ge 1$.

In 2000 Branciari [3] introduced the concept of generalized metric spaces (**gms**) (The triangular inequality $d(x,y) \le d(x,z) + d(z,y)$ is replaced by tetrahedral inequality $d(x,y) \le d(x,z) + d(z,w) + d(w,y)$). Starting with the paper of Branciari, some classical metric fixed point theorems have been transferred to **gms** (see [1], [4], [5], [6], [7], [10], [11], [12], [16], [17])

Recently L. Kikina and K. Kikina [9] introduced the concept of generalized **quasi-metric space** (**gqms**) replacing the tetrahedral inequality $d(x,y) \le d(x,z) + d(z,w) + d(w,y)$ with the quasi-tetrahedral inequality $d(x,y) \le k[d(x,z) + d(z,w) + d(w,y)]$. The metric spaces are a special case of generalized metric spaces and generalized metric spaces are a special case of generalized quasi-metric spaces (for $k = 1$). Also, every qms is a gqms, while the converse is not true [9].

Firstly, we will give some known definitions and notations.

Let (X, d) be a metric space. A mapping $T : X \to X$ is said to be a quasi-contraction if there exists $0 \le h < 1$ such that

$$d(Tx, Ty) \le h \max\{d(x, y), d(x, Tx), d(y, Ty), d(x, Ty), d(y, Tx)\}$$

for all $x, y \in X$. In 1974, Ciric [4] introduced these mappings and proved the following fixed point result:

Theorem 1.1 (Ciric [4]) Let T be a quasi-contraction on a metric space (X, d) and let X be T-orbitally complete metric space. Then

(a) T has a unique fixed point α in X,

(b) $\lim_{n\to\infty} T^n x = \alpha$, and

(c) $d(T^n x, \alpha) \le (h^n/(1-h))d(x, Tx)$ for every $x \in X$

In this paper we extend in generalized quasi-metric spaces the above theorem.

Definition 1.1 [3] Let X be a set and $d : X^2 \to R^+$ a mapping such that for all $x, y \in X$ and for all distinct points $z, w \in X$, each of them different from *x* and *y*, one has

(a) $d(x, y) = 0$ if and only if $x = y$,

(b) $d(x, y) = d(y, x)$,

(c) $d(x, y) \le d(x, z) + d(z, w) + d(w, y)$ (Tetrahedral inequality)

Then d is called *a generalized metric* and (X, d) is a *generalized metric space* (or shortly **gms**).

Definition 1.2 [9] Let *X* be a set. A nonnegative symmetric function d defined on $X \times X$ is called a *generalized quasi-distance* on X if and only if there exists a constant $k \ge 1$ such that for all $x, y \in X$ and for all distinct points $z, w \in X$, each of them different from *x* and *y* the following conditions hold:

(i) $d(x, y) = 0 \Leftrightarrow x = y$;

(ii) $d(x, y) = d(y, x)$;

(iii) $d(x, y) \le k[d(x, z) + d(z, w) + d(w, y)]$.

Inequality (3) is often called *quasi-tetrahedral inequality* and k is often called the *coefficient* of d. A pair (X, d) is called a *generalized quasi-metric space* (or shortly **gqms**) if *X* is a set and *d is* a generalized quasi-distance on *X*.

The set $B(a, r) = \{x \in X : d(x, a) < r\}$ is called “open” ball with center $a \in X$ and radius $r > 0$.

The family $\tau = \{Q \subset X : \forall a \in Q, \exists r > 0, B(a, r) \subset Q\}$ is a topology on X and it is called induced topology by the generalized quasi-distance d.

The following example illustrates the existence of the generalized quasi-metric space for an arbitrary constant $k \geq 1$:

Example 1.3 [9] Let $X = \left\{1 - \frac{1}{n} : n = 1, 2, ...\right\} \cup \{1, 2\}$, Define $d : X \times X \to R$ as follow:

$$d(x, y) = \begin{cases} 0 & for\ x = y \\ \frac{1}{n} & for\ x \in \{1, 2\}\ and\ y = 1 - \frac{1}{n}\ or\ y \in \{1, 2\}\ and\ x = 1 - \frac{1}{n}, x \neq y \\ 3k & for\ x, y \in \{1, 2\}, x \neq y \\ 1 & otherwise \end{cases}$$

Then it is easy to see that (X, d) is a generalized quasi-metric space and is not a generalized metric space (for $k > 1$).

Note that the sequence $\{x_n\} = \{1 - \frac{1}{n}\}$ converges to both 1 and 2 and it is not a Cauchy sequence: $d(x_n, x_m) = d(1 - \frac{1}{n}, 1 - \frac{1}{m}) = 1, \forall n, m \in N$

Since $B(1, r) \cap B(2, r) \neq \phi$ for all $r > 0$, the (X, d) is non a Hausdorff generalized quasi-metric space.

The function d is not continuous: $1 = \lim\limits_{n \to \infty} d(1 - \frac{1}{n}, \frac{1}{2}) \neq d(1, \frac{1}{2}) = \frac{1}{2}$.

The above example shows that: in a gqms (and for $k = 1$ in a gms), contrary to the case of a metric space, the "open" balls $B(a, r) = \{x \in X : d(x, a) < r\}$ are not always open sets and, moreover, the generalized quasi-metric d is not always necessarily continuous with respect to its variables. Also, the generalized quasi-metric space is not always a Hausdorff space and a convergent sequence $\{x_n\}$ in gqms is not always a Cauchy sequence. Under these circumstances, not every theorem of fixed points for metric spaces, can be extended in gqms as well. Even in the cases it may be done, the proof of theorem is more complicated and it may requires additional conditions.

In [9] is proved:

Proposition 1.4 If (X, d) is a quasi-metric space, then (X, d) is a generalized quasi-metric space. The converse proposition doesn't hold true.

Definition 1.5 A sequence $\{x_n\}$ in a generalized quasi-metric space (X, d) is called Cauchy sequence if $\lim\limits_{n,m \to \infty} d(x_n, x_m) = 0$.

Definition 1.6 Let (X, d) be a generalized quasi-metric space. Then:

(1) A sequence $\{x_n\}$ in X is said to be convergent to a point $x \in X$ (denoted by $\lim\limits_{n \to \infty} x_n = x$) if $\lim\limits_{n \to \infty} d(x_n, x) = 0$.

(2) It is called compact if every sequence contains a convergent subsequence.

Definition 1.7 A generalized quasi-metric space (X, d) is called complete, if every Cauchy sequence is convergent.

Definition 1.8 Let (X, d) be a gqms and the *coefficient* of d is k.

A map $T: X \to X$ is called contraction if there exists $0 \le c < \dfrac{1}{k}$ such that

$$d(Tx,Ty) \le cd(x,y) \quad \text{for all } x, y \in X .$$

Definition 1.9 Let $T: X \to X$ be a mapping where X is a gqms. For each $x \in X$, let

$$\mathrm{O(x)}=\{\mathrm{x,Tx,T}^2 x,...\}$$

which will be called the orbit of T at x. *The space* X is said to be T-orbitally complete if and only if every Cauchy sequence which is contained in $\mathrm{O(x)}$ converges to a point in X.

2. MAIN RESULTS

Similarly to Ciric definition of *quasi-contraction* on metric spaces [4], we introduce the concept of *quasi-contraction* in generalized quasi-metric spaces.

Definition 2.1. Let (X,d) be a generalized quasi-metric space and the *coefficient* of d is $k \ge 1$. The mapping $T: X \to X$ is said to be quasi-contraction if there exists a number $h, h \in [0, \frac{1}{k})$ such that

$$d(Tx,Ty) \le h \max\{d(x,y), d(x,Tx), d(y,Ty), d(x,Ty), d(y,Tx)\} \qquad (1)$$

holds for all $x, y \in X$.

Before stating the main fixed point theorem for quasi-contractions in gqms, we give three lemmas for these mappings.

First, let T as in the above definition. For each $x \in X$, let

$$O(x) = \{x, Tx, T^2 x, ...\}$$

the orbit of T at x and $O(x,n) = \{x, Tx, T^2 x, ..., T^n x\}$. We denote by $\delta(O(x))$ the diameter of the set $O(x)$:

$$\delta(O(x)) = \sup\{d(T^n x, T^m x) : n, m \in N\} \qquad (2)$$

and by $\delta(O(x,n))$ the diameter of the set $O(x,n)$.

To obtain the main theorem, we require the following lemmas.

Lemma 2.2. Let $T: X \to X$ be a quasi-contraction on generalized quasi-metric space (X,d). Then for each $x \in X$, $n \ge 1$ and $i, j \in \{1,2,...,n\}$ implies

$$d(T^i x, T^j x) \le h\delta(O(x,n)) \qquad (3)$$

and for each n, there exists $1 \le p \le n$ such that

$$d(x, T^p x) = \delta(O(x,n)) \qquad (4)$$

The proof is the same as in case of metric spaces (see [4]).

Lemma 2.3 If $T: X \to X$ is a quasi-contraction on generalized metric space (X,d) and the *coefficient* of d is k., then $\forall n \in N$ and $\forall x \in X$,

$$\delta(O(x,n)) \le \tfrac{k(1+h)}{1-kh^2} \max\{d(x,Tx), d(x,T^2 x)\} \qquad (5)$$

holds for all $x \in X$.

Moreover,

$$\delta(O(x)) \le \tfrac{k(1+h)}{1-kh^2} \max\{d(x,Tx), d(x,T^2 x)\} \qquad (6)$$

holds for all $x \in X$

Proof. From the Lemma 2.2, we have $d(x, T^p x) = \delta(O(x,n))$ for some p with $1 \le p \le n$.

If $p=1$ or $p=2$, then

$$(1-kh^2)\delta(O(x,n))=(1-kh^2)d(x,T^px)$$
$$\leq d(x,T^px)\leq k(1+h)\max\{d(x,Tx),d(x,T^2x)\}$$

Therefore,

$$\delta(O(x,n))\leq \tfrac{k(1+h)}{1-kh^2}\max\{d(x,Tx),d(x,T^2x)\}$$

Let p such that $3\leq p\leq n$. If $x=Tx,\ x=T^2x\ or\ Tx=T^2x$, then the result follows trivially. So we can assume that $x,Tx\ and\ T^2x$ are all distinct. Let T^px a point other than $Tx\ and\ T^2x$. Then from *quasi-tetrahedral inequality* and lemma 2.2 we have:

$$\delta(O(x,n))=d(x,T^px)\leq k[d(x,Tx)+d(Tx,T^2x)+d(T^2x,T^px)]$$
$$\leq kd(x,Tx)+kh\delta(O(x,2))+kd(TTx,T^{p-1}Tx)]$$
$$\leq kd(x,Tx)+kh\max\{d(x,Tx),d(x,T^2x)\}+kh\delta(O(Tx,p-1))$$
$$\leq k(1+h)\max\{d(x,Tx),d(x,T^2x)\}+khd(Tx,T^mTx),(1\leq m\leq p-1)$$
$$\leq k(1+h)\max\{d(x,Tx),d(x,T^2x)\}+kh^2\delta(O(x,m+1))$$
$$\leq k(1+h)\max\{d(x,Tx),d(x,T^2x)\}+kh^2\delta(O(x,n))$$

Therefore,

$$(1-kh^2)\delta(O(x,n))\leq k(1+h)\max\{d(x,Tx),d(x,T^2x)\}$$

Hence, since $(1-kh^2)>0$,

$$\delta(O(x,n))\leq \tfrac{k(1+h)}{1-kh^2}\max\{d(x,Tx),d(x,T^2x)\}$$

Moreover, since

$$\delta(O(x,1))\leq\delta(O(x,2))\leq\ldots\leq\delta(O(x,n))\leq\ldots$$

we can write

$$\delta(O(x))\leq \tfrac{k(1+h)}{1-kh^2}\max\{d(x,Tx),d(x,T^2x)\}$$

This completes the proof of the Lemma.

Remark 2.4 If T is a quasi-contraction, note that, in view of Lemma 2.3, $O(x)$ is bounded set: $\delta(O(x))<\infty,\forall x\in X$

Lemma 2.5 Let T be a quasi-contraction on generalized quasi-metric space (X,d). Then, for any $n\geq 1$, one has

$$\delta(O(T^nx))\leq h^n\delta(O(x))$$

where h is the constant associated with the quasi-contraction definition of T. Moreover, we have

$$d(T^nx,T^{n+m}x))\leq h^n\ \tfrac{k(1+h)}{1-kh^2}\max\{d(x,Tx),d(x,T^2x)\}$$

for any $n\geq 1$ and $m\in N$.

Proof. Let n and m $(n<m)$ be any positive integers. Since T is a quasi-contraction, by condition (1), we have

$$d(T^n x, T^m y) \le$$
$$\le h \max\{d(T^{n-1}x, T^{m-1}y), d(T^{n-1}x, T^n x), d(T^{m-1}y, T^m y), d(T^{n-1}x, T^m y), d(T^{m-1}y, T^n x)\} \quad (*)$$

From the remark to previous lemma we have $\delta(O(x)) < \infty, \forall x \in X$. Then it follows from (*) and (2) that

$$\delta(O(T^n x)) \le h\delta(O(T^{n-1}x)), n \in N$$

Inductively we get

$$\delta(O(T^n x)) \le h^n \delta(O(x))$$

Moreover, for any $n \ge 1$ and $m \in N$, we have

$$d(T^n x, T^{n+m} x)) \le \delta(O(T^n x)) \le h^n \delta(O(x))$$

And so, by (6), we get

$$d(T^n x, T^{n+m} x)) \le h^n \tfrac{k(1+h)}{1-kh^2} \max\{d(x,Tx), d(x,T^2 x)\}$$

This completes the proof of the Lemma.
Now we can state our main theorem.
Theorem 2.6 Let (X,d) be an *T*-orbitally complete gqms with the coefficient $k \ge 1$ and $T: X \to X$ a quasi-contraction with constant *h.* on a generalized quasi-metric space (X,d) with the coefficient *k* and (X,d) be *T*-orbitally complete. Then

- **(a)** T has a unique fixed point α in X,
- **(b)** $\lim_{n\to\infty} T^n x = \alpha$, for every $x \in X$ and
- **(c)** $d(T^n x, \alpha)) \le h^n \tfrac{k^2(1+h)}{1-kh^2} \max\{d(x,Tx), d(x,T^2 x)\}$, for all $n \in N$

Proof. Define the sequence $\{x_n\}$ as follows: $x_n = T^n x, n \in N$.
We divide the proof into two cases:
Case I: Suppose $x_p = x_q$ for some $p,q \in N, p \ne q.$ Let $p > q.$ Then $T^p x = T^{p-q} T^q x = T^q x$ i.e. $T^n \alpha = \alpha$ where $n = p - q$ and $T^q x = \alpha$. Now, if $n > 1$, then we have $\alpha = T^n \alpha = T^{rn} \alpha,\ r \in N$ and by Lemma 2.5, we get

$$d(\alpha, T\alpha) = d(T^n \alpha, T^{n+1} \alpha) = d(T^{rn} \alpha, T^{rn+1} \alpha) = d(T^{rn+q} x, T^{rn+q+1} x) \le$$
$$\le \delta(O(T^{rn+q} x)) \le h^{rn+q} \delta(O(x)), \forall r \in N$$

Since $\lim_{r\to\infty} h^{rn+q} = 0$, $d(\alpha, T\alpha) = 0$. So $T\alpha = \alpha$ and hence α is a fixed point of *T*.
Case II: Assume that $x_n \ne x_m$ for all $n \ne m$. Then $\{x_n\} = \{T^n x\}$ is a sequence of distinct point. By lemma 2.5, we have

$$d(x_n, x_{n+m}) = d(T^n x, T^{n+m} x) \le h^n \tfrac{k(1+h)}{1-kh^2} \max\{d(x,Tx), d(x,T^2 x)\}$$

Therefore,

$$\lim_{n\to\infty} d(x_n, x_{n+m}) = 0 \quad (7)$$

It implies that $\{x_n\}$ is a Cauchy sequence in *X*. Since (X,d) *is T-orbitally complete*, there exists a $\alpha \in X$ such that

$$\lim_{n\to\infty} x_n = \alpha \quad (8)$$

We now prove that the limit α is unique. Suppose to the contrary, that is $\alpha' \ne \alpha$ is also $\lim_{n\to\infty} x_n$.

Since $x_n \neq x_m$ for all $n \neq m$, there exists a subsequence $\{x_{n_p}\}$ of $\{x_n\}$ such that $x_{n_p} \neq \alpha$ and $x_{n_p} \neq \alpha'$ for all $p \in N$. Without loss of generality, assume that $\{x_n\}$ is this subsequence. Then, by *quasi-tetrahedral inequality*, we obtain

$$d(\alpha,\alpha') \leq k[d(\alpha,x_n) + d(x_n,x_{n+1}) + d(x_{n+1},\alpha')]$$

Letting *n* tend to infinity, by (7) and (8), we get $d(\alpha,\alpha') = 0$ and so $\alpha = \alpha'$.

Let we prove now that α is a fixed point of T. In contrary, if $\alpha \neq T\alpha$, then there exists a subsequence $\{x_{n_p}\}$ such that $x_{n_p} \neq T\alpha$ and $x_{n_p} \neq \alpha$ for all $p \in N$.

By *quasi-tetrahedral inequality*, we obtain

$$d(\alpha,T\alpha) \leq k[d(\alpha,x_{n_{p-1}}) + d(x_{n_{p-1}},x_{n_p}) + d(x_{n_p},T\alpha)]$$

Then, if $p \to \infty$, we get

$$d(\alpha,T\alpha) \leq k \overline{\lim_{p\to\infty}} d(x_{n_p},T\alpha) \tag{9}$$

From (1),

$$\begin{aligned} d(x_n,T\alpha) &= d(Tx_{n-1},T\alpha) \leq \\ &\leq h\max\{(d(x_{n-1},\alpha), d(x_{n-1},Tx_{n-1}), d(\alpha,T\alpha), d(x_{n-1},T\alpha), d(\alpha,Tx_{n-1})\} = \\ &= h\max\{(d(x_{n-1},\alpha), d(x_{n-1},x_n), d(\alpha,T\alpha), d(x_{n-1},T\alpha), d(\alpha,x_n))\} \end{aligned}$$

Letting *n* tend to infinity, by $\overline{\lim_{n\to\infty}} d(x_n,T\alpha) = \overline{\lim_{n\to\infty}} d(x_{n-1},T\alpha)$, we get

$$\overline{\lim_{n\to\infty}} d(x_n,T\alpha) \leq h\max\{(0,0,d(\alpha,T\alpha), \overline{\lim_{n\to\infty}} d(x_{n-1},T\alpha), 0\} \leq hd(\alpha,T\alpha) \tag{10}$$

From (9) and (10),

$$d(\alpha,T\alpha) \leq k \overline{\lim_{p\to\infty}} d(x_{n_p},T\alpha) \leq k \overline{\lim_{n\to\infty}} d(x_n,T\alpha) \leq khd(\alpha,T\alpha)$$

Since $0 \leq kh < 1$, we have $d(\alpha,T\alpha) = 0$. So α is a fixed point of *T*.

Let we prove now the uniqueness (for case I and II in the same time). Assume that $\alpha' \neq \alpha$ is also a fixed point *of* T. From (1) we get

$$d(\alpha,\alpha') = d(T\alpha,T\alpha') \leq h\max\{(d(\alpha,\alpha'),0,0,d(\alpha,\alpha'),d(\alpha',\alpha)\} \leq hd(\alpha,\alpha')$$

Since $0 \leq h < 1$, we have $\alpha = \alpha'$. So we have proved (a) and (b). By *quasi-tetrahedral inequality* and by Lemma 2.5 we obtain

$$\begin{aligned} d(x_n,\alpha) &\leq k[d(x_n,x_{n+m}) + d(x_{n+m},x_{n+m+1}) + d(x_{n+m+1},\alpha)] \leq \\ &\leq h^n \tfrac{k^2(1+h)}{1-kh^2} \max\{d(x,Tx), d(x,T^2x)\} + kd(x_{n+m},x_{n+m+1}) + kd(x_{n+m+1},\alpha) \end{aligned}$$

Letting *m* tend to infinity, by (7) and (8), we obtain the inequality (c).This completes the proof of the theorem.

Corollary 2.7 By the theorem 2.6, in special case $k = 1$, we obtain an extension of the Cirich's quasi-contraction principle in a generalized metric space presented by B. K. Lahiri and P. Das [12]. We note that in [12] the proof of the main theorem is not correct since it relies in the continuity of the generalized distance *d*, that it is not true always.

We end this paper with an example:

Example 2.8 Let $X=\{0,\frac{1}{2},\frac{3}{4},1\}$ and $T:X\to X$ be a mapping such that $T(\frac{1}{2})=1$ and $T(x)=0$ for $x\in X-\{\frac{1}{2}\}$.

In the ordinary metric space, the inequality (1) is not satisfied for $x=\frac{1}{2}$ and $y=0$:

$$1=d(T\tfrac{1}{2},T0)\le h\max\{d(\tfrac{1}{2},0),d(\tfrac{1}{2},T\tfrac{1}{2}),d(0,T0),d(\tfrac{1}{2},T0),d(0,T\tfrac{1}{2})\}=$$
$$=h\max\{\tfrac{1}{2},\tfrac{1}{2},0,\tfrac{1}{2},1\}=h$$

While for the mapping *T*, it can not be applied the Theorem Ciric [5], although there is unique fixed point, the Theorem 2.7 can be applied in gqms (X,d) with generalized quasi-distance as follows:

$$d(x,y)=\begin{cases}0 & for\ \ x=y\\ 6 & for\ \ x,y\in\{\frac{1}{2},1\},x\neq y\\ 1 & otherwise\end{cases}$$

Then it is easy to see that (X,d) is a generalized quasi-metric space and is not a metric space because it lacks the triangular *inequality:*

$6=d(\frac{1}{2},1)>d(\frac{1}{2},0)+d(0,1)=1+1=2$.

In this generalized quasi-metric with the *coefficient* $k=2$, the inequality (1) is satisfied for all $x,y\in X$:

If $x=y$ or $x,y\in X-\{\frac{1}{2}\}$, the left side of the inequality (1') is zero and consequently it is true for any $h\in[0,\frac{1}{2})$.

If $x=\frac{1}{2}$ and $y\neq\frac{1}{2}$, inequality (1') takes the form

$$1=d(T\tfrac{1}{2},Ty)\le h\max\{d(\tfrac{1}{2},y),d(\tfrac{1}{2},T\tfrac{1}{2}),d(y,Ty),d(\tfrac{1}{2},Ty),d(y,T\tfrac{1}{2})\}=$$
$$=h\max\{d(\tfrac{1}{2},y),6,d(y,Ty),d(\tfrac{1}{2},Ty),d(y,T\tfrac{1}{2})\}=h6$$

which is true for $h\in[\frac{1}{6},\frac{1}{2}=\frac{1}{k})$.

If $x\neq\frac{1}{2}$ and $y=\frac{1}{2}$, inequality (1') takes the form of above case.

All the conditions of Theorem 2.7 are satisfied with $h=[\frac{1}{6},\frac{1}{k}=\frac{1}{2})$. The mapping *T* has unique fixed point: $Fix(T)=\{0\}$ and, for any $x\in X$, the Picard iteration $\{x_n\}$ defined by $x_n=T^nx, n=1,2,\dots$, converges to 0.

The example given above, show that the Theorem 2.7 provides a larger class of mappings than the Theorem 1.1 (Ciric's Theorem [4]).

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J. COMPUTATIONAL ANALYSIS AND APPLICATIONS, VOL. 15, NO.7, 1266-1271, 2013, COPYRIGHT 2013 EUDOXUS PRESS, LLC

Explicit formulas on the second kind q-Euler numbers and polynomials

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Abstract : In [3], we introduced the second kind q-Euler numbers $E_{n,q}$ and polynomials $E_{n,q}(x)$. From these numbers and polynomials, we establish some interesting identities and explicit formulas.

Key words : the second kind Euler numbers and polynomials, the second kind q-Euler numbers and polynomials.

2000 Mathematics Subject Classification : 11B68, 11S40, 11S80

1. Introduction

Throughout this paper, we always make use of the following notations: $\mathbb{N} = \{1, 2, 3, \cdots\}$ denotes the set of natural numbers, $\mathbb{R}$ denotes the set of real numbers, $\mathbb{C}$ denotes the set of complex numbers, $\mathbb{Z}_p$ denotes the ring of p-adic rational integers, $\mathbb{Q}_p$ denotes the field of p-adic rational numbers, and $\mathbb{C}_p$ denotes the completion of algebraic closure of $\mathbb{Q}_p$.

Let ν_p be the normalized exponential valuation of $\mathbb{C}_p$ with $|p|_p = p^{-\nu_p(p)} = p^{-1}$. When one talks of q-extension, q is considered in many ways such as an indeterminate, a complex number $q \in \mathbb{C}$, or p-adic number $q \in \mathbb{C}_p$. If $q \in \mathbb{C}$ one normally assume that $|q| < 1$. If $q \in \mathbb{C}_p$, we normally assume that $|q-1|_p < p^{-\frac{1}{p-1}}$ so that $q^x = \exp(x \log q)$ for $|x|_p \leq 1$.Throughout this paper we use the notation:

$$[x]_q = \frac{1-q^x}{1-q}, \text{ cf. } [1,2,3,4,5] \ .$$

Hence, $\lim_{q\to 1}[x] = x$ for any x with $|x|_p \leq 1$ in the present p-adic case.

For

$$g \in UD(\mathbb{Z}_p) = \{g|g : \mathbb{Z}_p \to \mathbb{C}_p \text{ is uniformly differentiable function}\},$$

Kim[1] defined the p-adic integral on $\mathbb{Z}_p$ as follows:

$$I_1(g) = \int_{\mathbb{Z}_p} g(x)d\mu_{-1}(x) = \lim_{N\to\infty} \sum_{0\leq x<p^N} g(x)(-1)^x. \tag{1.1}$$

From (1.1), we obtain

$$I_{-1}(g_n) = (-1)^n I_{-1}(g) + 2\sum_{l=0}^{n-1}(-1)^{n-1-l}g(l), \text{ (see [1-3])}. \tag{1.2}$$

where $g_n(x) = g(x+n)$.

First, we introduce the second kind Euler numbers E_n and polynomials $E_n(x)$(see [4]). The second kind Euler numbers E_n are defined by the generating function:

$$F(t) = \frac{2e^t}{e^{2t}+1} = \sum_{n=0}^{\infty} E_n \frac{t^n}{n!}. \tag{1.3}$$

We introduce the second kind Euler polynomials $E_n(x)$ as follows:

$$F(x,t) = \frac{2e^t}{e^{2t}+1}e^{xt} = \sum_{n=0}^{\infty} E_n(x)\frac{t^n}{n!}. \tag{1.4}$$

In this paper, we give some interesting identities of the second kind q-Euler numbers and polynomials. By using the fermionic p-adic integral on $\mathbb{Z}_p$, we give recurrence identities the second kind q-Euler numbers and polynomials.

2. The second kind q-Euler numbers and polynomials

In this section, we introduce the second kind q-Euler numbers $E_{n,q}$ and polynomials $E_{n,q}(x)$ and investigate their properties. Let $\mathbb{Z}_+ = \mathbb{N} \cup \{0\}$. In [3], we introduced the second kind q-Euler numbers $E_{n,q}$ and polynomials $E_{n,q}(x)$.

For $q \in \mathbb{C}_p$ with $|1-q|_p < 1$, the second kind q-Euler numbers $E_{n,q}$ are defined by

$$E_{n,q} = \int_{\mathbb{Z}_p} [2x+1]_q^n d\mu_{-1}(x). \tag{2.1}$$

We consider the second kind q-Euler polynomials $E_{n,q}(x)$ as follows:

$$E_{n,q}(x) = \int_{\mathbb{Z}_p} [x+2y+1]_q^n d\mu_{-1}(y). \tag{2.2}$$

The following elementary properties of the second kind q-Euler numbers $E_{n,q}$ and polynomials $E_{n,q}(x)$ are readily derived from (2.1) and (2.2). We, therefore, choose to omit the details involved. More studies and results in this subject we may see references [3], [4].

Proposition 1. For $q \in \mathbb{C}_p$ with $|q-1|_p < 1$, we have

$$\begin{aligned} E_{n,q} &= 2\left(\frac{1}{1-q}\right)^n \sum_{l=0}^{n} \binom{n}{l} (-1)^l q^l \frac{1}{1+q^{2l}} \\ &= 2\sum_{m=0}^{\infty} (-1)^m [2m+1]_q^n. \end{aligned}$$

Proposition 2. For $q \in \mathbb{C}_p$ with $|q-1|_p < 1$ and $n \in \mathbb{Z}_+$, we have

$$\begin{aligned} E_{n,q}(x) &= \int_{\mathbb{Z}_p} [x+2y+1]_q^n d\mu_{-1}(y) \\ &= \sum_{l=0}^{n} \binom{n}{l} [x]_q^{n-l} q^{xl} E_{l,q} \\ &= ([x]_q + q^x E_q)^n \\ &= 2\sum_{m=0}^{\infty} (-1)^m [x+2m+1]_q^n, \end{aligned}$$

Proposition 3(Property of complement).

$$E_{n,q^{-1}}(-x) = (-1)^n q^n E_{n,q}(x)$$

Proposition 4. For $n \in \mathbb{Z}_+$, we have

$$E_{n,q^{-1}}(2) = (-1)^n q^n E_{n,q}(-2)$$

Proposition 5. For $n \in \mathbb{Z}_+$, we have

$$E_{n,q}(2) + E_{n,q} = 2.$$

Proposition 6. For $n \in \mathbb{Z}_+$, we have

$$\left(q^2 E_q + [2]_q\right)^n + E_{n,q} = 2,$$

with the usual convention of replacing $(E_q)^n$ by $E_{n,q}$.

3. Explicit formulas on the second kind q-Euler numbers and polynomials

In this section, we give some interesting identities of the second kind q-Euler numbers $E_{n,q}$ and polynomials $E_{n,q}(x)$.

From (2.1) and (1.1), we have

$$\begin{aligned} \int_{\mathbb{Z}_p} [1-2x]_{q^{-1}}^n d\mu_{-1}(x) &= (-1)^n q^n \int_{\mathbb{Z}_p} [2x-1]_q^n d\mu_{-1}(x) \\ &= (-1)^n q^n E_{n,q}(-2). \end{aligned} \tag{3.1}$$

Therefore, by (3.1) and Proposition 4, we obtain the following theorem.

Theorem 7. For $n \in \mathbb{Z}_+$, we get

$$\int_{\mathbb{Z}_p} [1-2x]_{q^{-1}}^n d\mu_{-1}(x) = E_{n,q^{-1}}(2).$$

Let $n \in \mathbb{N}$. Then, by Proposition 5 and Theorem 7, we obtain the following corollary.

Corollary 8. For $n \in \mathbb{N}$, we have

$$\begin{aligned} \int_{\mathbb{Z}_p} [1-2x]_{q^{-1}}^n d\mu_{-1}(x) &= E_{n,q^{-1}}(2) \\ &= 2 - E_{n,q^{-1}}. \end{aligned}$$

By Corollary 8, we get

$$\begin{aligned} &\int_{\mathbb{Z}_p} [2x+1]_q^k [1-2x]_{q^{-1}}^{n-k} d\mu_{-1}(x) \\ &= \sum_{l=0}^{k} \binom{k}{l} (-1)^{k-l} [2]_q^l q^{k-l} \int_{\mathbb{Z}_p} [1-2x]_{q^{-1}}^{n-l} d\mu_{-1}(x) \\ &= \sum_{l=0}^{k} \binom{k}{l} (-1)^{k-l} [2]_q^l q^{k-l} \left(2 - E_{n-l,q^{-1}}\right). \end{aligned} \tag{3.2}$$

Let $n, k \in \mathbb{Z}_+$ with $n > k$. Then, by (3.2) and Corollary 8, we have

$$\begin{aligned} &\int_{\mathbb{Z}_p} [2x+1]_q^k [1-2x]_{q^{-1}}^{n-k} d\mu_{-1}(x) \\ &= \sum_{l=0}^{n-k} \binom{n-k}{l} (-1)^{n-k-l} [2]_{q^{-1}}^l q^{k+l-n} \int_{\mathbb{Z}_p} [2x+1]_q^{n-l} d\mu_{-1}(x) \\ &= \sum_{l=0}^{n-k} \binom{n-k}{l} (-1)^{n-k-l} [2]_{q^{-1}}^l q^{k+l-n} E_{n-l,q}. \end{aligned} \tag{3.3}$$

Therefore, by comparing the coefficients on the both sides of (3.2) and (3.3), we obtain the following theorem.

Theorem 9. For $n, k \in \mathbb{Z}_+$ with $n > k$, we have

$$\begin{aligned}&\sum_{l=0}^{k}\binom{k}{l}(-1)^{k-l}[2]_q^l q^{k-l}\left(2-E_{n-l,q^{-1}}\right)\\&=\sum_{l=0}^{n-k}\binom{n-k}{l}(-1)^{n-k-l}[2]_{q^{-1}}^l q^{k+l-n}E_{n-l,q}.\end{aligned}$$

By Theorem 9, we have the following corollary.

Corollary 10. For $n, k \in \mathbb{Z}_+$ with $n > k$, we have

$$q^n\sum_{l=0}^{k}\binom{k}{l}(-1)^l[2]_{q^{-1}}^l\left(2-E_{n-l,q^{-1}}\right)=\sum_{l=0}^{n-k}\binom{n-k}{l}(-1)^{n+l}[2]_q^l E_{n-l,q}.$$

By Corollary 8, we have

$$\begin{aligned}&\int_{\mathbb{Z}_p}[2x+1]_q^k[1-2x]_{q^{-1}}^{n_1-k}[2x+1]_q^k[1-2x]_{q^{-1}}^{n_2-k}d\mu_{-1}(x)\\&=\int_{\mathbb{Z}_p}[2x+1]_q^{2k}[1-2x]_{q^{-1}}^{n_1+n_2-2k}d\mu_{-1}(x)\\&=\sum_{l=0}^{2k}\binom{2k}{l}(-1)^{2k-l}[2]_q^l q^{2k-l}\int_{\mathbb{Z}_p}[1-2x]_{q^{-1}}^{n_1+n_2-l}d\mu_{-1}(x)\\&=\sum_{l=0}^{2k}\binom{2k}{l}(-1)^l[2]_q^l q^{2k-l}\left(2-E_{n_1+n_2-l,q^{-1}}\right).\end{aligned}\tag{3.4}$$

Let $n_1, n_2, k \in \mathbb{Z}_+$ with $n_1 + n_2 > 2k$. Then we see that

$$\begin{aligned}&\int_{\mathbb{Z}_p}[2x+1]_q^k[1-2x]_{q^{-1}}^{n_1-k}[2x+1]_q^k[1-2x]_{q^{-1}}^{n_2-k}d\mu_{-1}(x)\\&=\int_{\mathbb{Z}_p}[2x+1]_q^{2k}[1-2x]_{q^{-1}}^{n_1+n_2-2k}d\mu_{-1}(x)\\&=\sum_{l=0}^{n_1+n_2-2k}\binom{n_1+n_2-2k}{l}(-1)^{n_1+n_2-l}[2]_{q^{-1}}^l q^{2k+l-n_1-n_2}\int_{\mathbb{Z}_p}[2x+1]_q^{n_1+n_2-l}d\mu_{-1}(x)\\&=\sum_{l=0}^{n_1+n_2-2k}\binom{n_1+n_2-2k}{l}(-1)^{n_1+n_2-l}[2]_{q^{-1}}^l q^{2k+l-n_1-n_2}E_{n_1+n_2-l,q}.\end{aligned}\tag{3.5}$$

By comparing the coefficients on the both sides of (3.4) and (3.5), we obtain the following theorem.

Theorem 11. Let $n_1, n_2, k \in \mathbb{Z}_+$ with $n_1 + n_2 > 2k$. Then we have

$$\begin{aligned}&q^{n_1+n_2}\sum_{l=0}^{2k}\binom{2k}{l}(-1)^l[2]_{q^{-1}}^l\left(2-E_{n_1+n_2-l,q^{-1}}\right)\\&=\sum_{l=0}^{n_1+n_2-2k}\binom{n_1+n_2-2k}{l}(-1)^{n_1+n_2+l}[2]_q^l E_{n_1+n_2-l,q}.\end{aligned}$$

Let $s \in \mathbb{N}$ with $s \geq 2$. For $n_1, n_2, \ldots, n_s, k \in \mathbb{Z}_+$ with $n_1 + \cdots + n_s > sk$, we have

$$\begin{aligned}
&\int_{\mathbb{Z}_p} \underbrace{[2x+1]_q^k [1-2x]_{q^{-1}}^{n_1-k} \cdots [2x+1]_q^k [1-2x]_{q^{-1}}^{n_s-k}}_{s-\text{times}} d\mu_{-1}(x) \\
&= \sum_{l=0}^{n_1+\cdots+n_s-sk} \binom{n_1+\cdots+n_s-sk}{l} (-1)^{n_1+\cdots+n_s-sk-l} [2]_{q^{-1}}^l q^{sk+l-n_1-\cdots-n_s} \\
&\qquad \times \int_{\mathbb{Z}_p} [2x+1]_q^{n_1+\cdots+n_s-l} d\mu_{-1}(x) \\
&= \sum_{l=0}^{n_1+\cdots+n_s-sk} \binom{n_1+\cdots+n_s-sk}{l} (-1)^{n_1+\cdots+n_s-sk-l} [2]_{q^{-1}}^l q^{sk+l-n_1-\cdots-n_s} \\
&\qquad \times E_{n_1+\cdots+n_s-l,q}.
\end{aligned} \tag{3.6}$$

From the binomial theorem, we note that

$$\begin{aligned}
&\int_{\mathbb{Z}_p} \underbrace{[2x+1]_q^k [1-2x]_{q^{-1}}^{n_1-k} \cdots [2x+1]_q^k [1-2x]_{q^{-1}}^{n_s-k}}_{s-\text{times}} d\mu_{-1}(x) \\
&= \int_{\mathbb{Z}_p} [2x+1]_q^{sk} [1-2x]_{q^{-1}}^{n_1+\cdots+n_s-sk} d\mu_{-1}(x) \\
&= \sum_{l=0}^{sk} \binom{sk}{l} (-1)^{sk-l} [2]_q^l q^{sk-l} \int_{\mathbb{Z}_p} [1-2x]_{q^{-1}}^{n_1+\cdots+n_s-l} d\mu_{-1}(x) \\
&= \sum_{l=0}^{sk} \binom{sk}{l} (-1)^{sk-l} [2]_q^l q^{sk-l} \left(2 - E_{n_1+\cdots+n_s-l,q^{-1}}\right).
\end{aligned} \tag{3.7}$$

Therefore, by (3.6) and (3.7), we obtain the following theorem.

Theorem 12. Let $s \in \mathbb{N}$ with $s \geq 2$. For $n_1, n_2, \ldots, n_s, k \in \mathbb{Z}_+$ with $n_1 + \cdots + n_s > sk$, we have

$$\begin{aligned}
&\sum_{l=0}^{n_1+\cdots+n_s-sk} \binom{n_1+\cdots+n_s-sk}{l} (-1)^{n_1+\cdots+n_s+l} [2]_q^l E_{n_1+\cdots+n_s-l,q} \\
&= q^{n_1+\cdots+n_s} \sum_{l=0}^{sk} \binom{sk}{l} (-1)^l [2]_{q^{-1}}^l \left(2 - E_{n_1+\cdots+n_s-l,q^{-1}}\right).
\end{aligned}$$

ACKNOWLEDGEMENT

This paper has been supported by the 2013 Hannam University Research Fund.

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J. COMPUTATIONAL ANALYSIS AND APPLICATIONS, VOL. 15, NO.7, 1272-1279, 2013, COPYRIGHT 2013 EUDOXUS PRESS, LLC

Second order α-univexity and duality for nondifferentiable minimax fractional programming *

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Abstract

In this paper, we introduce the concept of second order α-univexity by generalizing α-univexity and present a second-order dual for a nondifferentiable minimax fractional programming. Under the assumptions on the functions involving second order α-univexity, weak, strong and strict converse duality theorems are obtained in order to establish a connection between the primal problems and dual problems. Our results extend some existing dual results which were discussed previously in the literature [11, 12, 14, 15, 16].

Keywords. Nondifferentiable minimax fractional programming; Second order duality; second order α-univexity

MR(2000)Subject Classification: 49N15,90C30

1. Introduction

In this paper, we consider the following nondifferentiable minimax fractional programming problem:

$$(P) \qquad Minimize \quad \sup_{y\in Y} \frac{f(x,y)+(x^TBx)^{\frac{1}{2}}}{h(x,y)-(x^TDx)^{\frac{1}{2}}}$$
$$\text{s.t.} \qquad g(x)\le 0,\ x\in R^n,$$

where Y is a compact subset of R^m, $f,h : R^n \times R^m \to R$, $g : R^n \to R^p$ are twice continuously differentiable. B and D are $n\times n$ symmetric positive semidefinite matrices. It is assumed that for each (x,y) in $R^n \times R^m$, $f(x,y)+(x^TBx)^{\frac{1}{2}} \ge 0$ and $h(x,y)-(x^TDx)^{\frac{1}{2}} > 0$.

*Project supported by Soft Science Foundation of Hunan Provincial Scientific Department of China(No.2009CK3081)and Humanities and Social Sciences Foundation of Ministry of Education of China(No.12YJAZH173)

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Since Schmitendorf [1] introduced necessary and sufficient optimality conditions for generalized minimax programming, much attention has been paid to optimality conditions and duality theorems for the minimax fractional programming problems in recent years. Yadav and Mukherjee [2] formulated two dual models for (P) and derived duality theorems for the case of convex differentiable minimax fractional programming. Chandra and Kumar [3] pointed out some omissions in the dual formulation of Yadav and Mukherjee and constructed two modified dual problems for minimax fractional programming problem and proved duality results. Liu and Wu [12, 4], and Ahmad [5] obtained sufficient optimality conditions and duality theorems for (P) assuming the functions involved to be generalized convex.Yang and Hou [6] discussed optimality conditions and duality results for (P) involving generalized convexity assumptions. Bector et al [7] discussed second order duality results for minimax programming problems under generalized binvexity. Later on, Liu [8] extended these results involving second order generalized B-invexity. Husain et al [9] formulated two types of second order dual models for minimax fractional programming problems, and derived weak, strong and strict converse duality theorems under η-bonvexity assumptions. Lai and Lee [10] obtained duality theorems for two parameter-free dual models of nondifferentiable minimax fractional programming problem which involve pseudo-quasi convex functions by using optimality conditions given in [11]. Noor,M.A.[17], Noor,M.A. and Noor,K.I. [18], Mishra and Noor,M.A.[13] introduced some classes of α-invex function by relaxing the definition of an invex function. Mishra, Pant and Rautela [14] introduced the concept of strict pseudo α-invex and quasi α-invex functions. Pant and Rautela [19], and Rautela and Pant [20] introduced various generalizations of α-invex and α-univex functions. Recently, Mishra, Pant and Rautela [16] introduced the concepts of α-univex, pseudo α-univex, strict pseudo α-univex and quasi α-univex functions respectively by unifying the notions of α-invex and univex functions, and derived the sufficient optimality conditions and established duality theorems for three different dual models of problem (P).

In this paper, a new concept of second order α-univexity is introduced by generalizing α-univexity. Under the assumptions on the functions involving second order α-univexity, weak, strong and strict converse duality theorems about a second-order dual for a nondifferentiable minimax fractional programming are established. Our results extend some existing dual results which were discussed previously in the literature [11, 12, 14, 15, 16].

2. Preliminaries

Let $S = \{x \in R^n : g(x) \leq 0\}$ denote the set of all feasible solutions of (P). For each $(x, y) \in R^n \times R^m$, we define

$$J(x) = \{j \in M = \{1, 2, \cdots, m\} : g_j(x) = 0\},$$

$$Y(x) = \{y \in Y : \frac{f(x, y) + (x^T Bx)^{\frac{1}{2}}}{h(x, y) - (x^T Dx)^{\frac{1}{2}}} = \sup_{z \in Y} \frac{f(x, z) + (x^T Bx)^{\frac{1}{2}}}{h(x, z) - (x^T Dx)^{\frac{1}{2}}}\},$$

and

$$K(x) = \{(s, t, \widetilde{y}) \in N \times R^s_+ \times R^{ms} : 1 \leq s \leq n + 1, t = (t_1, t_2, \cdots, t_s) \in R^s_+,$$

$$\sum_{t=1}^{s} t_i = 1, \widetilde{y} = (\overline{y}_1, \overline{y}_2, \cdots, \overline{y}_s), \overline{y}_i \in Y(x), i = 1, 2, \cdots, s\}.$$

In our discussion we shall need the following generalized Schwartz inequality

$$\langle x, Av\rangle \leq \langle x, Ax\rangle^{\frac{1}{2}} \langle v, Av\rangle^{\frac{1}{2}}, \quad for \quad x, v \in R^n, \tag{2.1}$$

the equality holds when $Ax = \lambda Av$, for some $\lambda \geq 0$.

Let X(α-invex set) be a subset of R^n, $\eta : X \times X \to R^n$ be an n-dimensional vector-valued function and $\alpha(x, a) : X \times X \to R_+ \setminus \{0\}$ be a bifunction. Assume that $\phi_0, \phi_1 : R \to R$, $b_0, b_1 : X \times X \times [0, 1] \to R_+ \setminus \{0\}$, $b(x, a) = \lim_{\lambda \to 0} b(x, a, \lambda) \geq 0$, and b does not depend on λ if the function is differentiable.

In the sequel, we introduce a class of second order α-univexity.

Definition 2.1 *A twice differentiable function $f : X \to R$ is said to be second order α-univex at a with respect to b_0, ϕ_0, α and η if there exist functions b_0, ϕ_0, α and η such that, for every $x \in X$, $p \in R^n$, we have*

$$b_0(x, a)\phi_0[f(x) - f(a) + \frac{1}{2}p^T \nabla^2 f(a)p] \geq \langle \alpha(x, a)(\nabla f(a) + \nabla^2 f(a)p), \eta(x, a)\rangle.$$

Definition 2.2 *A twice differentiable function $f : R^n \to R$ over X is said to be second order (strictly) pseudo α-univex at a with respect to b_0, ϕ_0, α and η if there exist functions b_0, ϕ_0, α and η such that, for all $x \in X$, $p \in R^n$,*

$$\langle \alpha(x, a)(\nabla f(a) + \nabla^2 f(a)p), \eta(x, a)\rangle \geq 0 \Rightarrow b_0(x, a)\phi_0[f(x) - f(a) + \frac{1}{2}p^T \nabla^2 f(a)p] \geq (>)0.$$

Definition 2.3 *A twice differentiable function $f : R^n \to R$ over X is said to be second order quasi α-univex at x_0 with respect to b_0, ϕ_0, α and η if there exist functions b_0, ϕ_0, α and η such that, for all $x \in X$, $p \in R^n$,*

$$b_0(x, a)\phi_0[f(x) - f(a) + \frac{1}{2}p^T \nabla^2 f(a)p] > 0 \Rightarrow \langle \alpha(x, a)(\nabla f(a) + \nabla^2 f(a)p), \eta(x, a)\rangle > 0.$$

Remark 2.1 *It is obvious that the second order α-univexity generalizes the α-univexity in [16].*

The following theorem will be needed in the proofs of strong duality theorems:

Theorem 2.1 *(Necessary conditions)[11]Let x^* be a solution of (P) satisfying $x^{*T}Bx^* > 0$, $x^{*T}Dx^* > 0$, and let $\nabla g_j(x^*), j \in J(x^*)$ be linearly independent. There exist $(s^*, t^*, \overline{y}^*) \in K(x^*), \lambda_0 \in R_+$, $w, v \in R^n$ and $\mu^* \in R^p_+$ such that*

$$\sum_{i=1}^{s^*} t_i^* \{\nabla f(x^*, \overline{y}_i^*) + Bw - \lambda_0(\nabla h(x^*, \overline{y}_i^*) - Dv)\} + \nabla \sum_{j=1}^{m} \mu_j^* g_j(x^*) = 0,$$

$$f(x^*, \overline{y}_i^*) + (x^{*T}Bx^*)^{\frac{1}{2}} - \lambda_0(h(x^*, \overline{y}_i^*) - (x^{*T}Dx^*)^{\frac{1}{2}}) = 0, i = 1, 2, \cdots, s^*,$$

$$\sum_{j=1}^{m} \mu_j^* g_j(x^*) = 0,$$

$$t_i^* \geq 0, \sum_{i=1}^{s^*} t_i^* = 1, i = 1, 2, \cdots, s^*.$$

$$w^T B w \leq 1, \ v^T D v \leq 1,$$

$$(x^{*T} B x^*)^{\frac{1}{2}} = x^{*T} B w, \ (x^{*T} D x^*)^{\frac{1}{2}} = x^{*T} D v.$$

3. Second order duality

By utilizing the optimality conditions of the previous section, we formulate the second order dual to (P)as follows:

$$(D) \qquad \max_{(s,t,\overline{y}) \in K(z)} \sup_{(z,\mu,\lambda,w,v,p) \in H_1(s,t,\overline{y})} \lambda,$$

where $H_1(s,t,\overline{y})$ denotes the set of all $(z,\mu,\lambda,w,v,p) \in R^n \times R_+^m \times R_+ \times R^n \times R^n \times R^n$ satisfying

$$\begin{aligned} \sum_{i=1}^{s} t_i[\nabla f(z,\overline{y}_i) \quad &+ \quad Bw - \lambda(\nabla h(z,\overline{y}_i) - Dv)] + \nabla^2 \sum_{i=1}^{s} t_i(f(z,\overline{y}_i) - \lambda h(z,\overline{y}_i))p \\ &+ \quad \nabla \sum_{j=1}^{m} \mu_j g_j(z) + \nabla^2 \sum_{j=1}^{m} \mu_j g_j(z) p = 0, \end{aligned} \tag{3.1}$$

$$\sum_{i=1}^{s} t_i[f(z,\overline{y}_i) + z^T B w - \lambda(h(z,\overline{y}_i) - z^T D v)] - \tfrac{1}{2} p^T \nabla^2 \sum_{i=1}^{s} t_i(f(z,\overline{y}_i) - \lambda h(z,\overline{y}_i))p \geq 0, \tag{3.2}$$

$$\sum_{j=1}^{m} \mu_j g_j(z) - \tfrac{1}{2} p^T \nabla^2 \sum_{j=1}^{m} \mu_j g_j(z) p \geq 0, \tag{3.3}$$

$$w^T B w \leq 1, \ v^T D v \leq 1, \ (z^T B z)^{\frac{1}{2}} = z^T B w, \ (z^T D z)^{\frac{1}{2}} = z^T D v. \tag{3.4}$$

If, for a triplet $(s,t,\overline{y}) \in K(z)$, the set $H_1(s,t,\overline{y}) = \emptyset$, then we define the supremum over it to be $-\infty$. Let Z denote the set of all feasible solutions of (D). In this section, we denote $\psi(.) = \sum_{i=1}^{s} t_i[f(.,\overline{y}_i) + (.)^T B w - \lambda(h(.,\overline{y}_i) - (.)^T D v)]$.

Theorem 3.1 *(Weak Duality)Let x and $(z,\mu,\lambda,s,t,w,v,p)$ be feasible solutions of (P) and (D), respectively. If, for each $(z,\mu,\lambda,s,t,w,v,p) \in Z$, one of the following conditions holds:*
(i)$\mu^T g(.)$ is second order α-univex at z with respect to $b_1, \phi_1, \alpha, \eta$ and $\psi(.)$ is second order α-univex at z with respect to $b_0, \phi_0, \alpha, \eta$ with $\phi_0(V) \geq 0 \Rightarrow V \geq 0$ and $\phi_1(V) \leq V$,
(ii)$\mu^T g(.)$ is second order quasi α-univex at z with respect to $b_1, \phi_1, \alpha, \eta$ and $\psi(.)$ is second order pseudo α-univex at z with respect to $b_0, \phi_0, \alpha, \eta$ with $V < 0 \Rightarrow \phi_0(V) < 0$ and $V \leq 0 \Rightarrow \phi_1(V) \leq 0$,
(iii)$\mu^T g(.)$ is second order strictly pseudo α-univex at z with respect to $b_1, \phi_1, \alpha, \eta$ and $\psi(.)$ is second order quasi α-univex at z with respect to $b_0, \phi_0, \alpha, \eta$ with $V < 0 \Rightarrow \phi_0(V) < 0$ and $V \leq 0 \Rightarrow \phi_1(V) \leq 0$,
then

$$\sup_{y \in Y} \frac{f(x,y) + (x^T B x)^{\frac{1}{2}}}{h(x,y) - (x^T D x)^{\frac{1}{2}}} \geq \lambda.$$

Proof. Suppose the conclusion is not true, i.e.,

$$\sup_{y\in Y} \frac{f(x,y)+(x^TBx)^{\frac{1}{2}}}{h(x,y)-(x^TDx)^{\frac{1}{2}}} < \lambda.$$

Then, we have

$$f(x,y)+(x^TBx)^{\frac{1}{2}} - \lambda\{h(x,y)-(x^TDx)^{\frac{1}{2}}\} < 0,\ \forall y \in Y.$$

That is

$$t_i[f(x,\overline{y}_i)+(x^TBx)^{\frac{1}{2}} - \lambda\{h(x,\overline{y}_i)-(x^TDx)^{\frac{1}{2}}\}] \leq 0,\ i=1,2,\cdots,s.$$

From (2.1),(3.4) and the above inequality, we obtain

$$\begin{aligned}
\sum_{i=1}^{s} t_i[f(x,\overline{y}_i)+x^TBw-\lambda\{h(x,\overline{y}_i)-x^TDv\}] &\leq \sum_{i=1}^{s} t_i[f(x,\overline{y}_i)+(x^TBx)^{\frac{1}{2}}-\lambda\{h(x,\overline{y}_i)-(x^TDx)^{\frac{1}{2}}\}] \\
&< 0 \\
&\leq \sum_{i=1}^{s} t_i[f(z,\overline{y}_i)+z^TBw-\lambda\{h(z,\overline{y}_i)-z^TDv\}] \\
&- \tfrac{1}{2}p^T\nabla^2 \sum_{i=1}^{s} t_i(f(z,\overline{y}_i)-\lambda h(z,\overline{y}_i))p.
\end{aligned}$$

That is

$$\psi(x) < \psi(z) - \tfrac{1}{2}p^T\nabla^2\psi(z)p. \tag{3.5}$$

If condition (i) holds, then

$$\begin{aligned}
b_0(x,z)\phi_0[\psi(x)-\psi(z)+\tfrac{1}{2}p^T\nabla^2\psi(z)p] &\geq \langle\alpha(x,z)(\nabla\psi(z)+\nabla^2\psi(z)p),\eta(x,z)\rangle \\
&= \langle\alpha(x,z)(-\nabla\mu^Tg(z)-\nabla^2\mu^Tg(z)p),\eta(x,z)\rangle \\
&\geq -b_1(x,z)\phi_1[\mu^Tg(x)-\mu^Tg(z)+\tfrac{1}{2}p^T\nabla^2\mu^Tg(z)p] \\
&\geq \mu^Tg(z)-\mu^Tg(x)-\tfrac{1}{2}p^T\nabla^2\mu^Tg(z)p \geq 0
\end{aligned} \tag{3.6}$$

Since $\phi_0(V) \geq 0 \Rightarrow V \geq 0$ and $b_0 > 0$, we have

$$\psi(x) \geq \psi(z) - \frac{1}{2}p^T\nabla^2\psi(z)p,$$

which contradicts with (3.5). Hence, the assertion is true.

If condition (ii) holds, by the positivity of b_0 and $V < 0 \Rightarrow \phi_0(V) < 0$, then from (3.5), we get

$$b_0(x,z)\phi_0[\psi(x)-\psi(z)+\frac{1}{2}p^T\nabla^2\psi(z)p] < 0.$$

Using the second order pseudo α-univexity, we can deduce the following inequality

$$\langle\alpha(x,z)(\nabla\psi(z)+\nabla^2\psi(z)p),\eta(x,z)\rangle < 0. \tag{3.7}$$

Taking into account (3.1), (3.7) and the positivity of $\alpha(x,z)$, we have

$$\langle(\nabla\mu^Tg(z)+\nabla^2\mu^Tg(z)p),\eta(x,z)\rangle > 0. \tag{3.8}$$

According to $\mu^T g(x) \leq 0$, (3.3), the positivity of $b_1(x,z)$ and $V \leq 0 \Rightarrow \phi_1(V) \leq 0$, we have

$$b_1(x,z)\phi_1[\mu^T g(x) - \mu^T g(z) + \tfrac{1}{2}p^T \nabla^2 \mu^T g(z)p] \leq 0. \tag{3.9}$$

By the second order quasi α-univexity of $\mu^T g(.)$ and the above inequality, we get

$$\langle \alpha(x,z)(\nabla \mu^T g(z) + \nabla^2 \mu^T g(z)p), \eta(x,z)\rangle \leq 0.$$

That is,

$$\langle (\nabla \mu^T g(z) + \nabla^2 \mu^T g(z)p), \eta(x,z)\rangle \leq 0,$$

which contradicts with(3.8).

For condition (iii), the proof is similar to that of condition (ii).

Remark 3.1 *If we take $\phi_0,\ \phi_1$ as identity maps, $b_0 = b_1 = 1,\ \alpha_0 = \alpha_1 = 1,\ p = 0$ and $\eta(x_1, x_0) = x_1 - x_0$ in the above theorem, we get Theorem 4.1 in [11]. If we take $\phi_0,\ \phi_1$ as identity maps, $b_0 = b_1 = 1,\ \alpha_0 = \alpha_1 = 1,\ p = 0$ and remove the quadratic terms from the numerator and denominator of objective function and from the constraints in the above theorem, we get Theorem 3.1 in [12]. If we take $\phi_0,\ \phi_1$ as identity maps, $b_0 = b_1 = 1,\ p = 0$, we get Theorem 4.1 in [14]. If we take $\alpha_0 = \alpha_1 = 1,\ p = 0$ in the above theorem, we get Theorem 2 in [15]. If we take $p = 0$ in the above theorem, we get Theorem 4.1 in [16].*

Theorem 3.2 *(Strong Duality)Let x^* be an optimal solution of (P) and $\nabla g_j(x^*), j \in J(x^*)$ be linearly independent, then there exist $(s^*, t^*, \overline{y}^*) \in K(x^*)$ and $(x^*, u^*, \lambda^*, w^*, v^*, p^* = 0) \in H_1(s^*, t^*, \overline{y}^*)$ such that $(x^*, u^*, \lambda^*, s^*, t^*, w^*, v^*, p^* = 0)$ is feasible for (D), and the corresponding objective values of (P) and (D) are equal. If, in addition, the assumptions of Weak Duality hold for all feasible solutions of (P) and (D), then $(x^*, u^*, \lambda^*, s^*, t^*, w^*, v^*, p^* = 0)$ is an optimal solution of (D).*

Proof. By Theorem 2.1, there exist $(s^*, t^*, \overline{y}^*) \in K(x^*)$ and $(x^*, u^*, \lambda^*, w^*, v^*, p^* = 0) \in H_1(s^*, t^*, \overline{y}^*)$ such that $(x^*, u^*, \lambda^*, s^*, t^*, w^*, v^*, p^* = 0)$ is feasible for (D) and

$$\lambda^* = \frac{f(x^*, \overline{y}_i^*) + (x^{*T} B x^*)^{\frac{1}{2}}}{g(x^*, \overline{y}_i^*) - (x^{*T} D x^*)^{\frac{1}{2}}}.$$

The optimality of the feasible solution for (D)can be derived from Theorem 3.1.

Remark 3.2 *If we take $\phi_0,\ \phi_1$ as identity maps, $b_0 = b_1 = 1,\ \alpha_0 = \alpha_1 = 1,\ p^* = 0$ and $\eta(x_1, x_0) = x_1 - x_0$ in the above theorem, we get Theorem 4.2 in [11]. If we take $\phi_0,\ \phi_1$ as identity maps, $b_0 = b_1 = 1,\ \alpha_0 = \alpha_1 = 1,\ p^* = 0$ and remove the quadratic terms from the numerator and denominator of objective function and from the constraints in the above theorem, we get Theorem 3.2 in [12]. If we take $\phi_0,\ \phi_1$ as identity maps, $b_0 = b_1 = 1,\ p^* = 0$, we get Theorem 4.2 in [14]. If we take $\alpha_0 = \alpha_1 = 1,\ p^* = 0$ in the above theorem, we get Theorem 3 in [15]. If we take $p^* = 0$ in the above theorem, we get Theorem 4.2 in [16].*

Theorem 3.3 *(Strict Converse Duality) Let x^* and $(\overline{z}, \overline{\mu}, \overline{\lambda}, \overline{s}, \overline{t}, \overline{w}, \overline{v}, \overline{p})$ be optimal solutions of (P) and (D), respectively. Assume that the hypothesis of Theorem 3.2 is fulfilled, if one of the following conditions holds:*
(i)$)\overline{\mu}^T g(.)$ is second order strictly α-univex at $\overline{z}$ with respect to $b_1, \phi_1, \alpha, \eta$ and $\sum_{i=1}^{\overline{s}} \overline{t}_i[f(., \overline{y}_i) + \langle ., B\overline{w}\rangle - \overline{\lambda}(h(., \overline{y}_i) + \langle ., D\overline{v}\rangle)]$ is second order strictly α-univex at $\overline{z}$ with respect to $b_0, \phi_0, \alpha, \eta$ with $\phi_0(V) \geq 0 \Rightarrow V \geq 0$ and $\phi_1(V) \leq V$;
(ii)$\overline{\mu}^T g(.)$ is second order quasi α-univex at $\overline{z}$ with respect to $b_1, \phi_1, \alpha, \eta$ and $\sum_{i=1}^{\overline{s}} \overline{t}_i[f(., \overline{y}_i) + \langle ., B\overline{w}\rangle - \overline{\lambda}(h(., \overline{y}_i) + \langle ., D\overline{v}\rangle)]$ is second order strictly pseudo α-univex at $\overline{z}$ with respect to $b_0, \phi_0, \alpha, \eta$ with $V < 0 \Rightarrow \phi_0(V) < 0$ and $V \leq 0 \Rightarrow \phi_1(V) \leq 0$.
Then $x^* = \overline{z}$, that is, $\overline{z}$ is an optimal solution for (P) and

$$\sup_{y\in Y} \frac{f(\overline{z}, y) + (\overline{z}^T B\overline{z})^{\frac{1}{2}}}{h(\overline{z}, y) - (\overline{z}^T D\overline{z})^{\frac{1}{2}}} = \overline{\lambda}.$$

Proof. Suppose that $x^* \neq \overline{z}$. From Theorem 3.2, we know that there exist $(s^*, t^*, \overline{y}^*) \in K(x^*)$ and $(x^*, u^*, \lambda^*, w^*, v^*, p^* = 0) \in H_1(s^*, t^*, \overline{y}^*)$ such that $(x^*, u^*, \lambda^*, s^*, t^*, w^*, v^*, p^* = 0)$ is optimal for (D) and

$$\lambda^* = \sup_{y\in Y} \frac{f(x^*, y) + (x^{*T} Bx^*)^{\frac{1}{2}}}{g(x^*, y) - (x^{*T} Dx^*)^{\frac{1}{2}}} = \overline{\lambda}. \tag{3.10}$$

The remaining part of the proof is similar to that of Theorem 3.1 in which x is replaced by x^* and $(z, \mu, \lambda, s, t, w, v, p)$ by $(\overline{z}, \overline{\mu}, \overline{\lambda}, \overline{s}, \overline{t}, \overline{w}, \overline{v}, \overline{p})$, and we get

$$\sup_{y\in Y} \frac{f(x^*, y) + (x^{*T} Bx^*)^{\frac{1}{2}}}{g(x^*, y) - (x^{*T} Dx^*)^{\frac{1}{2}}} > \overline{\lambda},$$

which contradicts with (3.10). Therefore, we conclude that $x^* = \overline{z}$.

Remark 3.3 *If we take $\phi_0,\ \phi_1$ as identity maps, $b_0 = b_1 = 1,\ \alpha_0 = \alpha_1 = 1,\ \overline{p} = 0$ and $\eta(x_1, x_0) = x_1 - x_0$ in the above theorem, we get Theorem 4.3 in [11]. If we take $\phi_0,\ \phi_1$ as identity maps, $b_0 = b_1 = 1,\ \alpha_0 = \alpha_1 = 1,\ \overline{p} = 0$ and remove the quadratic terms from the numerator and denominator of objective function and from the constraints in the above theorem, we get Theorem 3.3 in [12]. If we take $\phi_0,\ \phi_1$ as identity maps, $b_0 = b_1 = 1,\ \overline{p} = 0$, we get Theorem 4.3 in [14]. If we take $\alpha_0 = \alpha_1 = 1,\ \overline{p} = 0$ in the above theorem, we get Theorem 4 in [15]. If we take $\overline{p} = 0$ in the above theorem, we get Theorem 4.3 in [16].*

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J. COMPUTATIONAL ANALYSIS AND APPLICATIONS, VOL. 15, NO.7, 1280-1290, 2013, COPYRIGHT 2013 EUDOXUS PRESS, LLC

SOME PROPERTIES OF THE INTERVAL-VALUED GENERALIZED FUZZY INTEGRAL WITH RESPECT TO A FUZZY MEASURE BY MEANS OF AN INTERVAL-REPRESENTABLE GENERALIZED TRIANGULAR NORM

LEE-CHAE JANG

Abstract. We consider the generalized fuzzy integral introduced by Fang[4] and use the concept of interval-valued functions which is used for representing uncertain functions. In this paper, we define the interval-valued generalized fuzzy integral with respect to a fuzzy measure by means of an interval-representable generalized triangular norm of measurable interval-valued functions and investigate some characterizations and convergence properties of them.

1. Introduction

Fang[4], Wu-Wang-Ma[22], and Wu-Ma-Song-Zhang[23] introduced the theory of the generalized fuzzy integral(for short, the (G) fuzzy integral) by means of a generalized triangular norm. Many researchers[5,16,17,20,22-26] have been studying fuzzy measure and fuzzy integral theory used in the decision making and information theory.

The main idea of this study is the concept of interval-valued functions which is used for representing uncertain functions. Aubin[1], Aumann[2], Beliakov et al.[3], Jang et al.[6-12], Schjear-Jacoben[18], Weichselberger[21], and Zhang et al.[24-26] have been researching various integrals of uncertain functions, for examples, the Lebesgue integral, the fuzzy integral, and the Choquet integral of interval-valued functions, the calculation of economic uncertain, and the theory of interval-probability, etc.

In this paper, we define the interval-valued generalized fuzzy integral (for short, (IG) fuzzy integral) with respect to a fuzzy measure by means of an interval-representable generalized triangular norm of measurable interval-valued functions and investigate some characterizations and convergence properties of them.

In section 2, we list definitions and basic properties of a fuzzy measure, a generalized triangular norm, and the (G) fuzzy integral with respect to a fuzzy measure by means of a generalized triangular norm of measurable functions. In section 3, we define the (IG) fuzzy integral of interval-valued functions by means of an interval-representable generalized triangular norm of measurable interval-valued functions and investigate some characterizations of them. In section 4, we investigate some convergence properties of the (IG) fuzzy integral with respect to a fuzzy measure by means of an interval-representable generalized triangular norm of measurable interval-valued functions. In section 5, we give a brief summary results and some conclusions.

1991 *Mathematics Subject Classification.* 28E10, 28E20, 03E72, 26E50 11B68.

Key words and phrases. fuzzy measure, interval-representable generalized triangular norm, generalized fuzzy integral, interval-valued function, convergence theorem.

2. Definitions and Preliminaries

In this section, we first introduce some definitions and basic properties of a fuzzy measure, the (G) fuzzy integral with respect to a fuzzy measure by means of a generalized triangular norm of measurable functions. Let X be a set, $\mathcal{B}$ be a $\sigma-$algebra of subsets of X, and $(X, \mathcal{B})$ be a measurable space. Denote $\mathcal{F}(X)$ by the set of all nonnegative measurable functions on $(X, \mathcal{B})$ and $\mathbb{N} = \{1, 2, 3, \cdots\}$.

Definition 2.1. ([3-26]) (1) A set function $\mu : \mathcal{B} \to [0, \infty]$ is called a fuzzy measure if
(i) $\mu(\emptyset) = 0$ and
(ii) $A, B \in \mathcal{B}$ and $A \subset B$ implies $\mu(A) \leq \mu(B)$.

It is easy to see that if m is the Lebesgue measure on X and we define $\mu = m^2$, then μ is a fuzzy measure which satisfies the two conditions of Definition 2.1. Since μ is not additive, we can see that this fuzzy measure is not a classical measure.

Definition 2.2. ([22,23]) Let $D = [0, \infty]^2 \setminus \{(0, \infty), (\infty, 0)\}$. The mapping $T : D \to [0, \infty]$ is said to be a generalized triangular norm if it satisfies the following conditions
(i) $T[0, x] = 0$ for all $x \in [0, \infty)$ and exists an $e \in (0, \infty]$ such that $T[x, e] = x$ for each $x[0, \infty]$. In this case, e is said to be the unit element of T,
(ii) $T[x, y] = T[y, x]$ for all $(x, y) \in D$,
(iii) $T[a, b] \leq T[c, d]$ whenever $a \leq c,\ b \leq d$, and
(iv) if $\{(x_n, y_n)\} \in D$, $(x, y) \in D$, $x_n \searrow x$, and $y_n \nearrow y$, then $T[x_n, y_n] \longrightarrow T[x, y]$.

Remark 2.1. $T_1[x, y] = \min\{x, y\}$ and $T_2[x, y] = kxy(k > 0)$ are generalized triangular norms and the identities of T_1 and T_2 are ∞ and $\frac{1}{k}$, respectively (see [4]).

Definition 2.3. ([22,23]) Let $(X, \mathcal{B}, \mu)$ be a fuzzy measure space and T be a generalized triangular norm. If $A \in \mathcal{B}$ and $f \in \mathcal{F}(X)$, then the (G) fuzzy integral with respect to μ by means of T of f on A is defined by

$$(G) \int_A f d\mu = \sup_{\alpha > 0} T[\alpha, \mu_{A,f}(\alpha)], \tag{1}$$

where $\mu_{A,f}(\alpha) = \mu(A \cap \{x \in X \mid f(x) \geq \alpha\})$ for all $\alpha \in [0, \infty)$.

We remark that the Sugeno integral defined by M. Sugeno[20] and the (N) fuzzy integrals defined by N. Shilkret[19] are the special kinds of (G) fuzzy integrals and the corresponding generalized triangular norms are $T[x, y] = \min\{x, y\}$ and $T[x, y] = xy$, respectively. Recall that

$$\overline{\lim}_{n \to \infty} f_n = \inf_{k \geq 1} \sup_{n \geq k} \{f_n\}, \tag{2}$$

for all $\{f_n\} \subset \mathcal{F}(X)$. In [4], the authors have shown the following theorems which are convergence properties of the (G) fuzzy integral.

Theorem 2.1. ([4]) *Let $\{f_n\} \subset \mathcal{F}(X)$, $f \in \mathcal{F}(X)$, $A \in \mathcal{B}$, and $f_n \searrow f$ on A. Then we have*

$$\lim_{n\to\infty}(G)\int_A f_n d\mu = (G)\int_A f d\mu \tag{3}$$

if and only if the following conditions are satisfied

(i) *for any given $\varepsilon > 0$ there exists $n_0 \in \mathbb{N}$ such that*

$$\mu_{A,f_{n_0}}(c_0+\varepsilon) < \infty, \tag{4}$$

where $c_0 = sup\{a > 0 : T[a,\infty] \leq (G)\int_A f d\mu\}$ and $\mu_{A,f_{n_0}}(c_0+\varepsilon) = \mu(A \cap \{x \in X \mid f_{n_0}(x) \geq c_0+\varepsilon\})$ and

(ii) *for any $\{\alpha_n\}$ with $\alpha_n \nearrow \infty$ or $\alpha_n \searrow 0$,*

$$\overline{\lim}_{n\to\infty} T[\alpha_n, \mu_{A,f_n}(\alpha_n)] \leq (G)\int_A f d\mu \tag{5}$$

where $\mu_{A,f_n}(\alpha) = \mu(A \cap \{x \in X \mid f_n(x) \geq \alpha\})$ for all $n \in \mathbb{N}$ and $\alpha \in \mathbb{R}^+$.

Theorem 2.2. ([4]) *Let $\{f_n\} \subset \mathcal{F}(X)$, $f \in \mathcal{F}(X)$, $\mu(A) < \infty$, and $f_n \searrow f$. Then we have*

$$\lim_{n\to\infty}(G)\int_A f_n d\mu = (G)\int_A f d\mu \tag{6}$$

if and only if for any $\{\alpha_n\}$ with $\alpha_n \nearrow \infty$,

$$\overline{\lim}_{n\to\infty} T[\alpha_n, \mu_{A,f_n}(\alpha_n)] \leq (G)\int_A f d\mu, \tag{7}$$

where $\mu_{A,f_n}(\alpha) = \mu(A \cap \{x \in X \mid f_n(x) \geq \alpha\})$ for all $n \in \mathbb{N}$ and $\alpha \in \mathbb{R}^+$.

3. The (IG) fuzzy integral of measurable interval-valued functions

In this section, we consider the intervals and define an interval-valued generalized triangular norm. Let $I(Y)$ be the set of all bounded closed intervals (intervals, for short) in Y as follows:

$$I(Y) = \{\overline{a} = [a_l, a_r] \mid a_l, a_r \in Y \text{ and } a_l \leq a_r\}, \tag{8}$$

where Y is $[0,\infty)$ or $[0,\infty]$. For any $a \in \mathbb{R}^+$, we define $a = [a,a]$. Obviously, $a \in I(\mathbb{R}^+)$ (see[3, 9-12, 18, 21, 24-26]).

Definition 3.1. If $\overline{a} = [a_l,a_r], \overline{b} = [b_l,b_r], \overline{a}_n = [a_{n,l},a_{n,r}], \overline{a}_\alpha = [a_{\alpha,l},a_{\alpha,r}] \in I(Y)$ for all $n \in \mathbb{N}$ and $\alpha \in [0,\infty)$, and $k \in [0,\infty)$, then we define arithmetic, maximum, minimum, order, inclusion, supremum, and infinimum operations as follows:

(1) $\overline{a}+\overline{b} = [a_l+b_l, a_r+b_r]$,
(2) $k\overline{a} = [ka_l, ka_r]$,
(3) $\overline{a}\overline{b} = [a_lb_l, a_rb_r]$,
(4) $\overline{a} \vee \overline{b} = [a_l \vee b_l, a_r \vee b_r]$,
(5) $\overline{a} \wedge \overline{b} = [a_l \wedge b_l, a_r \wedge b_r]$,
(6) $\overline{a} \leq \overline{b}$ if and only if $a_l \leq b_l$ and $a_r \leq b_r$,
(7) $\overline{a} < \overline{b}$ if and only if $\overline{a} \leq \overline{b}$ and $\overline{a} \neq \overline{b}$,
(8) $\overline{a} \subset \overline{b}$ if and only if $b_l \leq a_l$ and $a_r \leq b_r$,

(9) $\sup_n \overline{a}_n = [\sup_n a_{n,l}, \sup_n a_{n,r}]$,
(10) $\inf_n \overline{a}_n = [\inf_n a_{n,l}, \inf_n a_{n,r}]$,
(11) $\sup_\alpha \overline{a}_\alpha = [\sup_\alpha a_{\alpha,l}, \sup_\alpha a_{\alpha,r}]$, and
(12) $\inf_\alpha \overline{a}_\alpha = [\inf_\alpha a_{\alpha,l}, \inf_\alpha a_{\alpha,r}]$.

Note that if a mapping $d_H : I(Y) \times I(Y) \to [0,\infty]$ is defined by

$$d_H(A,B) = max\left\{\sup_{x\in A}\inf_{y\in B}|x-y|, \sup_{y\in B}\inf_{x\in A}|x-y|\right\}, \tag{9}$$

for all $A, B \in I(Y)$, then d_H is called a Hausdorff metric and $(I(Y), d_H)$ is a metric space. It is well-known that for every $\overline{a} = [a_l, a_r], \overline{b} = [b_l, b_r] \in I(Y)$,

$$d_H(\overline{a},\overline{b}) = max\left\{|a_l - b_l|, |a_r - b_r|\right\}. \tag{10}$$

For a sequence of intervals $\{\overline{a}_n\}$, we say that $\{\overline{a}_n\}$ converges in the Hausdorff metric to $\overline{a}$, in symbols, $d_H - \lim_{n\to\infty} \overline{a}_n = \overline{a}$ if $\lim_{n\to\infty} d_H(\overline{a}_n, \overline{a}) = 0$. Then, it is easy to see that

$$d_H - \lim_{n\to\infty} \overline{a}_n = \overline{a} \text{ if and only if } \lim_{n\to\infty} a_{n,l} = a_l \text{ and } \lim_{n\to\infty} a_{n,r} = a_r. \tag{11}$$

Now, we consider an interval-representable generalized triangular norm as follows(see [3]):

Definition 3.2. Let $\overline{D} = I([0,\infty])^2 \setminus \{(0,\infty),(\infty,0)\}$. The mapping $\overline{T} : \overline{D} \to I([0,\infty])$ is called an interval-representable generalized triangular norm if there are two generalized triangular norm T_l and T_r such that $T_l \leq T_l$ and $\mathfrak{T} = [T_l, T_r]$.

Theorem 3.1. *If we take $\mathfrak{T}_1[\overline{x},\overline{y}] = \min\{\overline{x},\overline{y}\}$ and $\mathfrak{T}_2[\overline{x},\overline{y}] = k\overline{x}\,\overline{y}(k>0)$, then $\mathfrak{T}_1$ and $\mathfrak{T}_2$ are interval-representable generalized triangular norms.*

Proof. If we define $T_{1,l}[x,y] = \min\{x,y\}$ and $T_{1,r}[x,y] = \min\{x,y\}$, then, by Remark 2.1, $T_{1,l}$ and $T_{1,r}$ are generalized triangular norms. Thus, by Definition 3.2, we see that $\mathfrak{T}_1 = [T_{1,l}, T_{1,r}$ is an interval-representable generalized triangular norm. Similarly, if we define $T_{2,l}[x,y] = T_{2,r}[x,y] = k\overline{xy}(k>0)$, then, by Remark 2.1, $T_{2,l}$ and $T_{2,r}$ are generalized triangular norms. By Definition 3.2, we see that $\mathfrak{T}_2 = [T_{2,l}, T_{2,r}$ is an interval-representable generalized triangular norm.

Let $\mathcal{IF}(X)$ the set of all measurable interval-valued functions $\overline{f} : X \to I([0,\infty)) \setminus \{\emptyset\}$. Then we define the (IG) fuzzy integral with respect to a fuzzy measure by means of an interval-representable generalized triangular norm of interval-valued functions as follows.

Definition 3.3. Let $(X, \mathcal{B}, \mu)$ be a fuzzy measure space, $\mathfrak{T} = [T_l, T_r]$ be an interval-representable generalized triangular norm, $A \in \mathcal{B}$, and $\overline{f} = [f_l, f_r] \in \mathcal{IF}(X)$.

(1) An interval-valued function $\overline{f}$ is said to be measurable if for any open set $O \subset [0,\infty)$,

$$\overline{f}^{-1}(O) = \{x \in X \mid \overline{f}(x) \cap O \neq \emptyset\} \in \mathcal{B}. \tag{12}$$

(2) The (IG) fuzzy integral with respect to μ by means of $\mathfrak{T}$ of $\overline{f}$ on A is defined by

$$(IG)\int_A \overline{f} d\mu = \sup_{\alpha>0} \mathfrak{T}[\alpha, \mu_{A,\overline{f}}(\alpha)], \tag{13}$$

where $\mu_{A,\overline{f}}(\alpha) = [\mu_{A,f_l}(\alpha), \mu_{A,f_r}(\alpha)]$ for all $\alpha \in [0,\infty)$.

(3) $\overline{f}$ is said to be integrable on A if

$$(IG)\int_A \overline{f}d\mu \in \mathcal{P}([0,\infty))\setminus\{\emptyset\}, \tag{14}$$

where $\mathcal{P}([0,\infty))$ is the set of all subsets of $[0,\infty)$.

Let $\mathcal{IF}^*(X)$ be the set of all integrable interval-valued functions. We can obtain the following basic characterizations of the (IG) fuzzy integral with respect to a fuzzy measure by means of an interval-representable generalized triangular norm of interval-valued functions.

Theorem 3.2. *Let $(X,\mathcal{B},\mu)$ be a fuzzy measure space and $\mathfrak{T}=[T_l,T_r]$ be an interval-representable generalized triangular norm.* (1) *If $\overline{f},\overline{g}\in\mathcal{IF}^*(X)$ and $\overline{f}\le\overline{g}$, then we have*

$$(IG)\int_A \overline{f}d\mu \le (IG)\int_A \overline{g}d\mu. \tag{15}$$

(2) *If $A\in\mathcal{B}$ and $\overline{a}\in I([0,\infty))$, then we have*

$$(IG)\int_A \overline{a}d\mu = \mathfrak{T}[a_l,\mu(A)]\vee\mathfrak{T}[a_r,[0,\mu(A)]]. \tag{16}$$

Proof. (1) Since $\overline{f}\le\overline{g}$, $f_l\le g_l$ and $f_r\le g_r$. Thus, we have

$$\mu_{A,f_l}(\alpha)\le\mu_{A,g_l}(\alpha) \text{ and } \mu_{A,f_r}(\alpha)\le\mu_{A,g_r}(\alpha)$$

for all $\alpha\in[0,\infty)$. By Definition 3.2,

$$\begin{aligned}\mathfrak{T}[\alpha,\mu_{A,\overline{f}}(\alpha)] &= [T_l[\alpha,\mu_{A,f_l}(\alpha)],T_r[\alpha,\mu_{A,f_l}(\alpha)]\\ &\le [T_l[\alpha,\mu_{A,g_l}(\alpha)],T_r[\alpha,\mu_{A,g_l}(\alpha)]\\ &= \mathfrak{T}[\alpha,\mu_{A,\overline{g}}(\alpha)].\end{aligned} \tag{17}$$

for all $\alpha\in[0,\infty)$. Therefore we obtain

$$\begin{aligned}(IG)\int_A \overline{f}d\mu &= \sup_{\alpha>0}\mathfrak{T}[\alpha,\mu_{A,\overline{f}}(\alpha)]\\ &\le \sup_{\alpha>0}\mathfrak{T}[\alpha,\mu_{A,\overline{g}}(\alpha)] = (IG)\int_A \overline{g}d\mu.\end{aligned}$$

(2) Note that if μ is a fuzzy measure and $\overline{a}=[a_l,a_r]\in[0,\infty)$, then we have

$$\begin{aligned}\mu_{A,\overline{a}}(\alpha) &= [\mu_{A,a_l}(\alpha),\mu_{A,a_r}(\alpha)]\\ &= \begin{cases}[\mu(A),\mu(A)] & \text{if } \alpha\in(0,a_l]\\ [0,\mu(A)] & \text{if } \alpha\in(a_l,a_r]\\ 0 & \text{if } \alpha\in(a_r,\infty).\end{cases}\end{aligned}$$

Thus, by Definition 3.1 (11) and Definition 3.3(2), we have

$$\begin{aligned}&(IG)\int_A \overline{a}d\mu\\ =\ & \sup_{\alpha>0}\mathfrak{T}[\alpha,\mu_{A,\overline{a}}(\alpha)]\\ =\ & \sup_{\alpha>0}[T_l[\alpha,\mu_{A,a_l}(\alpha),T_r[\alpha,\mu_{A,a_r}(\alpha)]\\ =\ & \left[\sup_{\alpha>0}T_l[\alpha,\mu_{A,a_l}(\alpha),\sup_{\alpha>0}T_r[\alpha,\mu_{A,a_r}(\alpha)\right]\\ =\ & \left[\sup_{0<\alpha\le a_l}T_l[\alpha,\mu(A)],\max\{\sup_{0<\alpha\le a_l}T_r[\alpha,\mu(A)],\sup_{a_l<\alpha\le a_r}T_r[\alpha,\mu(A)]\}\right]\\ =\ & [T_l[a_l,\mu(A)],T_r[a_r,\mu(A)]].\end{aligned}$$

Finally, we obtain the following important theorem which is used in the next section and give a simple example for the (IG) fuzzy integral.

Theorem 3.3. *Let T_l, T_r be generalized triangular norms and $\mathfrak{T}[\overline{x}, \overline{y}] = [T_l[x_l, y_l], T_r[x_r, y_r]]$ for all $\overline{x} = [x_l, x_r], \overline{y} = [y_l, y_r] \in I([0, \infty))$ be an interval-representable generalized triangular norm. If $\overline{f} = [f_l, f_r] \in \mathcal{IF}^*(X)$, and $A \in \mathcal{B}$, then we have*

$$(IG) \int_A \overline{f} d\mu = \left[(G) \int_A f_l d\mu, (G) \int_A f_r d\mu \right], \tag{18}$$

where $(G) \int_A f_u d\mu$ is the (G) fuzzy integral with respect to a fuzzy measure by means of a generalized triangular norm T_u of a measurable function f_u for $u = l, r$.

Proof. For any $\overline{f} = [f_l, f_r] \in \mathcal{IF}^*(X)$, we have

$$\begin{aligned}
(IG) \int_A \overline{f} d\mu &= \sup_{\alpha > 0} \mathfrak{T}[\alpha, \mu_{A, \overline{f}}(\alpha)] \\
&= \sup_{\alpha > 0} \mathfrak{T}[\alpha, [\mu_{A, f_l}(\alpha), \mu_{A, f_r}(\alpha)]] \\
&= \sup_{\alpha > 0} [T_l[\alpha, \mu_{A, f_l}(\alpha)], T_r[\alpha, \mu_{A, f_r}(\alpha)]] \\
&= [\sup_{\alpha > 0} T_r[\alpha, \mu_{A, f_l}(\alpha)], \sup_{\alpha > 0} T_r[\alpha, \mu_{A, f_r}(\alpha)]] \\
&= \left[(G) \int_A f_l d\mu, (G) \int_A f_r d\mu \right],
\end{aligned}$$

where $(G) \int_A f_u d\mu$ is the (G) fuzzy integral with respect to a fuzzy measure by means of a generalized triangular norm T_u of a measurable function f_u for $u = l, r$.

Example 3.1. Let $T_l[x_l, y_l] = \min\{\min\{x_l, y_l\}, x_l \cdot y_l\}$ and $T_r[x_r, y_r] = \max\{\min\{x_r, y_r\}, x_r \cdot y_r\}$, and $\mathfrak{T}[\overline{x}, \overline{y}] = [T_l[x_l, y_l], T_r[x_l, y_r]]$ be an interval-valued generalized triangular norm for all $\overline{x} = [x_l, x_r], \overline{y} = [y_l, y_r] \in I([0, \infty))$, and m be the Lebesgue measure on $[0, \infty)$. Note that if $\overline{x}, \overline{y} \subset [0, 1]$, then we have

$$T_l[x_l, y_l] = x_l \cdot y_l \text{ and } T_r[x_r, y_r] = \min\{x_r, y_r\}$$

If we take $X = [0, 1]$ and $\overline{f} : X \longrightarrow I([0, \infty)) \backslash \emptyset$ by $\overline{f} = \left[\frac{1}{4}x, 2x\right]$ for all $x \in X$ is an interval-valued function,and $\mu = m^2$, then we have

$$\begin{aligned}
(IG) \int \bar{f} d\mu &= \sup_{\alpha > 0} [T_l[\alpha, \mu_{f_l}(\alpha)], T_r[\alpha, \mu_{f_r}(\alpha)]] \\
&= \left[\sup_{0 < \alpha \leq \frac{1}{4}} \{\alpha \cdot (1 - 4\alpha)^2\}, \sup_{0 < \alpha \leq 2} \min\{\alpha, (1 - \frac{1}{2}\alpha)^2\} \right] \\
&= \left[\frac{1}{27}, 3 - \sqrt{5} \right].
\end{aligned}$$

4. Convergence properties for the (IG) fuzzy integral by means of an interval-representable generalized triangular norm

In this section, we consider monotone convergent sequences of measurable interval-valued functions in the Hausdorff metric and investigate some convergence properties of the (IG) fuzzy integral with respect to a fuzzy measure by means of an interval-representable generalized triangular norm of measurable interval-valued functions.

Definition 4.1. If $\{\overline{f}_n\}$ be a sequence of measurable interval-valued functions and $\{\overline{f}\} \in \mathcal{IF}(X)$ and $A \in \mathcal{B}$.

(1) $\overline{f}_n \nearrow \overline{f}$ on A in the Hausdorff metric if $\{\overline{f}_n\}$ is an increasing sequence of interval-valued functions and $\lim_{n\to\infty} d_H(\overline{f}_n(x), \overline{f}(x)) = 0$, in symbols

$$d_H - \lim_{n\to\infty} \bar{f}_n(x) = \bar{f}(x), \tag{19}$$

for all $x \in X$.

(2) $\overline{f}_n \searrow \overline{f}$ on A in the Hausdorff metric if $\{\overline{f}_n\}$ is an decreasing sequence of interval-valued functions and $d_H - \lim_{n\to\infty} \overline{f}_n(x) = \overline{f}(x)$.

By using Definition 4.1, we obtain the following theorem under an interval-representable generalized triangular norm which is an extension of Theorem 2.1.

Theorem 4.1. *Let T_l, T_r be generalized triangular norms and*

$$\mathfrak{T}[\overline{x}, \overline{y}] = [T[x_l, y_l], T[x_r, y_r]] \tag{20}$$

be an interval-representable generalized triangular norm for all $\overline{x} = [x_l, x_r], \overline{y} = [y_l, y_r] \in I([0,\infty))$. If $\{f_n\} \subset \mathcal{IF}^*(X)$ and $\overline{f} \in \mathcal{IF}^*(X)$, and $A \in \mathcal{B}$, and $\overline{f}_n \searrow \overline{f}$ on A in the Hausdorff metric, then we have

$$d_H - \lim_{n\to\infty} (IG) \int_A \overline{f}_n d\mu = (IG) \int_A \overline{f} d\mu, \tag{21}$$

if and only if the following conditions are satisfied

(i) *for any given $\varepsilon > 0$ there exists a $n_0 \in \mathbb{N}$ such that*

$$\overline{\mu}_{A,\overline{f}_{n_0}}(c_0 + \varepsilon) < \infty, \tag{22}$$

where $c_0 = \max\left\{\sup\{a > 0 \ : \ T_l[a,\infty] \leq (G)\int_A f_l d\mu\}, \sup\{a > 0 \ : \ T_r[a,\infty] \leq (G)\int_A f_r d\mu\}\right\}$ *and*

(ii) *for any α_n with $\alpha_n \nearrow \infty$ or $\alpha_n \searrow 0$,*

$$\overline{\lim}_{n\to\infty} \mathfrak{T}[\alpha_n, \mu_{A,\overline{f}_n}(\alpha_n)] \leq (IG) \int_A \overline{f} d\mu. \tag{23}$$

Proof. By Theorem 3.3, we have

$$(IG) \int_A \overline{f} d\mu = \left[(G) \int_A f_{n,l} d\mu, (G) \int_A f_{n,r} d\mu\right] \tag{24}$$

for all $n \in \mathbb{N}$ and

$$(IG) \int_A \overline{f} d\mu = \left[(G) \int_A f_l d\mu, (G) \int_A f_r d\mu\right], \tag{25}$$

where where $(G)\int_A f_{n,u} d\mu$ and $(G)\int_A f_u d\mu$ are the (G) fuzzy integrals with respect to a fuzzy measure by means of a generalized triangular norm T_u for $u = l, r$. By (11),(18),(24) and (25), (21) implies that

$$\lim_{n\to\infty} (G) \int_A f_{n,l} d\mu = (G) \int_A f_l d\mu, \tag{26}$$

and

$$\lim_{n\to\infty}(G)\int_A f_{n,r}d\mu = (G)\int_A f_r d\mu. \tag{27}$$

By Theorem 2.1, (26) holds if and only if the following conditions are satisfied

(i) for any given $\varepsilon > 0$ there exists a $n_1 \in \mathbb{N}$ such that

$$\mu_{A,f_{n_1,l}}(c_1+\varepsilon) < \infty,$$

where $c_1 = \sup\{a > 0 \ : \ T_l[a,\infty] \le (G)\int_A f_l d\mu\}$.

(ii) For any $\{\alpha_n\}$ with $\alpha_n \nearrow \infty$ or $\alpha_n \searrow 0$,

$$\overline{\lim}_{n\to\infty} T_l[\alpha_n, \mu_{A,f_{n,l}}(\alpha_n)] \le (G)\int_A f_l d\mu. \tag{28}$$

and (27) holds if and only if the following conditions are satisfied

(i) for any given $\varepsilon > 0$ there exists a $n_2 \in \mathbb{N}$ such that

$$\mu_{A,f_{n_2,r}}(c_2+\varepsilon) < \infty,$$

where $c_2 = \sup\{a > 0 \ : \ T_l[a,\infty] \le (G)\int_A f_r d\mu\}$ and

(ii) for any $\{\alpha_n\}$ with $\alpha_n \nearrow \alpha$ or $\alpha_n \searrow 0$,

$$\overline{\lim}_{n\to\infty} T[\alpha_n, \mu_{A,f_{rn}}(\alpha_n)] \le (GF)\int_A f_r d\mu. \tag{29}$$

Without loss of the generality, we assume that $n_1 \ge n_2$ and $c_1 \le c_2$. Thus, $f_{n_1,l} \le f_{n_2,l}$ and $f_{n_1,r} \le f_{n_2,r}$ and hence

$$\mu_{A,f_{n_1,l}}(c_2+\varepsilon) \le \mu_{A,f_{n_1,l}}(c_1+\varepsilon), \tag{30}$$

and

$$\mu_{A,f_{n_1,r}}(c_2+\varepsilon) \le \mu_{A,f_{n_2,r}}(c_2+\varepsilon). \tag{31}$$

If we take $c_0 = \max\{c_1, c_2\}$, then (30) and (31) implies that for any given $\varepsilon > 0$, there exists a $n_0 = n_1 \in \mathbb{N}$ such that

$$\begin{aligned}
&\mu_{A,\overline{f}_{n_0}}(c_0+\varepsilon)\\
&\le \mu_{A,\overline{f}_{n_1}}(c_2+\varepsilon)\\
&= [\mu_{A,f_{n_1,l}}(c_2+\varepsilon), \mu_{A,f_{n_1,r}}(c_2+\varepsilon)]\\
&\le [\mu_{A,f_{n_1,l}}(c_1+\varepsilon), \mu_{A,f_{n_2,r}}(c_2+\varepsilon)]\\
&< [\infty,\infty] = \infty.
\end{aligned}$$

Thus, the condition (22) holds. For any $\{\alpha_n\}$ with $\alpha_n \nearrow \infty$ or $\alpha_n \searrow 0$, by Theorem 2.1, we have

$$\overline{\lim}_{n\to\infty} T_l[\alpha_n, \mu_{A,f_{n,l}}(\alpha_n)] \le (G)\int_A f_l d\mu, \tag{32}$$

and

$$\overline{\lim}_{n\to\infty} T_r[\alpha_n, \mu_{A,f_{n,r}}(\alpha_n)] \le (G)\int_A f_r d\mu. \tag{33}$$

By (32) and (33) and (20) and Theorem 3.3,

$$\begin{aligned}
&\overline{\lim}_{n\to\infty} \overline{T}\,[\alpha_n, [\mu_{A,f_l}(\alpha), \mu_{A,f_r}(\alpha)]]\\
&\overline{\lim}_{n\to\infty}\,[T_l[\alpha_n, [\mu_{A,f_l}(\alpha)], T_r[\alpha_n, \mu_{A,f_r}(\alpha)]]\\
&[\overline{\lim}_{n\to\infty} T_l[\alpha_n, [\mu_{A,f_l}(\alpha)], \overline{\lim}_{n\to\infty} T_r[\alpha_n, \mu_{A,f_r}(\alpha)]]\\
&\le \left[(G)\int_A f_l d\mu, (G)\int_A f_r d\mu\right]
\end{aligned}$$

$$= (IG)\int_A \overline{f} d\mu.$$

Thus, the condition (23) holds. Similarly, we can derive the converse that (22) and (23) implies (21).

Theorem 4.2. *Let T_l, T_r be generalized triangular norms and $\mathfrak{T}[\overline{x},\overline{y}] = [T_l[x_l, y_l], T_r[x_r, y_r]]$ be an interval-representable generalized triangular norm for all $\overline{x} = [x_l, x_r], \overline{y} = [x_l, x_r] \in I([0 < \infty))$. If $\{\overline{f}_n\} \subset \mathcal{IF}^*(X)$ and $\overline{f} \in \mathcal{IF}^*(X)$, and $A \in \mathcal{B}$, and $\overline{f}_n \searrow \overline{f}$ on A in the Hausdorff metric, then we have*

$$d_H - \overline{\lim}_{n\to\infty}\overline{f}_n(x) = (IG)\int_A \overline{f} d\mu, \tag{34}$$

if and only if for any $\{\alpha_n\}$ with $\alpha_n \nearrow \infty$,

$$\overline{\lim}_{n\to\infty}\mathfrak{T}[\alpha_n, \mu_{A,\overline{f}_n}(\alpha_n)] \leq (IG)\int_A \overline{f} d\mu. \tag{35}$$

Proof. By (11),(18),(24) and (25), (34) implies the following two equations:

$$\lim_{n\to\infty}(G)\int_A f_{n,l} d\mu = (G)\int_A f_l d\mu, \tag{36}$$

and

$$\lim_{n\to\infty}(GF)\int_A f_{rn} d\mu = (GF)\int_A f_r d\mu. \tag{37}$$

By Theorem 2.2, (36) and (37) hold if and only if for any $\{\alpha_n\}$ with $\alpha_n \nearrow \infty$,

$$\overline{\lim_{n\to\infty}} T_l[\alpha_n, \mu_{A,f_{n,l}}(\alpha_n)] \leq (G)\int_A f_l d\mu, \tag{38}$$

and

$$\overline{\lim_{n\to\infty}} T_r[\alpha_n, \mu_{A,f_{n,r}}(\alpha_n)] \leq (G)\int_A f_r d\mu. \tag{39}$$

By (38),(39) and Definition 3.1 (9) and (10), we have

$$\begin{aligned}
& \overline{\lim_{n\to\infty}}\mathfrak{T}[\alpha_n, \mu_{A,\overline{f}_n}(\alpha_n)] \\
= \ & \overline{\lim_{n\to\infty}}[T_l[\alpha_n, \mu_{A,f_{n,l}}(\alpha_n)], T_r[\alpha_n, \mu_{A,f_{n,r}}(\alpha_n)]] \\
= \ & [\overline{\lim_{n\to\infty}} T_l[\alpha_n, \mu_{A,f_{n,l}}(\alpha_n)], \overline{\lim_{n\to\infty}} T_r[\alpha_n, \mu_{A,f_{n,r}}(\alpha_n)]] \\
\leq \ & [(G)\int_A f_l d\mu,\ (G)\int_A f_r d\mu] \\
= \ & (IG)\int_A \overline{f} d\mu.
\end{aligned}$$

Thus, the condition (35) holds. Similarly, we can derive the converse that (35) implies (34).

5. Conclusions

In this paper, we considered the concept of an interval-representable generalized triangular norm (see [3] and Definition 3.2)and studied some characterizations and convergence properties of the (IG) fuzzy integral with respect to a fuzzy measure by means of an interval-representable generalized triangular norms of measurable interval-valued functions (see Definition 3.3) which is an extension of the (G)fuzzy integral with respect to a fuzzy measure by means of a generalized triangular norm of measurable functions by Fang[4].

From Theorems 3.1 and 3.2, we investigated some characterizations of the (IVG) fuzzy integral with respect to a fuzzy measure on the space of measurable interval-valued functions. Theorem 3.3 are used in the proof of Theorems 4.1 and 4.2. From Theorems 4.1 and 4.2, we discussed some convergence properties of the (IG) fuzzy integral with respect to a fuzzy measure of measurable interval-valued functions.

Acknowledgement This paper was supported by Konkuk University in 2013.

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J. COMPUTATIONAL ANALYSIS AND APPLICATIONS, VOL. 15, NO.7, 1291-1299, 2013, COPYRIGHT 2013 EUDOXUS PRESS, LLC

Soft rough sets and their properties

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Abstract

Molodtsov initiated the concept of soft set theory, which can be used as a generic mathematical tool for dealing with uncertainty. However, it has been pointed out that classical soft sets are not appropriate to deal with imprecise and fuzzy parameters. In this paper, the notion of the soft rough set theory is proposed. Soft rough set theory is a combination of a rough theory and a soft set theory. The complement, relative complement, union, restricted union, intersection, restricted intersection, "and" and "or" operations are defined on the soft rough sets. The basic properties of the soft rough sets are also presented and discussed.

Keywords: Rough sets; Soft sets; Soft rough sets; Properties

MR2000: 08A99.

1 Introduction

Soft set theory was firstly proposed by Molodtsov in 1999 [7]. It is different from traditional tools for dealing with uncertainties, such as the theory of probability [13], the theory of fuzzy sets [16], the theory of rough sets [12]. It has been demonstrated that soft set theory brings about a rich potential for applications in many fields such as function smoothness, Riemann integration, decision making, measurement theory, game theory, etc.

Soft set theory has received much attention since its introduction by Molodtsov. The concept and basic properties of soft set theory are presented in [9,7]. Chen et al. [2] presented a new definition of soft set parameterization reduction and compared this definition with the related concept of knowledge reduction in the rough set theory. In fact, the soft set model can also be combined with other mathematical models [15]. For example, by amalgamating the soft sets and algebra, Aktas and Cagman [1] introduce the basic properties of soft sets, compare soft sets to the related concepts of fuzzy sets [16] and rough sets [12], point out that every fuzzy set and every rough set may be considered a soft set, and give a definition of soft groups. Feng et al. [4] defined soft semirings and several related notions to establish a connection between soft sets and semirings. Maji et al. [11] presented the concept of the fuzzy soft set which is based on a combination of the fuzzy set and soft set models. Xu et al. [14] introduce the notion of vague soft sets which is an extension to the soft sets and is based on a combination of the vague set [5] and soft set models. Majumdar and Samanta [8] further generalized the concept of fuzzy soft sets as introduced by Maji et al. [10], in other words, a degree is attached with the parameterization of fuzzy sets while defining a fuzzy soft set. Jiang et al. [6] presented the concept of the interval-valued intuitionistic fuzzy soft sets by combining the interval-valued intuitionistic fuzzy set and soft set models.

The purpose of this paper is to combine the rough sets and soft sets, from which we can obtain a new soft set model: soft rough set theory.

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The rest of this paper is organized as follows. The following section briefly reviews some background on soft sets, rough sets. At the same time some operations of rough sets are defined. In Section 3, we propose the concepts and operations of soft rough sets and discuss their properties in detail. Finally, in Section 4, we draw the conclusion and present some topics for future research.

2 Preliminaries

Given a non-empty universe U, by $\mathscr{P}(U)$ we will denote the power-set on U. If ρ is an equivalence relation on U then foe every $x \in U$, $[x]_\rho$ denotes the equivalence class of ρ determined by x. For any $X \subseteq U$, we write X^c to denote the complementation of X in U, that is the set $U - X$.

Definition 2.1 [3]. A pair (U, ρ) where $U \neq \emptyset$ and ρ is an equivalence relation on U, is called an approximation space.

Definition 2.2 [3]. For an approximation space (U, ρ), by a rough approximation in (U, ρ) we mean a mapping $\rho : \mathscr{P}(U) \to \mathscr{P}(U) \times \mathscr{P}(U)$ defined by for every $X \in \mathscr{P}(U)$, $\rho(X) = (\underline{\rho}(X), \overline{\rho}(X))$, where

$$\underline{\rho}(X) = \{x \in X | [x]_\rho \subseteq X\}, \quad \overline{\rho}(X) = \{x \in X | [x]_\rho \cap X \neq \emptyset\}.$$

$\underline{\rho}(X)$ is called a lower rough approximation of X in (U, ρ), where as $\overline{\rho}(X)$ is called a upper rough approximation of X in (U, ρ).

Definition 2.3 [3]. Given an approximation space (U, ρ), a pair $(A, B) \in \mathscr{P}(U) \times \mathscr{P}(U)$ is called a rough set in (U, ρ) iff $(A, B) = \rho(X)$ for some $X \in \mathscr{P}(U)$.

Definition 2.4. Let $\rho(X)$ be is a rough set over U with respect to an equivalence relation ρ, then the complement of $\rho(X)$ is denoted by $\rho^c(X) = (\underline{\rho}^c(X), \overline{\rho}^c(X))$, is a rough set, where $\underline{\rho}^c(X) = \left\{x \in X^c | [x]_\rho \subseteq X^c\right\}$, $\overline{\rho}^c(X) = \left\{x \in X^c | [x]_\rho \cap X^c \neq \phi\right\}$.

By the definition of rough set, obviously, $\rho^c(X) = \rho(X^c)$.

Definition 2.5. Let $\rho(X)$ and $\rho(Y)$ be two rough sets over U with respect to an equivalence relation ρ, then union of $\rho(X)$ and $\rho(Y)$ denoted by $\rho(X) \cup \rho(Y)$, is a rough set $\rho(Z)$, where

$$\underline{\rho}(Z) = \{x \in X \cup Y | [x]_\rho \subseteq (X \cup Y)\}, \overline{\rho}(Z) = \{x \in X \cup Y | [x]_\rho \cap (X \cup Y) \neq \emptyset\}.$$

By the definition of rough set, obviously, $\rho(Z) = \rho(X \cup Y)$.

Definition 2.6. Let $\rho(X)$ and $\rho(Y)$ be two rough sets over U with respect to an equivalence relation ρ, then intersection of $\rho(X)$ and $\rho(Y)$ denoted by $\rho(X) \cap \rho(Y)$, is a rough set $\rho(Z)$, where

$$\underline{\rho}(Z) = \{x \in X \cap Y | [x]_\rho \subseteq (X \cap Y)\}, \overline{\rho}(Z) = \{x \in X \cap Y | [x]_\rho \cap (X \cap Y) \neq \emptyset\}.$$

By the definition of rough set, obviously, $\rho(Z) = \rho(X \cap Y)$.

Molodtsov [7] defined the soft set in the following way. Let U be an initial universe of objects and E the set of parameters in relation to objects in U. Parameters are often attributes, characteristics, or properties of objects. Let $\mathscr{P}(U)$ denote the power set of U and $A \subseteq E$.

Definition 2.7. A pair $\langle F, A \rangle$ is called a soft set over U, where F is a mapping given by $F : A \to \mathscr{P}(U)$.

In other words, the soft set is not a kind of set, but a parameterized family of subsets of the set U. For any parameter $\varepsilon \in A$, $F(\varepsilon)$ may be considered as the set of ε-approximate elements of the soft set $\langle F, A \rangle$.

3 Soft rough sets and their properties

Definition 3.1. Let U be an initial universe and E be a set of parameters. $RS(U)$ denotes the set of all rough sets of U with respect to an equivalence relation ρ. Let $A \subseteq E$. A pair $\langle F, A \rangle$ is a soft rough set over U, where F is a mapping given by $F : A \to RS(U)$.

In other words, a soft rough set is a parameterized family of rough subsets of U, thus, its universe is the set of all rough sets of U, i.e., $RS(U)$. A soft rough set is also a special case of a soft set because it is still a mapping from parameters to $RS(U)$.

Definition 3.2. The union of two soft rough sets $\langle F,A\rangle$ and $\langle G,B\rangle$ over a common universe U with respect to an equivalence relation ρ is a soft rough set $\langle H,C\rangle$, where $C = A \cup B$ and $\forall\varepsilon \in C$,

$$H(\varepsilon) = \begin{cases} F(\varepsilon), & if\ \varepsilon \in A - B, \\ G(\varepsilon), & if\ \varepsilon \in B - A, \\ F(\varepsilon) \cup G(\varepsilon), & if\ \varepsilon \in A \cap B. \end{cases}$$

We write $\langle F,A\rangle \widetilde{\cup} \langle G,B\rangle = \langle H,C\rangle$.

Definition 3.3. The intersection of two soft rough sets $\langle F,A\rangle$ and $\langle G,B\rangle$ over a common universe U with respect to an equivalence relation ρ is a soft rough set $\langle H,C\rangle$, where $C = A \cup B$ and $\forall\varepsilon \in C$,

$$H(\varepsilon) = \begin{cases} F(\varepsilon), & if\ \varepsilon \in A - B, \\ G(\varepsilon), & if\ \varepsilon \in B - A, \\ F(\varepsilon) \cap G(\varepsilon), & if\ \varepsilon \in A \cap B. \end{cases}$$

We write $\langle F,A\rangle \widetilde{\cap} \langle G,B\rangle = \langle H,C\rangle$.

Definition 3.4. Let $E = \{e_1, e_2, \ldots, e_n\}$ be a parameter set. The not set of E denoted by $\rceil E$ is defined by $\rceil E = \{\rceil e_1, \rceil e_2, \ldots, \rceil e_n\}$ where $\rceil e_i = not\ e_i$.

Definition 3.5. Let $\langle F,A\rangle$ be a soft rough set over a common universe U with respect to an equivalence relation ρ, then complement of $\langle F,A\rangle$ denoted by $\langle F,A\rangle^c = \langle F^c, \rceil A\rangle$ is a soft rough set, and $\forall \rceil\varepsilon \in \rceil A$, $F^c(\rceil\varepsilon) = \rho^c(X) = \rho(X^c)$, where $F(\varepsilon) = \rho(X)$.

Definition 3.6. Let $\langle F,A\rangle$ and $\langle G,B\rangle$ be two soft rough sets over a common universe U with respect to an equivalence relation ρ such that $A \cap B \neq \emptyset$. The restricted union of $\langle F,A\rangle$ and $\langle G,B\rangle$ is denoted by $\langle F,A\rangle \Cup \langle G,B\rangle$, and is defined as $\langle F,A\rangle \Cup \langle G,B\rangle = \langle H,C\rangle$, where $C = A \cap B$ and $\forall\varepsilon \in C$, $H(\varepsilon) = F(\varepsilon) \cup G(\varepsilon)$.

Definition 3.7. Let $\langle F,A\rangle$ and $\langle G,B\rangle$ be two soft rough sets over a common universe U with respect to an equivalence relation ρ such that $A \cap B \neq \emptyset$. The restricted intersection of $\langle F,A\rangle$ and $\langle G,B\rangle$ is denoted by $\langle F,A\rangle \Cap \langle G,B\rangle$, and is defined as $\langle F,A\rangle \Cap \langle G,B\rangle = \langle H,C\rangle$, where $C = A \cap B$ and $\forall\varepsilon \in C$, $H(\varepsilon) = F(\varepsilon) \cap G(\varepsilon)$.

Definition 3.8. Let $\langle F,A\rangle$ be a soft rough set over a common universe U with respect to an equivalence relation ρ, then restricted complement of $\langle F,A\rangle$ denoted by $\langle F,A\rangle^r = \langle F^r, A\rangle$ is a soft rough set, and $\forall\varepsilon \in A$, $F^r(\varepsilon) = \rho^c(X) = \rho(X^c)$, where $F(\varepsilon) = \rho(X)$.

Definition 3.9. A soft rough set $\langle F,A\rangle$ over U with respect to an equivalence relation ρ is said to be a null soft rough set denoted by $\emptyset_A$, if $\varepsilon \in A$, $F(\varepsilon) = \rho(\emptyset)$.

Definition 3.10. A soft rough set $\langle F,A\rangle$ over U with respect to an equivalence relation ρ is said to be a absolute soft rough set denoted by Σ_A, if $\varepsilon \in A$, $F(\varepsilon) = \rho(U)$.

Theorem 3.11. Let E be a set of parameters, $A \subseteq E$. If $\emptyset_A$ is a null soft rough set, Σ_A a absolute soft rough set, and $\langle FA\rangle$ and $\langle F,E\rangle$ two soft rough sets over a common universe U with respect to an equivalence relation ρ, then

(1) $\langle F,A\rangle \widetilde{\cup} \langle F,A\rangle = \langle F,A\rangle$;

(2) $\langle F,A\rangle \widetilde{\cap} \langle F,A\rangle = \langle F,A\rangle$;

(3)$\langle F,E\rangle \widetilde{\cup} \emptyset_A = \langle F,E\rangle$;

(4)$\langle F,E\rangle \widetilde{\cap} \emptyset_E = \emptyset_E$;

(5)$\langle F,E\rangle \widetilde{\cup} \Sigma_E = \Sigma_E$;

(6)$\langle F,E\rangle \widetilde{\cap} \Sigma_A = \langle F,E\rangle$.

Proof. It is easily obtained from Definitions above.

Theorem 3.12. Let $\langle F,A\rangle$ and $\langle G,B\rangle$ be two soft rough sets over a common universe U with respect to an equivalence relation ρ such that $A \cap B \neq \emptyset$. Then

(1) $(\langle F,A\rangle \Cup \langle G,B\rangle)^r = \langle F,A\rangle^r \Cap \langle G,B\rangle^r$;
(2) $(\langle F,A\rangle \Cap \langle G,B\rangle)^r = \langle F,A\rangle^r \Cup \langle G,B\rangle^r$.
Proof. For $\forall\varepsilon \in A\cap B$, let $F(\varepsilon) = \rho(X)$, $G(\varepsilon) = \rho(Y)$, and $\langle F,A\rangle \Cup \langle G,B\rangle = \langle H,C\rangle$. According to definition, $H(\varepsilon) = F(\varepsilon)\cup G(\varepsilon) = \rho(X)\cup\rho(Y) = \rho(X\cup Y)$, and then $H^r(\varepsilon) = \rho^c(X\cup Y) = \rho(X^c\cap Y^c)$.

Now $\langle F,A\rangle^r \Cap \langle G,B\rangle^r = \langle F^r,A\rangle \Cap \langle G^r,B\rangle = \langle K,C\rangle$, where $C = A\cap B$. So by definition, we have $K(\varepsilon) = F^r(\varepsilon)\cap G^r(\varepsilon) = \rho^c(X)\cap\rho^c(Y) = \rho(X^c\cap Y^c) = H^r(\varepsilon)\ \ \forall\varepsilon\in C$.

Hence $(\langle F,A\rangle \Cup \langle G,B\rangle)^r = \langle F,A\rangle^r \Cap \langle G,B\rangle^r$.
(2) Let $\langle F,A\rangle \Cap \langle G,B\rangle = \langle H,C\rangle$ where $C = A\cap B \neq \emptyset$, thus $H(\varepsilon) = F(\varepsilon)\cap G(\varepsilon) = \rho(X)\cap\rho(Y) = \rho(X\cap Y)$ for all $\varepsilon\in C$.

Since $(\langle F,A\rangle \Cap \langle G,B\rangle)^r = \langle H,C\rangle^r = \langle H^r,C\rangle$, by definition, $H^r(\varepsilon) = \rho((X\cap Y)^c) = \rho(X^c\cup Y^c)$.

Now $\langle F,A\rangle^r \Cup \langle G,B\rangle^r = \langle F^r,A\rangle \Cup \langle G^r,B\rangle = \langle K,C\rangle$, where $C = A\cap B$. So by definition, we have $K(\varepsilon) = F^r(\varepsilon)\cup G^r(\varepsilon) = \rho^c(X)\cup\rho^c(Y) = \rho(X^c\cup Y^c) = H^r(\varepsilon)\ \ \forall\varepsilon\in C$.

Hence $(\langle F,A\rangle \Cap \langle G,B\rangle)^r = \langle F,A\rangle^r \Cup \langle G,B\rangle^r$.
Theorem 3.13. Let $\langle F,A\rangle$ and $\langle G,B\rangle$ be two soft rough sets over a common universe U with respect to an equivalence relation ρ. Then we have the following:
(1) $(\langle F,A\rangle\widetilde{\cup}\langle G,B\rangle)^c = \langle F,A\rangle^c\widetilde{\cap}\langle G,B\rangle^c$;
(2) $(\langle F,A\rangle\widetilde{\cap}\langle G,B\rangle)^c = \langle F,A\rangle^c\widetilde{\cup}\langle G,B\rangle^c$.
Proof. (1) For the convenience, we do following assumptions, $\forall\varepsilon\in A\cup B$:
if $\varepsilon\in A-B$, then $F(\varepsilon) = \rho(X)$;
if $\varepsilon\in B-A$, then $G(\varepsilon) = \rho(Y)$;
if $\varepsilon\in A\cap B$, then $F(\varepsilon) = \rho(Z), G(\varepsilon) = \rho(W)$.

Suppose that $\langle F,A\rangle\widetilde{\cup}\langle G,B\rangle = \langle H,A\cup B\rangle$. Then $(\langle F,A\rangle\widetilde{\cup}\langle G,B\rangle)^c = \langle H,A\cup B\rangle^c = \langle H^c,\rceil(A\cup B)\rangle = \langle H^c,\rceil A\cup\rceil B\rangle)$. For $\forall\varepsilon\in A\cup B$, we have

$$H(\varepsilon) = \begin{cases} F(\varepsilon) = \rho(X), & if\ \varepsilon\in A-B,\\ G(\varepsilon) = \rho(Y), & if\ \varepsilon\in B-A,\\ F(\varepsilon)\cup G(\varepsilon) = \rho(Z\cup W), & if\ \varepsilon\in A\cap B.\end{cases}$$

Thus

$$H^c(\rceil\varepsilon) = \begin{cases} \rho(X^c), & if\ \rceil\varepsilon\in\rceil A-\rceil B,\\ \rho(Y^c), & if\ \rceil\varepsilon\in\rceil B-\rceil A,\\ \rho(Z^c\cap W^c), & if\ \rceil\varepsilon\in\rceil A\cap\rceil B.\end{cases}$$

Moreover, let $\langle F,A\rangle^c\widetilde{\cap}\langle G,B\rangle^c = \langle F^c,\rceil A\rangle\widetilde{\cap}\langle G^{c\rceil},\rceil B\rangle = \langle K,\rceil A\cup\rceil B\rangle$. Then

$$K(\rceil\varepsilon) = \begin{cases} F^c(\rceil\varepsilon) = \rho(X^c), & if\ \rceil\varepsilon\in\rceil A-\rceil B,\\ G^c(\rceil\varepsilon) = \rho(Y^c), & if\ \rceil\varepsilon\in\rceil B-\rceil A,\\ F^c(\rceil\varepsilon)\cap G^c(\rceil\varepsilon) = \rho(Z^c\cap W^c), & if\ \varepsilon\in\rceil A\cap\rceil B.\end{cases}$$

Since H^c and K are indeed the same rough-set-valued mapping, we conclude that $(\langle F,A\rangle\widetilde{\cup}\langle G,B\rangle)^c = \langle F,A\rangle^c\widetilde{\cap}\langle G,B\rangle^c$ as required.
(2) The proof is similar to that of (1).
Definition 3.14. Let $\langle F,A\rangle$ and $\langle G,B\rangle$ be two soft rough sets over a common universe U with respect to an equivalence relation ρ. Then "$\langle F,A\rangle$ and $\langle G,B\rangle$" is a soft rough set denoted by $\langle F,A\rangle\wedge\langle G,B\rangle$, is defined as $\langle F,A\rangle\wedge\langle G,B\rangle = \langle H,A\times B\rangle$, where $H(\alpha,\beta) = F(\alpha)\cap G(\beta), \forall(\alpha,\beta)\in A\times B$.
Definition 3.15. Let $\langle F,A\rangle$ and $\langle G,B\rangle$ be two soft rough sets over a common universe U with respect to an equivalence relation ρ. Then "$\langle F,A\rangle$ or $\langle G,B\rangle$" is a soft rough set denoted by $\langle F,A\rangle\vee\langle G,B\rangle$, is defined as $\langle F,A\rangle\vee\langle G,B\rangle = \langle O,A\times B\rangle$, where $O(\alpha,\beta) = F(\alpha)\cup G(\beta), \forall(\alpha,\beta)\in A\times B$.
Theorem 3.16. Let $\langle F,A\rangle$ and $\langle G,B\rangle$ be two soft rough sets over a common universe U with respect to an

equivalence relation ρ. Then we have the following:
(1) $(\langle F, A\rangle \wedge \langle G, B\rangle)^c = \langle F, A\rangle^c \vee \langle G, B\rangle^c$;
(2) $(\langle F, A\rangle \vee \langle G, B\rangle)^c = \langle F, A\rangle^c \wedge \langle G, B\rangle^c$.
Proof. (1) Suppose that $\langle F, A\rangle \wedge \langle G, B\rangle = \langle H, A\times B\rangle$. Then $(\langle F, A\rangle \wedge \langle G, B\rangle)^c = \langle H, A\times B\rangle^c = \langle H^c, \rceil(A\times B)\rangle$. For $\forall\rceil(\alpha,\beta) \in\rceil(A\times B)$, let $F(\alpha) = \rho(X)$, $G(\beta) = \rho(Y)$. By definition, $H(\alpha,\beta) = F(\alpha) \cap G(\beta) = \rho(X \cap Y)$. Thus $H^c(\rceil(\alpha,\beta)) = \rho^c(X \cap Y) = \rho((X \cap Y)^c) = \rho(X^c \cup Y^c)$.

Since $\langle F, A\rangle^c = \langle F^c, \rceil A\rangle$ and $\langle G, B\rangle^c = \langle G^c, \rceil B\rangle$, then $\langle F, A\rangle^c \vee \langle G, B\rangle^c = \langle F^c, \rceil A\rangle \vee \langle G^c, \rceil B\rangle$. Assume that $\langle F^c, \rceil A\rangle \vee \langle G^c, \rceil B\rangle = \langle O, \rceil A\times\rceil B\rangle = \langle O, \rceil(A \times B)\rangle$, where $\forall(\rceil\alpha, \rceil\beta) \in\rceil A\times\rceil B$, by definition, $O(\rceil\alpha, \rceil\beta) = F^c(\rceil\alpha) \cup G^c(\rceil\beta) = \rho^c(X) \cup \rho^c(Y) = \rho(X^c \cup \rho(Y^c) = \rho(X^c \cup Y^c)$.

Consequently, H^c and O are the same operators. Thus, $(\langle F, A\rangle \wedge \langle G, B\rangle)^c = \langle F, A\rangle^c \vee \langle G, B\rangle^c$.
(2) The proof is similar to that of (1).
Theorem 3.17. Let $\langle F, A\rangle$, $\langle G, B\rangle$ and $\langle H, C\rangle$ be three soft rough sets over a common universe U with respect to an equivalence relation ρ. Then we have the following:
(1) $\langle F, A\rangle \wedge (\langle G, B\rangle \wedge \langle H, C\rangle) = (\langle F, A\rangle \wedge \langle G, B\rangle) \wedge \langle H, C\rangle$;
(2) $\langle F, A\rangle \vee (\langle G, B\rangle \vee \langle H, C\rangle) = (\langle F, A\rangle \vee \langle G, B\rangle) \vee \langle H, C\rangle$.
Proof. (1) Assume that $\langle G, B\rangle \wedge \langle H, C\rangle = \langle I, B \times C\rangle$. For $\forall(\alpha,\beta) \in B \times C$, let $G(\alpha) = \rho(Y)$, $H(\beta) = \rho(Z)$. By definition, $I(\alpha,\beta) = G(\alpha) \cap H(\beta) = \rho(Y \cap Z)$.

Since $\langle F, A\rangle\wedge(\langle G, B\rangle\wedge\langle H, C\rangle) = \langle F, A\rangle\wedge\langle I, B\times C\rangle$, we suppose that $\langle F, A\rangle\wedge\langle I, B\times C\rangle = \langle K, A\times(B\times C)\rangle$. For $\forall(\delta, \alpha, \beta) \in A \times (B \times C)$, let $F(\delta) = \rho(X)$, by definition, $K(\delta, \alpha, \beta) = F(\delta) \cap I(\alpha, \beta) = \rho(X) \cap \rho(Y \cap Z) = \rho(X \cap Y \cap Z)$.

On the other hand, we take $(\delta, \alpha) \in A \times B$. Suppose that $\langle F, A\rangle \wedge \langle G, B\rangle = \langle J, A \times B\rangle$, by definition, $J(\delta, \alpha) = F(\delta) \cap G(\alpha) = \rho(X \cap Y)$.

Since $(\langle F, A\rangle\wedge\langle G, B\rangle)\wedge\langle H, C\rangle = \langle J, A\times B\rangle\wedge\langle H, C\rangle$, we suppose that $\langle J, A\times B\rangle\wedge\langle H, C\rangle = \langle O, (A\times B)\times C)\rangle$, where $O(\delta, \alpha, \beta) = J(\delta, \alpha) \cap H(\beta) = \rho(X \cap Y \cap Z)$, $(\delta, \alpha, \beta) \in (A \times B) \times C = A \times B \times C$.

Consequently, K and O are the same operators. Thus, $\langle F, A\rangle \wedge (\langle G, B\rangle \wedge \langle H, C\rangle) = (\langle F, A\rangle \wedge \langle G, B\rangle) \wedge \langle H, C\rangle$.
Theorem 3.18. Let $\langle F, A\rangle$, $\langle G, B\rangle$ and $\langle H, C\rangle$ be three soft rough sets over a common universe U with respect to an equivalence relation ρ such that $A \cap B \cap C \neq \emptyset$. Then we have the following:
(1) $\langle F, A\rangle \Cap (\langle G, B\rangle \Cap \langle H, C\rangle) = (\langle F, A\rangle \Cap \langle G, B\rangle) \Cap \langle H, C\rangle$;
(2) $\langle F, A\rangle \Cup (\langle G, B\rangle \Cup \langle H, C\rangle) = (\langle F, A\rangle \Cup \langle G, B\rangle) \Cup \langle H, C\rangle$;
(3) $\langle F, A\rangle \Cap (\langle G, B\rangle \Cup \langle H, C\rangle) = (\langle F, A\rangle \Cap \langle G, B\rangle) \Cup (\langle F, A\rangle \Cap \langle H, C\rangle)$;
(4) $\langle F, A\rangle \Cup (\langle G, B\rangle \Cap \langle H, C\rangle) = (\langle F, A\rangle \Cup \langle G, B\rangle) \Cap (\langle F, A\rangle \Cup \langle H, C\rangle)$.
Proof. In the following, we shall prove (1) and (3); (2) and (4) are proved analogously.

For the convenience, we do following assumptions, $\forall\varepsilon \in A \cup B \cup C$,
if $\varepsilon \in A - B - C$, then $F(\varepsilon) = \rho(X_1)$;
if $\varepsilon \in B - A - C$, then $G(\varepsilon) = \rho(X_2)$;
if $\varepsilon \in C - A - B$, then $H(\varepsilon) = \rho(X_3)$;
if $\varepsilon \in A \cap B - C$, then $F(\varepsilon) = \rho(X_4), G(\varepsilon) = \rho(X_5)$;
if $\varepsilon \in A \cap C - B$, then $F(\varepsilon) = \rho(X_6), H(\varepsilon) = \rho(X_7)$;
if $\varepsilon \in B \cap C - A$, then $G(\varepsilon) = \rho(X_8), H(\varepsilon) = \rho(X_9)$;
if $\varepsilon \in A \cap B \cap C$, then $F(\varepsilon) = \rho(X_{10}), G(\varepsilon) = \rho(X_{11}), H(\varepsilon) = \rho(X_{12})$.
(1) Suppose that $\langle G, B\rangle\Cap\langle H, C\rangle = \langle I, D\rangle$, where $D = B\cap C$. For $\forall\varepsilon \in D$, by definition, $I(\varepsilon) = F(\varepsilon)\cap H(\varepsilon) = \rho(X_8 \cap X_9)$ *or* $\rho(X_{11} \cap X_{12})$.

Since $\langle F, A\rangle \Cap (\langle G, B\rangle \Cap \langle H, C\rangle) = \langle F, A\rangle \Cap \langle I, D\rangle$, we assume that $\langle F, A\rangle \Cap \langle I, D\rangle = \langle J, S\rangle$, where $S = A \cap D$. By definition, for $\forall\varepsilon \in S$, $J(\varepsilon) = F(\varepsilon) \cap I(\varepsilon) = \rho(X_{10}) \cap \rho(X_{11} \cap X_{12}) = \rho(X_{10} \cap X_{11} \cap X_{12})$.

On the other hand, assume that $\langle F, A\rangle \Cap \langle G, B\rangle = \langle K, V\rangle$, where $V = A \cap B$. For $\forall\varepsilon \in V$, $K(\varepsilon) = F(\varepsilon) \cap G(\varepsilon) = \rho(X_4 \cap X_5)$ *or* $\rho(X_{10} \cap X_{11})$.

Since $(\langle F,A\rangle \Cap (\langle G,B\rangle) \Cap \langle H,C\rangle = \langle K,V\rangle \Cap \langle H,C\rangle$, we assume that $\langle K,V\rangle \Cap \langle H,C\rangle = \langle L,W\rangle$, where $W = V \cap C = A \cap B \cap C$. By definition, for $\forall\varepsilon \in W$, $L(\varepsilon) = K(\varepsilon) \cap H(\varepsilon) = \rho(X_{10} \cap X_{11}) \cap \rho(X_{12}) = \rho(X_{10} \cap X_{11} \cap X_{12})$.

Therefore, $L(\varepsilon) = J(\varepsilon)$ for all $\forall\varepsilon \in A \cap B \cap C$. That is, J and L are the same operators. Thus, $\langle F,A\rangle \Cap (\langle G,B\rangle \Cap \langle H,C\rangle) = (\langle F,A\rangle \Cap \langle G,B\rangle) \Cap \langle H,C\rangle$.

(3) Let $\langle G,B\rangle \Cup \langle H,C\rangle = \langle I,D\rangle$, where $D = B \cap C$. For $\forall\varepsilon \in D$, by definition, $I(\varepsilon) = G(\varepsilon) \cup H(\varepsilon) = \rho(X_8 \cup X_9)$ *or* $\rho(X_{11} \cup X_{12})$.

Since $\langle F,A\rangle \Cap (\langle G,B\rangle \Cup \langle H,C\rangle) = \langle F,A\rangle \Cap \langle I,D\rangle$, we assume that $\langle F,A\rangle \Cap \langle I,D\rangle = \langle K,V\rangle$, where $V = A \cap D$. For $\forall\varepsilon \in V = A \cap B \cap C$, $K(\varepsilon) = F(\varepsilon) \cap I(\varepsilon) = \rho(X_{10}) \cap \rho(X_{11} \cup X_{12}) = \rho(X_{10} \cap (X_{11} \cup X_{12}))$.

On the other hand, suppose $\langle F,A\rangle \Cap \langle G,B\rangle = \langle J,M\rangle$ and $\langle F,A\rangle \Cap \langle H,C\rangle = \langle L,W\rangle$, where $M = A \cap B$, $W = A\cap C$. Since $(\langle F,A\rangle \Cap \langle G,B\rangle) \Cup (\langle F,A\rangle \Cap \langle H,C\rangle) = \langle J,M\rangle \Cup \langle L,W\rangle$, assume that $\langle J,M\rangle \Cup \langle L,W\rangle = \langle O,N\rangle$, where $N = M\cap W$. For $\forall \in N = A\cap B\cap C$, by definition, $O(\varepsilon) = J(\varepsilon) \cup L(\varepsilon) = (F(\varepsilon)\cap G(\varepsilon)) \cup (F(\varepsilon)\cap H(\varepsilon)) = (\rho(X_{10}) \cap \rho(X_{11})) \cup (\rho(X_{10}) \cap \rho(X_{12})) = \rho(X_{10} \cap X_{11}) \cup \rho(X_{10} \cap X_{12}) = \rho((X_{10} \cap X_{11}) \cup (X_{10} \cap X_{12})) = \rho(X_{10} \cap (X_{11} \cup X_{12}))$.

Therefore, $K(\varepsilon) = O(\varepsilon)$ for all $\forall\varepsilon \in A \cap B \cap C$. That is, K and O are the same operators. Thus, $\langle F,A\rangle \Cap (\langle G,B\rangle \Cup \langle H,C\rangle) = (\langle F,A\rangle \Cap \langle G,B\rangle) \Cup (\langle F,A\rangle \Cap \langle H,C\rangle)$.

Theorem 3.19. Let $\langle F,A\rangle$, $\langle G,B\rangle$ and $\langle H,C\rangle$ be three soft rough sets over a common universe U with respect to an equivalence relation ρ. Then we have the following:

(1) $\langle F,A\rangle \widetilde{\cap} (\langle G,B\rangle \widetilde{\cap} \langle H,C\rangle) = (\langle F,A\rangle \widetilde{\cap} \langle G,B\rangle) \widetilde{\cap} \langle H,C\rangle$;

(2) $\langle F,A\rangle \widetilde{\cup} (\langle G,B\rangle \widetilde{\cup} \langle H,C\rangle) = (\langle F,A\rangle \widetilde{\cup} \langle G,B\rangle) \widetilde{\cup} \langle H,C\rangle$;

Proof. In the following, we shall prove (1), (2) is proved analogously.

For the convenience, $\forall\varepsilon \in A \cup B \cup C$, we do following assumptions:

if $\varepsilon \in A - B - C$, then $F(\varepsilon) = \rho(X_1)$;
if $\varepsilon \in B - A - C$, then $G(\varepsilon) = \rho(X_2)$;
if $\varepsilon \in C - A - B$, then $H(\varepsilon) = \rho(X_3)$;
if $\varepsilon \in A \cap B - C$, then $F(\varepsilon) = \rho(X_4), G(\varepsilon) = \rho(X_5)$;
if $\varepsilon \in A \cap C - B$, then $F(\varepsilon) = \rho(X_6), H(\varepsilon) = \rho(X_7)$;
if $\varepsilon \in B \cap C - A$, then $G(\varepsilon) = \rho(X_8), H(\varepsilon) = \rho(X_9)$;
if $\varepsilon \in A \cap B \cap C$, then $F(\varepsilon) = \rho(X_{10}), G(\varepsilon) = \rho(X_{11}), H(\varepsilon) = \rho(X_{12})$.

(1) Suppose that $\langle G,B\rangle \widetilde{\cap} \langle H,C\rangle = \langle I,D\rangle$, where $D = B \cup C$. For $\forall\varepsilon \in D$, by definition,

$$I(\varepsilon) = \begin{cases} G(\varepsilon) = \rho(X_2) \;\; or \;\; \rho(X_5), & \varepsilon \in B - C, \\ H(\varepsilon) = \rho(X_3) \;\; or \;\; \rho(X_7), & \varepsilon \in C - B, \\ G(\varepsilon) \cap H(\varepsilon) = \rho(X_8 \cap X_9) \;\; or \;\; \rho(X_{11} \cap X_{12}), & \varepsilon \in B \cap C. \end{cases}$$

Since $\langle F,A\rangle \widetilde{\cap} (\langle G,B\rangle \widetilde{\cap} \langle H,C\rangle) = \langle F,A\rangle \widetilde{\cap} \langle I,D\rangle$, we assume that $\langle F,A\rangle \widetilde{\cap} \langle I,D\rangle = \langle J,S\rangle$, where $S = A\cup D$. By definition, for $\forall\varepsilon \in S$,

$$J(\varepsilon) = \begin{cases} F(\varepsilon) = \rho(X_1), & \varepsilon \in A - D, \\ I(\varepsilon) = \rho(X_2) \;\; or \;\; \rho(X_3) \;\; or \;\; \rho(X_8 \cap X_9), & \varepsilon \in D - A, \\ F(\varepsilon) \cap I(\varepsilon) = \rho(X_4 \cap X_5) \;\; or \;\; \rho(X_{10} \cap X_{11} \cap X_{12}) \;\; or \;\; \rho(X_6 \cap X_7), & \varepsilon \in A \cap D. \end{cases}$$

$$= \begin{cases} \rho(X_1), & \varepsilon \in A - B - C, \\ \rho(X_2), & \varepsilon \in B - A - C, \\ \rho(X_3), & \varepsilon \in C - B - A, \\ \rho(X_8 \cap X_9), & \varepsilon \in B \cap C - A, \\ \rho(X_4 \cap X_5), & \varepsilon \in A \cap B - C, \\ \rho(X_6 \cap X_7), & \varepsilon \in A \cap C - B, \\ \rho(X_{10} \cap X_{11} \cap X_{12}), & \varepsilon \in A \cap B \cap C. \end{cases}$$

On the other hand, assume that $\langle F,A\rangle\widetilde{\cup}\langle G,B\rangle = \langle K,V\rangle$, where $V = A\cup B$. For $\forall\varepsilon\in V$,

$$K(\varepsilon)=\begin{cases} F(\varepsilon)=\rho(X_1)\ \ or\ \ \rho(X_6), & \varepsilon\in A-B,\\ G(\varepsilon)=\rho(X_2)\ \ or\ \ \rho(X_8), & \varepsilon\in B-A,\\ F(\varepsilon)\cap G(\varepsilon)=\rho(X_4\cap X_5)\ \ or\ \ \rho(X_{10}\cap X_{11}), & \varepsilon\in A\cap B.\end{cases}$$

Since $(\langle F,A\rangle\widetilde{\cap}(\langle G,B\rangle)\widetilde{\cap}\langle H,C\rangle = \langle K,V\rangle\widetilde{\cap}\langle H,C\rangle$, we assume that $\langle K,V\rangle\widetilde{\cap}\langle H,C\rangle = \langle L,W\rangle$, where $W = V\cup C = A\cup B\cup C$. By definition, for $\forall\varepsilon\in W$,

$$L(\varepsilon)=\begin{cases} K(\varepsilon)=\rho(X_1)\ \ or\ \ \rho(X_2)\ \ or\ \ \rho(X_4\cap X_5), & \varepsilon\in V-C,\\ H(\varepsilon)=\rho(X_3), & \varepsilon\in C-V,\\ K(\varepsilon)\cap H(\varepsilon)=\rho(X_8\cap X_9)\ \ or\ \ \rho(X_{10}\cap X_{11}\cap X_{12})\ \ or\ \ \rho(X_6\cap X_7), & \varepsilon\in C\cap V.\end{cases}$$

$$=\begin{cases} \rho(X_1), & \varepsilon\in A-B-C,\\ \rho(X_2), & \varepsilon\in B-A-C,\\ \rho(X_3), & \varepsilon\in C-B-A,\\ \rho(X_8\cap X_9), & \varepsilon\in B\cap C-A,\\ \rho(X_4\cap X_5), & \varepsilon\in A\cap B-C,\\ \rho(X_6\cap X_7), & \varepsilon\in A\cap C-B,\\ \rho(X_{10}\cap X_{11}\cap X_{12}), & \varepsilon\in A\cap B\cap C.\end{cases}$$

Therefore, $L(\varepsilon) = J(\varepsilon)$ for all $\forall\varepsilon\in A\cup B\cup C$. That is, J and L are the same operators. Thus, $\langle F,A\rangle\widetilde{\cap}(\langle G,B\rangle\widetilde{\cap}\langle H,C\rangle) = (\langle F,A\rangle\widetilde{\cap}\langle G,B\rangle)\widetilde{\cap}\langle H,C\rangle$.

The following example shows that if $\Cap$ and $\Cup$ of assertions (3) and (4) of theorem 3.18 are replaced by $\widetilde{\cap}$ and $\widetilde{\cup}$ respectively, then assertions (3) and (4) of theorem 3.18 do not hold, i.e., $\langle F,A\rangle\widetilde{\cap}(\langle G,B\rangle\widetilde{\cup}\langle H,C\rangle) = (\langle F,A\rangle\widetilde{\cap}\langle G,B\rangle)\widetilde{\cup}(\langle F,A\rangle\widetilde{\cap}\langle H,C\rangle)$ and $\langle F,A\rangle\widetilde{\cup}(\langle G,B\rangle\widetilde{\cap}\langle H,C\rangle) = (\langle F,A\rangle\widetilde{\cup}\langle G,B\rangle)\widetilde{\cap}(\langle F,A\rangle\widetilde{\cup}\langle H,C\rangle)$ are both incorrect.

Example. Let $U = \{x_1,x_2,x_3,x_4,x_5,x_6\}$ be an initial universe and $E = \{e_1,e_2,e_3,e_4\}$ be a set of parameters. Let ρ be an equivalence relation on U such that $\rho-$equivalence classes are the subsets $\{x_1,x_3\}$, $\{x_2,x_4,x_5\}$ and $\{x_6\}$. $\langle F,A\rangle$, $\langle G,B\rangle$ and $\langle H,C\rangle$ are three soft rough sets over U with respect to an equivalence relation ρ. Here $A = \{e_1,e_2,e_3\}$, $B = \{e_1,e_2,e_4\}$, $C = \{e_1,e_3,e_4\}$.

We take $X_1 = \{x_1,x_3\}$, $X_2 = \{x_1,x_6\}$, $X_3 = \{x_2,x_4,x_5\}$, $X_4 = \{x_2,x_5\}$, $X_5 = \{x_1,x_4\}$, $X_6 = \{x_1,x_2,x_3\}$, $X_7 = \{x_3,x_6\}$, $X_8 = \{x_4,x_6\}$ and $X_9 = \{x_1,x_3,x_6\}$. Let
$F(e_1) = \rho(X_1) = (\{x_1,x_3\},\{x_1,x_3\})$;
$F(e_2) = \rho(X_2) = (\{x_6\},\{x_1,x_3,x_6\})$;
$F(e_3) = \rho(X_3) = (\{x_2,x_4,x_5\},\{x_2,x_4,x_5\})$;
$G(e_1) = \rho(X_4) = (\emptyset,\{x_2,x_4,x_5\})$;
$G(e_2) = \rho(X_5) = (\emptyset,\{x_1,x_2,x_3,x_4,x_5\})$;
$G(e_4) = \rho(X_6) = (\{x_1,x_3\},\{x_1,x_2,x_3,x_4,x_5\})$;
$H(e_1) = \rho(X_7) = (\{x_6\},\{x_1,x_3,x_6\})$;
$H(e_3) = \rho(X_8) = (\{x_6\},\{x_2,x_4,x_5,x_6\})$;
$H(e_4) = \rho(X_9) = (\{x_1,x_3,x_6\},\{x_1,x_3,x_6\})$.

Suppose that $\langle G,B\rangle\widetilde{\cup}\langle H,C\rangle = \langle I,D\rangle$, where $D = B\cup C$. By definition,
$I(e_1) = G(e_1)\cup H(e_1) = \rho(X_4)\cup\rho(X_7) = \rho(X_4\cup X_7) = \rho(\{x_2,x_3,x_5,x_6\}) = (\{x_6\},\{x_1,x_2,x_3,x_4,x_5,x_6\})$;
$I(e_2) = G(e_2) = \rho(X_5) = (\emptyset,\{x_1,x_2,x_3,x_4,x_5\})$;
$I(e_3) = H(e_3) = \rho(X_8) = (\{x_6\},\{x_2,x_4,x_5,x_6\})$;
$I(e_4) = G(e_4)\cup H(e_4) = \rho(X_6)\cup\rho(X_9) = \rho(X_6\cup X_9) = \rho(\{x_1,x_2,x_3,x_6\})$
$= (\{x_1,x_3,x_6\},\{x_1,x_2,x_3,x_4,x_5,x_6\})$.

Since $\langle F,A\rangle\widetilde{\cap}(\langle G,B\rangle\widetilde{\cup}\langle H,C\rangle) = \langle F,A\rangle\widetilde{\cap}\langle I,D\rangle$, we assume that $\langle F,A\rangle\widetilde{\cap}\langle I,D\rangle = \langle J,S\rangle$, where $S = A\cup D$. By definition,
$J(e_1) = F(e_1)\cap I(e_1) = \rho(X_1)\cap\rho(X_4\cup X_7) = \rho(X_1\cap(X_4\cup X_7)) = \rho(\{x_3\}) = (\emptyset,\{x_1,x_3\})$;
$J(e_2) = F(e_2)\cap I(e_2) = \rho(X_2)\cap\rho(X_5) = \rho(X_2\cap X_5) = \rho(\{x_1\}) = (\emptyset,\{x_1,x_3\})$;
$J(e_3) = F(e_3)\cap I(e_3) = \rho(X_3)\cap\rho(X_8) = \rho(X_3\cap X_8) = \rho(\{x_4\}) = (\emptyset,\{x_2,x_4,x_5\})$;
$J(e_4) = I(e_4) = \rho(X_6\cup X_9) = \rho(\{x_1,x_2,x_3,x_6\})$
$= (\{x_1,x_3,x_6\},\{x_1,x_2,x_3,x_4,x_5,x_6\})$.

On the other hand, suppose that $\langle F,A\rangle\widetilde{\cap}\langle G,B\rangle = \langle K,V\rangle$ and $\langle F,A\rangle\widetilde{\cap}\langle H,C\rangle = \langle L,W\rangle$, where $V = A\cup B$, $W = A\cup C$. Since $(\langle F,A\rangle\widetilde{\cap}\langle G,B\rangle)\widetilde{\cup}(\langle F,A\rangle\widetilde{\cap}\langle H,C\rangle) = \langle K,V\rangle\widetilde{\cup}\langle L,W\rangle$, assume that $\langle K,V\rangle\widetilde{\cup}\langle L,W\rangle = \langle O,N\rangle$, where $N = V\cup W$. By definition,
$O(e_1) = K(e_1)\cup L(e_1) = (F(e_1)\cap G(e_1))\cup(F(e_1)\cap H(e_1)) = \rho(X_1\cap X_4)\cup\rho(X_1\cap X_7) = \rho(\emptyset)\cup\rho(\{X_3\}) = \rho(\{x_3\}) = (\emptyset,\{x_1,x_3\})$;
$O(e_2) = K(e_2)\cup L(e_2) = (F(e_2)\cap G(e_2))\cup F(e_2) = (\rho(X_2)\cap\rho(X_5))\cup\rho(X_2) = \rho((X_2\cap X_5)\cup X_2) = \rho(X_2) = \rho(\{x_1,x_6\}) = (\{x_6\},\{x_1,x_3,x_6\})$;
$O(e_3) = K(e_3)\cup L(e_3) = F(e_3)\cup(F(e_3)\cap H(e_3)) = \rho(X_3)\cup\rho(X_3\cap X_8) = \rho(X_3\cup(X_3\cap X_8)) = \rho(X_3) = \rho(\{x_2,x_4,x_5\}) = (\{x_2,x_4,x_5\},\{x_2,x_4,x_5\})$;
$O(e_4) = K(e_4)\cup L(e_4) = G(e_4)\cup H(e_4) = \rho(X_6)\cup\rho(X_9) = \rho(X_6\cup X_9) = \rho(\{x_1,x_2,x_3,x_6\})$
$= (\{x_1,x_3,x_6\},\{x_1,x_2,x_3,x_4,x_5,x_6\})$.

Since $J(e_2)\neq O(e_2)$ and $J(e_3)\neq O(e_3)$. That is, J and O are not the same operators. Thus,
$\langle F,A\rangle\widetilde{\cap}(\langle G,B\rangle\widetilde{\cup}\langle H,C\rangle) \neq (\langle F,A\rangle\widetilde{\cap}\langle G,B\rangle)\widetilde{\cup}(\langle F,A\rangle\widetilde{\cap}\langle H,C\rangle)$.

Likewise, we may show that $\langle F,A\rangle\widetilde{\cup}(\langle G,B\rangle\widetilde{\cap}\langle H,C\rangle) = (\langle F,A\rangle\widetilde{\cup}\langle G,B\rangle)\widetilde{\cap}(\langle F,A\rangle\widetilde{\cup}\langle H,C\rangle)$ is incorrect.

4 Conclusion

In this paper, the notion of the soft rough set theory is proposed. soft rough set theory is a combination of a rough set theory and a soft set theory. The complement, restricted complement, union, restricted union, intersection, restricted intersection, "and" and "or" operations are defined on the soft rough sets. The basic properties of the soft rough sets are also presented and discussed. This new extension not only provides a significant addition to existing theories for handling uncertainties, but also leads to potential areas of further field research and pertinent applications. Our work in this paper is completely theoretical. As far as future directions are concerned, these will include the parameterization reduction of the soft rough sets. It is also desirable to further explore the applications of using the soft rough set approach to solve real world problems such as decision making, forecasting, and data analysis.

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J. COMPUTATIONAL ANALYSIS AND APPLICATIONS, VOL. 15, NO.7, 1300-1309, 2013, COPYRIGHT 2013 EUDOXUS PRESS, LLC

Rate of convergence of some multivariate neural network operators to the unit, revisited

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Abstract

This paper deals with the determination of the rate of convergence to the unit of some multivariate neural network operators, namely the the normalized "bell" and "squashing" type operators. This is given through the multidimensional modulus of continuity of the involved multivariate function or its partial derivatives of specific order that appear in the right-hand side of the associated multivariate Jackson type inequalitiy.

2010 AMS Mathematics Subject Classification: 41A17, 41A25, 41A30, 41A36.

Keywords and Phrases: Neural Networks, Positive operators.

1 Introduction

The multivariate Cardaliaguet-Euvrard operators were first introduced and studied thoroughly in [3], where the authors among many other interesting things proved that these multivariate operators converge uniformly on compacta, to the unit over continuous and bounded multivariate functions. Our multivariate normalized "bell" and "squashing" type operators (1) and (16) were motivated and inspired by the "bell" and "squashing" functions of [3].

The work in [3] is qualitative where the used multivariate bell-shaped function is general. However, though our work is greatly motivated by [3], it is quantitative and the used multivariate "bell-shaped" and "squashing" functions are of compact support.

This paper is the continuation and simplification of [1] and [2], in the multidimensional case.We produce a set of multivariate inequalities giving close upper bounds to the errors in approximating the unit operator by the above multidimensional neural network induced operators. All appearing constants there are well determined. These are mainly pointwise estimates involving the first multivariate modulus of continuity of the engaged multivariate continuous function or its partial derivatives of some fixed order.

2 Convergence with rates of multivariate neural network operators

We need the following (see [3]) definitions.

Definition 1 *A function $b : \mathbb{R} \to \mathbb{R}$ is said to be bell-shaped if b belongs to L^1 and its integral is nonzero, if it is nondecreasing on $(-\infty, a)$ and nonincreasing on $[a, +\infty)$, where a belongs to $\mathbb{R}$. In particular $b(x)$ is a nonnegative number and at a, b takes a global maximum; it is the center of the bell-shaped function. A bell-shaped function is said to be centered if its center is zero.*

Definition 2 *(see [3]) A function $b : \mathbb{R}^d \to \mathbb{R}$ $(d \geq 1)$ is said to be a d-dimensional bell-shaped function if it is integrable and its integral is not zero, and for all $i = 1, ..., d$,*

$$t \to b(x_1, ..., t, ..., x_d)$$

is a centered bell-shaped function, where $\overrightarrow{x} := (x_1, ..., x_d) \in \mathbb{R}^d$ arbitrary.

Example 3 *(from [3]) Let b be a centered bell-shaped function over $\mathbb{R}$, then $(x_1, ..., x_d) \to b(x_1) ... b(x_d)$ is a d-dimensional bell-shaped function.*

Assumption 4 *Here $b(\overrightarrow{x})$ is of compact support $\mathcal{B} := \prod_{i=1}^{d} [-T_i, T_i]$, $T_i > 0$ and it may have jump discontinuities there. Let $f : \mathbb{R}^d \to \mathbb{R}$ be a continuous and bounded function or a uniformly continuous function.*

In this paper, we study the pointwise convergence with rates over $\mathbb{R}^d$, to the unit operator, of the "normalized bell" multivariate neural network operators

$$M_n(f)(\overrightarrow{x}) :=$$

$$\frac{\sum_{k_1=-n^2}^{n^2} \cdots \sum_{k_d=-n^2}^{n^2} f\left(\frac{k_1}{n}, ..., \frac{k_d}{n}\right) b\left(n^{1-\alpha}\left(x_1 - \frac{k_1}{n}\right), ..., n^{1-\alpha}\left(x_d - \frac{k_d}{n}\right)\right)}{\sum_{k_1=-n^2}^{n^2} \cdots \sum_{k_d=-n^2}^{n^2} b\left(n^{1-\alpha}\left(x_1 - \frac{k_1}{n}\right), ..., n^{1-\alpha}\left(x_d - \frac{k_d}{n}\right)\right)}, \tag{1}$$

where $0 < \alpha < 1$ and $\overrightarrow{x} := (x_1, ..., x_d) \in \mathbb{R}^d$, $n \in \mathbb{N}$. Clearly M_n is a positive linear operator.

The terms in the ratio of multiple sums (1) can be nonzero iff simultaneously

$$\left| n^{1-\alpha}\left(x_i - \frac{k_i}{n}\right) \right| \leq T_i, \quad \text{all } i = 1, ..., d,$$

i.e., $\left|x_i - \frac{k_i}{n}\right| \leq \frac{T_i}{n^{1-\alpha}}$, all $i = 1, ..., d$, iff

$$nx_i - T_i n^\alpha \leq k_i \leq nx_i + T_i n^\alpha, \quad \text{all } i = 1, ..., d. \tag{2}$$

To have the order

$$-n^2 \leq nx_i - T_i n^\alpha \leq k_i \leq nx_i + T_i n^\alpha \leq n^2, \tag{3}$$

we need $n \geq T_i + |x_i|$, all $i = 1, ..., d$. So (3) is true when we take

$$n \geq \max_{i \in \{1,...,d\}} (T_i + |x_i|). \tag{4}$$

When $\overrightarrow{x} \in \mathcal{B}$ in order to have (3) it is enough to assume that $n \geq 2T^*$, where $T^* := \max\{T_1, ..., T_d\} > 0$. Consider

$$\widetilde{I}_i := [nx_i - T_i n^\alpha, nx_i + T_i n^\alpha], \;\; i = 1, ..., d, \;\; n \in \mathbb{N}.$$

The length of $\widetilde{I}_i$ is $2T_i n^\alpha$. By Proposition 1 of [1], we get that the cardinality of $k_i \in \mathbb{Z}$ that belong to $\widetilde{I}_i := card\,(k_i) \geq \max(2T_i n^\alpha - 1, 0)$, any $i \in \{1, ..., d\}$. In order to have $card\,(k_i) \geq 1$, we need $2T_i n^\alpha - 1 \geq 1$ iff $n \geq T_i^{-\frac{1}{\alpha}}$, any $i \in \{1, ..., d\}$.

Therefore, a sufficient condition in order to obtain the order (3) along with the interval $\widetilde{I}_i$ to contain at least one integer for all $i = 1, ..., d$ is that

$$n \geq \max_{i \in \{1,...,d\}} \left\{T_i + |x_i|, T_i^{-\frac{1}{\alpha}}\right\}. \tag{5}$$

Clearly as $n \to +\infty$ we get that $card\,(k_i) \to +\infty$, all $i = 1, ..., d$. Also notice that $card\,(k_i)$ equals to the cardinality of integers in $[\lceil nx_i - T_i n^\alpha \rceil, [nx_i + T_i n^\alpha]]$ for all $i = 1, ..., d$. Here, $[\cdot]$ denotes the integral part of the number while. $\lceil \cdot \rceil$ denotes its ceiling.

From now on, in this article we will assume (5). Furthermore it holds

$$(M_n(f))(\overrightarrow{x}) = \frac{\sum_{k_1=\lceil nx_1 - T_1 n^\alpha \rceil}^{[nx_1+T_1n^\alpha]} \cdots \sum_{k_d=\lceil nx_d - T_d n^\alpha \rceil}^{[nx_d+T_dn^\alpha]} f\left(\frac{k_1}{n}, ..., \frac{k_d}{n}\right)}{V(\overrightarrow{x})}\cdot \tag{6}$$

$$b\left(n^{1-\alpha}\left(x_1 - \frac{k_1}{n}\right), ..., n^{1-\alpha}\left(x_d - \frac{k_d}{n}\right)\right)$$

all $\overrightarrow{x} := (x_1, ..., x_d) \in \mathbb{R}^d$, where

$$V(\overrightarrow{x}) :=$$

$$\sum_{k_1=\lceil nx_1 - T_1 n^\alpha \rceil}^{[nx_1+T_1n^\alpha]} \cdots \sum_{k_d=\lceil nx_d - T_d n^\alpha \rceil}^{[nx_d+T_dn^\alpha]} b\left(n^{1-\alpha}\left(x_1 - \frac{k_1}{n}\right), ..., n^{1-\alpha}\left(x_d - \frac{k_d}{n}\right)\right).$$

Denote by $\|\cdot\|_{\infty}$ the maximum norm on $\mathbb{R}^d$, $d \geq 1$. So if $\left|n^{1-\alpha}\left(x_i - \frac{k_i}{n}\right)\right| \leq T_i$, all $i = 1, ..., d$, we get that

$$\left\|\overrightarrow{x} - \frac{\overrightarrow{k}}{n}\right\|_{\infty} \leq \frac{T^*}{n^{1-\alpha}},$$

where $\overrightarrow{k} := (k_1, ..., k_d)$.

Definition 5 *Let $f : \mathbb{R}^d \to \mathbb{R}$. We call*

$$\omega_1(f, h) := \sup_{\substack{all\ \overrightarrow{x}, \overrightarrow{y}: \\ \|\overrightarrow{x} - \overrightarrow{y}\|_{\infty} \leq h}} |f(\overrightarrow{x}) - f(\overrightarrow{y})|, \tag{7}$$

where $h > 0$, the first modulus of continuity of f.

Here is our first main result.

Theorem 6 *Let $\overrightarrow{x} \in \mathbb{R}^d$; then*

$$|(M_n(f))(\overrightarrow{x}) - f(\overrightarrow{x})| \leq \omega_1\left(f, \frac{T^*}{n^{1-\alpha}}\right). \tag{8}$$

Inequality (8) is attained by constant functions.

Inequality (8) gives $M_n(f)(\overrightarrow{x}) \to f(\overrightarrow{x})$, pointwise with rates, as $n \to +\infty$, where $\overrightarrow{x} \in \mathbb{R}^d$, $d \geq 1$.

Proof. Next, we estimate

$$|(M_n(f))(\overrightarrow{x}) - f(\overrightarrow{x})| \overset{(6)}{=}$$

$$\left|\sum_{k_1=\lceil nx_1 - T_1 n^{\alpha}\rceil}^{[nx_1 + T_1 n^{\alpha}]} \cdots \sum_{k_d=\lceil nx_d - T_d n^{\alpha}\rceil}^{[nx_d + T_d n^{\alpha}]} f\left(\frac{k_1}{n}, ..., \frac{k_d}{n}\right) \cdot \right.$$

$$\left.\frac{b\left(n^{1-\alpha}\left(x_1 - \frac{k_1}{n}\right), ..., n^{1-\alpha}\left(x_d - \frac{k_d}{n}\right)\right)}{V(\overrightarrow{x})} - f(\overrightarrow{x})\right| =$$

$$\left|\frac{\sum_{\overrightarrow{k}=\lceil n\overrightarrow{x} - \overrightarrow{T} n^{\alpha}\rceil}^{[n\overrightarrow{x} + \overrightarrow{T} n^{\alpha}]} \left(f\left(\frac{\overrightarrow{k}}{n}\right) - f(\overrightarrow{x})\right) b\left(n^{1-\alpha}\left(\overrightarrow{x} - \frac{\overrightarrow{k}}{n}\right)\right)}{V(\overrightarrow{x})}\right| \leq$$

$$\sum_{\overrightarrow{k}=\lceil n\overrightarrow{x} - \overrightarrow{T} n^{\alpha}\rceil}^{[n\overrightarrow{x} + \overrightarrow{T} n^{\alpha}]} \frac{\left|f\left(\frac{\overrightarrow{k}}{n}\right) - f(\overrightarrow{x})\right|}{V(\overrightarrow{x})} b\left(n^{1-\alpha}\left(\overrightarrow{x} - \frac{\overrightarrow{k}}{n}\right)\right) \leq$$

$$\sum_{\overrightarrow{k}=\lceil n\overrightarrow{x}-\overrightarrow{T}n^{\alpha}\rceil}^{[n\overrightarrow{x}+\overrightarrow{T}n^{\alpha}]} \frac{\omega_1\left(f, \left\|\overrightarrow{x}-\frac{\overrightarrow{k}}{n}\right\|_\infty\right)}{V\left(\overrightarrow{x}\right)} b\left(n^{1-\alpha}\left(\overrightarrow{x}-\frac{\overrightarrow{k}}{n}\right)\right).$$

That is

$$|(M_n(f))(\overrightarrow{x}) - f(\overrightarrow{x})| \leq \frac{\omega_1\left(f, \frac{T^*}{n^{1-\alpha}}\right)}{V\left(\overrightarrow{x}\right)} \cdot$$

$$\sum_{k_1=\lceil nx_1-T_1n^{\alpha}\rceil}^{[nx_1+T_1n^{\alpha}]} \cdots \sum_{k_d=\lceil nx_d-T_dn^{\alpha}\rceil}^{[nx_d+T_dn^{\alpha}]} b\left(n^{1-\alpha}\left(x_1-\frac{k_1}{n}\right), ..., n^{1-\alpha}\left(x_d-\frac{k_d}{n}\right)\right)$$

$$= \omega_1\left(f, \frac{T^*}{n^{1-\alpha}}\right), \tag{9}$$

proving the claim. ■

Our second main result follows.

Theorem 7 *Let $\overrightarrow{x} \in \mathbb{R}^d$, $f \in C^N\left(\mathbb{R}^d\right)$, $N \in \mathbb{N}$, such that all of its partial derivatives $f_{\widetilde{\alpha}}$ of order N, $\widetilde{\alpha} : |\widetilde{\alpha}| = N$, are uniformly continuous or continuous are bounded. Then,*

$$|(M_n(f))(\overrightarrow{x}) - f(\overrightarrow{x})| \leq \tag{10}$$

$$\left\{\sum_{j=1}^{N} \frac{(T^*)^j}{j!n^{j(1-\alpha)}}\left(\left(\sum_{i=1}^{d}\left|\frac{\partial}{\partial x_i}\right|\right)^j f\left(\overrightarrow{x}\right)\right)\right\} +$$

$$\frac{(T^*)^N d^N}{N!n^{N(1-\alpha)}} \cdot \max_{\widetilde{\alpha}:|\widetilde{\alpha}|=N} \omega_1\left(f_{\widetilde{\alpha}}, \frac{T^*}{n^{1-\alpha}}\right).$$

Inequality (10) is attained by constant functions. Also, (10) gives us with rates the pointwise convergence of $M_n(f) \to f$ over $\mathbb{R}^d$, as $n \to +\infty$.

Proof. Set

$$g_{\frac{\overrightarrow{k}}{n}}(t) := f\left(\overrightarrow{x} + t\left(\frac{\overrightarrow{k}}{n} - \overrightarrow{x}\right)\right), \quad 0 \leq t \leq 1.$$

Then

$$g_{\frac{\overrightarrow{k}}{n}}^{(j)}(t) =$$

$$\left[\left(\sum_{i=1}^{d}\left(\frac{k_i}{n} - x_i\right)\frac{\partial}{\partial x_i}\right)^j f\right]\left(x_1 + t\left(\frac{k_1}{n} - x_1\right), ..., x_d + t\left(\frac{k_d}{n} - x_d\right)\right)$$

and $g_{\frac{\overrightarrow{k}}{n}}(0) = f\left(\overrightarrow{x}\right)$. By Taylor's formula, we get

$$f\left(\frac{k_1}{n}, ..., \frac{k_d}{n}\right) = g_{\frac{\overrightarrow{k}}{n}}(1) = \sum_{j=0}^{N} \frac{g_{\frac{\overrightarrow{k}}{n}}^{(j)}(0)}{j!} + R_N\left(\frac{\overrightarrow{k}}{n}, 0\right),$$

where

$$R_N\left(\frac{\overrightarrow{k}}{n},0\right)=\int_0^1\left(\int_0^{t_1}...\left(\int_0^{t_{N-1}}\left(g_{\frac{\overrightarrow{k}}{n}}^{(N)}(t_N)-g_{\frac{\overrightarrow{k}}{n}}^{(N)}(0)\right)dt_N\right)...\right)dt_1.$$

Here we denote by

$$f_{\widetilde{\alpha}}:=\frac{\partial^{\widetilde{\alpha}}f}{\partial x^{\widetilde{\alpha}}},\quad \widetilde{\alpha}:=(\alpha_1,...,\alpha_d)\,,\ \alpha_i\in\mathbb{Z}^+,$$

$i=1,...,d$, such that $|\widetilde{\alpha}|:=\sum_{i=1}^d\alpha_i=N$. Thus,

$$\frac{f\left(\frac{\overrightarrow{k}}{n}\right)b\left(n^{1-\alpha}\left(\overrightarrow{x}-\frac{\overrightarrow{k}}{n}\right)\right)}{V\left(\overrightarrow{x}\right)}=$$

$$\sum_{j=0}^N\frac{g_{\frac{\overrightarrow{k}}{n}}^{(j)}(0)}{j!}\frac{b\left(n^{1-\alpha}\left(\overrightarrow{x}-\frac{\overrightarrow{k}}{n}\right)\right)}{V\left(\overrightarrow{x}\right)}+\frac{b\left(n^{1-\alpha}\left(\overrightarrow{x}-\frac{\overrightarrow{k}}{n}\right)\right)}{V\left(\overrightarrow{x}\right)}\cdot R_N\left(\frac{\overrightarrow{k}}{n},0\right).$$

Therefore

$$\left(M_n(f)\right)\left(\overrightarrow{x}\right)-f\left(\overrightarrow{x}\right)=$$

$$\sum_{\overrightarrow{k}=\lceil n\overrightarrow{x}-\overrightarrow{T}n^{\alpha}\rceil}^{\lceil n\overrightarrow{x}+\overrightarrow{T}n^{\alpha}\rceil}\frac{f\left(\frac{\overrightarrow{k}}{n}\right)}{V\left(\overrightarrow{x}\right)}b\left(n^{1-\alpha}\left(\overrightarrow{x}-\frac{\overrightarrow{k}}{n}\right)\right)-f\left(\overrightarrow{x}\right)=$$

$$\sum_{j=1}^N\frac{1}{j!}\left(\sum_{\overrightarrow{k}=\lceil n\overrightarrow{x}-\overrightarrow{T}n^{\alpha}\rceil}^{\lceil n\overrightarrow{x}+\overrightarrow{T}n^{\alpha}\rceil}g_{\frac{\overrightarrow{k}}{n}}^{(j)}(0)\frac{b\left(n^{1-\alpha}\left(\overrightarrow{x}-\frac{\overrightarrow{k}}{n}\right)\right)}{V\left(\overrightarrow{x}\right)}\right)+R^*,$$

where

$$R^*:=\sum_{\overrightarrow{k}=\lceil n\overrightarrow{x}-\overrightarrow{T}n^{\alpha}\rceil}^{\lceil n\overrightarrow{x}+\overrightarrow{T}n^{\alpha}\rceil}\frac{b\left(n^{1-\alpha}\left(\overrightarrow{x}-\frac{\overrightarrow{k}}{n}\right)\right)}{V\left(\overrightarrow{x}\right)}\cdot R_N\left(\frac{\overrightarrow{k}}{n},0\right).$$

Consequently, we obtain

$$\left|\left(M_n(f)\right)\left(\overrightarrow{x}\right)-f\left(\overrightarrow{x}\right)\right|\leq$$

$$\sum_{j=1}^N\frac{1}{j!}\left(\sum_{\overrightarrow{k}=\lceil n\overrightarrow{x}-\overrightarrow{T}n^{\alpha}\rceil}^{\lceil n\overrightarrow{x}+\overrightarrow{T}n^{\alpha}\rceil}\frac{\left|g_{\frac{\overrightarrow{k}}{n}}^{(j)}(0)\right|b\left(n^{1-\alpha}\left(\overrightarrow{x}-\frac{\overrightarrow{k}}{n}\right)\right)}{V\left(\overrightarrow{x}\right)}\right)+|R^*|=:\Theta.$$

Noyice that

$$\left|g_{\frac{\overrightarrow{k}}{n}}^{(j)}(0)\right|\leq\left(\frac{T^*}{n^{1-\alpha}}\right)^j\left(\left(\sum_{i=1}^d\left|\frac{\partial}{\partial x_i}\right|\right)^j f\left(\overrightarrow{x}\right)\right)$$

6

and

$$\Theta \leq \left\{ \sum_{j=1}^{N} \frac{1}{j!} \left(\frac{T^*}{n^{1-\alpha}} \right)^j \left(\left(\sum_{i=1}^{d} \left| \frac{\partial}{\partial x_i} \right| \right)^j f(\overrightarrow{x}) \right) \right\} + |R^*| . \tag{11}$$

That is, by (11), we get

$$|(M_n(f))(\overrightarrow{x}) - f(\overrightarrow{x})| \leq$$
$$\left\{ \sum_{j=1}^{N} \frac{(T^*)^j}{j! n^{j(1-\alpha)}} \left(\left(\sum_{i=1}^{d} \left| \frac{\partial}{\partial x_i} \right| \right)^j f(\overrightarrow{x}) \right) \right\} + |R^*| . \tag{12}$$

Next, we need to estimate $|R^*|$. For that, we observe $(0 \leq t_N \leq 1)$

$$\left| g_{\frac{\overrightarrow{k}}{n}}^{(N)}(t_N) - g_{\frac{\overrightarrow{k}}{n}}^{(N)}(0) \right| =$$
$$\left| \left(\sum_{i=1}^{d} \left(\frac{k_i}{n} - x_i \right) \frac{\partial}{\partial x_i} \right)^N f\left(\overrightarrow{x} + t_N \left(\frac{\overrightarrow{k}}{n} - \overrightarrow{x} \right) \right) - \right.$$
$$\left. \left(\sum_{i=1}^{d} \left(\frac{k_i}{n} - x_i \right) \frac{\partial}{\partial x_i} \right)^N f(\overrightarrow{x}) \right|$$
$$\leq \frac{(T^*)^N d^N}{n^{N(1-\alpha)}} \cdot \max_{\widetilde{\alpha}:|\widetilde{\alpha}|=N} \omega_1 \left(f_{\widetilde{\alpha}}, \frac{T^*}{n^{1-\alpha}} \right).$$

Thus,

$$\left| R_N \left(\frac{\overrightarrow{k}}{n}, 0 \right) \right| \leq \int_0^1 \left(\int_0^{t_1} \ldots \left(\int_0^{t_{N-1}} \left| g_{\frac{\overrightarrow{k}}{n}}^{(N)}(t_N) - g_{\frac{\overrightarrow{k}}{n}}^{(N)}(0) \right| dt_N \right) \ldots \right) dt_1$$
$$\leq \frac{(T^*)^N d^N}{N! n^{N(1-\alpha)}} \cdot \max_{\widetilde{\alpha}:|\widetilde{\alpha}|=N} \omega_1 \left(f_{\widetilde{\alpha}}, \frac{T^*}{n^{1-\alpha}} \right).$$

Therefore,

$$|R^*| \leq \sum_{\overrightarrow{k} = \lceil n\overrightarrow{x} - \overrightarrow{T} n^{\alpha} \rceil}^{[n\overrightarrow{x} + \overrightarrow{T} n^{\alpha}]} \frac{b\left(n^{1-\alpha} \left(\overrightarrow{x} - \frac{\overrightarrow{k}}{n} \right) \right)}{V(\overrightarrow{x})} \left| R_N \left(\frac{\overrightarrow{k}}{n}, 0 \right) \right|$$
$$\leq \frac{(T^*)^N d^N}{N! n^{N(1-\alpha)}} \cdot \max_{\widetilde{\alpha}:|\widetilde{\alpha}|=N} \omega_1 \left(f_{\widetilde{\alpha}}, \frac{T^*}{n^{1-\alpha}} \right). \tag{13}$$

By (12) and (13) we get (10). ■

Corollary 8 *Here, additionally assume that b is continuous on $\mathbb{R}^d$. Let*

$$\Gamma := \prod_{i=1}^{d} [-\gamma_i, \gamma_i] \subset \mathbb{R}^d, \ \ \gamma_i > 0,$$

and take

$$n \geq \max_{i \in \{1,\ldots,d\}} \left(T_i + \gamma_i, T_i^{-\frac{1}{\alpha}} \right).$$

Consider $p \geq 1$. Then,

$$\|M_n f - f\|_{p,\Gamma} \leq \omega_1 \left(f, \frac{T^*}{n^{1-\alpha}} \right) 2^{\frac{d}{p}} \prod_{i=1}^{d} \gamma_i^{\frac{1}{p}}, \tag{14}$$

attained by constant functions. From (14), we get the L_p convergence of $M_n f$ to f with rates.

Proof. By (8). ■

Corollary 9 *Same assumptions as in Corollary 8. Then*

$$\|M_n f - f\|_{p,\Gamma} \leq \left\{ \sum_{j=1}^{N} \frac{(T^*)^j}{j! n^{j(1-\alpha)}} \left\| \left(\sum_{i=1}^{d} \left| \frac{\partial}{\partial x_i} \right| \right)^j f \right\|_{p,\Gamma} \right\} +$$

$$\frac{(T^*)^N d^N}{N! n^{N(1-\alpha)}} \cdot \max_{\widetilde{\alpha}: |\widetilde{\alpha}| = N} \omega_1 \left(f_{\widetilde{\alpha}}, \frac{T^*}{n^{1-\alpha}} \right) 2^{\frac{d}{p}} \prod_{i=1}^{d} \gamma_i^{\frac{1}{p}}, \tag{15}$$

attained by constants. Here, from (15), we get again the L_p convergence of $M_n(f)$ to f with rates.

Proof. By the use of (10). ■

3 The multivariate "normalized squashing type operators" and their convergence to the unit with rates

We give the following definition

Definition 10 *Let the nonnegative function $S : \mathbb{R}^d \to \mathbb{R}$, $d \geq 1$, S has compact support $\mathcal{B} := \prod_{i=1}^{d} [-T_i, T_i]$, $T_i > 0$ and is nondecreasing there for each coordinate. S can be continuous only on either $\prod_{i=1}^{d}(-\infty, T_i]$ or $\mathcal{B}$ and can have jump discontinuities. We call S the multivariate "squashing function" (see also [3]).*

Example 11 *Let $\widehat{S}$ as above when $d=1$. Then,*

$$S\left(\overrightarrow{x}\right) := \widehat{S}\left(x_1\right) ... \widehat{S}\left(x_d\right), \quad \overrightarrow{x} := \left(x_1, ..., x_d\right) \in \mathbb{R}^d,$$

is a multivariate "squashing function".

Let $f : \mathbb{R}^d \to \mathbb{R}$ be either uniformly continuous or continuous and bounded function.

For $\overrightarrow{x} \in \mathbb{R}^d$, we define the multivariate "normalized squashing type operator",

$$L_n\left(f\right)\left(\overrightarrow{x}\right) :=$$

$$\frac{\sum_{k_1=-n^2}^{n^2} ... \sum_{k_d=-n^2}^{n^2} f\left(\frac{k_1}{n}, ..., \frac{k_d}{n}\right) S\left(n^{1-\alpha}\left(x_1 - \frac{k_1}{n}\right), ..., n^{1-\alpha}\left(x_d - \frac{k_d}{n}\right)\right)}{W\left(\overrightarrow{x}\right)}, \tag{16}$$

where $0 < \alpha < 1$ and $n \in \mathbb{N}$:

$$n \geq \max_{i \in \{1,...,d\}} \left\{T_i + \left|x_i\right|, T_i^{-\frac{1}{\alpha}}\right\}, \tag{17}$$

and

$$W\left(\overrightarrow{x}\right) := \sum_{k_1=-n^2}^{n^2} ... \sum_{k_d=-n^2}^{n^2} S\left(n^{1-\alpha}\left(x_1 - \frac{k_1}{n}\right), ..., n^{1-\alpha}\left(x_d - \frac{k_d}{n}\right)\right). \tag{18}$$

Obviously L_n is a positive linear operator. It is clear that

$$\left(L_n\left(f\right)\right)\left(\overrightarrow{x}\right) = \sum_{\overrightarrow{k} = \left\lceil n\overrightarrow{x} - \overrightarrow{T} n^{\alpha}\right\rceil}^{\left[n\overrightarrow{x} + \overrightarrow{T} n^{\alpha}\right]} \frac{f\left(\frac{\overrightarrow{k}}{n}\right)}{\Phi\left(\overrightarrow{x}\right)} S\left(n^{1-\alpha}\left(\overrightarrow{x} - \frac{\overrightarrow{k}}{n}\right)\right), \tag{19}$$

where

$$\Phi\left(\overrightarrow{x}\right) := \sum_{\overrightarrow{k} = \left\lceil n\overrightarrow{x} - \overrightarrow{T} n^{\alpha}\right\rceil}^{\left[n\overrightarrow{x} + \overrightarrow{T} n^{\alpha}\right]} S\left(n^{1-\alpha}\left(\overrightarrow{x} - \frac{\overrightarrow{k}}{n}\right)\right). \tag{20}$$

Here, we study the pointwise convergence with rates of $\left(L_n\left(f\right)\right)\left(\overrightarrow{x}\right) \to f\left(\overrightarrow{x}\right)$, as $n \to +\infty$, $\overrightarrow{x} \in \mathbb{R}^d$.

This is given by the next result.

Theorem 12 *Under the above terms and assumptions, we find that*

$$\left|\left(L_n\left(f\right)\right)\left(\overrightarrow{x}\right) - f\left(\overrightarrow{x}\right)\right| \leq \omega_1\left(f, \frac{T^*}{n^{1-\alpha}}\right). \tag{21}$$

Inequality (21) is attained by constant functions.

Proof. Similar to (8). ∎

We also give

Theorem 13 *Let $\overrightarrow{x} \in \mathbb{R}^d$, $f \in C^N\left(\mathbb{R}^d\right)$, $N \in \mathbb{N}$, such that all of its partial derivatives $f_{\widetilde{\alpha}}$ of order N, $\widetilde{\alpha} : |\widetilde{\alpha}| = N$, are uniformly continuous or continuous are bounded. Then,*

$$\left|\left(L_n\left(f\right)\right)\left(\overrightarrow{x}\right) - f\left(\overrightarrow{x}\right)\right| \leq \tag{22}$$

$$\left\{\sum_{j=1}^{N} \frac{\left(T^*\right)^j}{j! n^{j(1-\alpha)}} \left(\left(\sum_{i=1}^{d}\left|\frac{\partial}{\partial x_i}\right|\right)^j f\left(\overrightarrow{x}\right)\right)\right\} +$$

$$\frac{\left(T^*\right)^N d^N}{N! n^{N(1-\alpha)}} \cdot \max_{\widetilde{\alpha}:|\widetilde{\alpha}|=N} \omega_1\left(f_{\widetilde{\alpha}}, \frac{T^*}{n^{1-\alpha}}\right).$$

Inequality (22) is attained by constant functions. Also, (22) gives us with rates the pointwise convergence of $L_n\left(f\right) \to f$ over $\mathbb{R}^d$, as $n \to +\infty$.

Proof. Similar to (10). ∎

Note 14 *We see that*

$$M_n\left(1\right) = L_n\left(1\right) = 1.$$

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NEW APPROACH TO THE ANALOGUE OF LEBESGUE-RADON-NIKODYM THEOREM WITH RESPECT TO WEIGHTED p-ADIC q-MEASURE ON $\mathbb{Z}_p$

JOO-HEE JEONG, JIN-WOO PARK, AND SEOG-HOON RIM

Abstract In this paper we reprove the result in Kim et al 2011 by using Mahler expansion of uniformly differentiable function over $\mathbb{C}_p$. This result is related with Frobenius-Euler numbers.

1. Introduction

Let p be a fixed odd prime number. Throughout this paper, the symbols $\mathbb{Z}_p$, $\mathbb{Q}_p$, and $\mathbb{C}_p$ denote the ring of p-adic integers, the field of p-adic rational numbers, and the p-adic completion of the algebraic closure of $\mathbb{Q}_p$, respectively. Let ν_p be the normalized exponential valuation of $\mathbb{C}_p$ with $|p| = p^{-\nu_p(p)} = \frac{1}{p}$ and $\nu_p(0) = \infty$. When one speaks of q-extension, q can be regarded as an indeterminate, a complex $q \in \mathbb{C}$, or a p-adic number $q \in \mathbb{C}_p$. In this paper we assume that $q \in \mathbb{C}_p$ with $|1-q| < 1$ and we use the notations of q-numbers as follows:

$$[x]_q = [x:q] = \frac{1-q^x}{1-q}, \text{ and } [x]_{-q} = \frac{1-(-q)^x}{1+q}. \tag{1.1}$$

For any positive integer N, let

$$a + p^N\mathbb{Z}_p = \left\{x \in \mathbb{Z}_p | x \equiv a \pmod{p^N}\right\}, \tag{1.2}$$

where $a \in \mathbb{Z}$ satisfies the condition $0 \le a < p^N$(see [1-8]).

It is known that the fermionic p-adic q-measure on $\mathbb{Z}_p$ is given by Kim as follows:

$$\mu_{-q}(a + p^N\mathbb{Z}_p) = \frac{(-q)^a}{[p^N]_{-q}} = \frac{1+q}{1+q^{p^N}}(-q)^a, \text{ (see [7, 12, 13, 14, 15])}. \tag{1.3}$$

Let $C(\mathbb{Z}_p)$ be the space of continuous functions on $\mathbb{Z}_p$. From (1.3), the fermionic p-adic q-integral on $\mathbb{Z}_p$ is defined by Kim as follows:

$$I_{-q}(f) = \int_{\mathbb{Z}_p} f(x)d\mu_{-q}(x) = \lim_{N\to\infty} \frac{1}{[p^N]_{-q}} \sum_{x=0}^{p^N-1} f(x)(-q)^x, \tag{1.4}$$

$f \in C(\mathbb{Z}_p)$(see [1, 7, 12, 13, 14, 15]). From (1.4) we have the following integral equation.

$$qI_{-q}(f_1) + I_{-q}(f) = [2]_q f(x) \tag{1.5}$$

where $f_1(x) = f(x+1)$.

Let us take $f(x) = e^{tx}$ in (1.5), we have

$$(qe^t + 1)\int_{\mathbb{Z}_p} e^{xt}d\mu_{-q}(x) = [2]_q. \tag{1.6}$$

2010 Mathematics Subject Classification : 11B68, 11S80.

Key words and phrases : p-adic q-measure, Lebesgue-Radon-Nikodym .

Thus

$$\begin{aligned}\int_{\mathbb{Z}_p} e^{xt} d\mu_{-q}(x) &= \frac{1+q}{qe^t+1} = \frac{1+q^{-1}}{e^t+q^{-1}} \\ &= \sum_{n=0}^{\infty} H_n(-q^{-1}) \frac{t^n}{n!}\end{aligned} \tag{1.7}$$

where $H_n(-q^{-1})$ is well-known the nth Frobenius-Euler number(see [3]). Thus

$$\int_{\mathbb{Z}_p} x^n d\mu_{-q}(x) = H_n(-q^{-1}). \tag{1.8}$$

The relation between Frobenius-Euler numbers $H_n(q)$ and q-Euler numbers $\tilde{\varepsilon}_{n,q}$ are given as follows(see, [3])

$$\frac{[2]_q}{2} \tilde{\varepsilon}_{n,q} = H_n(-q^{-1}).$$

We will reprove the analogue of the Lebesgue-Radon-Nikodym theorem with respect to weighted p-adic q-measure on $\mathbb{Z}_p$. We use Mahler expansion of uniformly differentiable function over $\mathbb{C}_p$, this result is related with Frobenius-Euler numbers. In special case, the weight q^x is 1, we can derive the same result as Kim et al, 2011(see [10]). And if $q = 1$, we have the same result as Kim, 2012(see [4]).

2. Lebesgue-Radon-Nikodym's type theorem with respect to weighted p-adic q-measure on $\mathbb{Z}_p$

For any positive integer a and n with $a < p^n$, and $f \in C(\mathbb{Z}_p)$, let us define

$$\tilde{\mu}_{f,-q}(a + p^n\mathbb{Z}_p) = \int_{a+p^n\mathbb{Z}_p} q^{-x} f(x) d\mu_{-q}(x), \tag{2.1}$$

where the integral is the fermionic p-adic q-integral on $\mathbb{Z}_p$.

From (1.3), (1.4) and (2.1), we note that

$$\begin{aligned}\tilde{\mu}_{f,-q}(a + p^n\mathbb{Z}_p) &= \lim_{m\to\infty} \frac{1}{[p^{m+n}]_{-q}} \sum_{x=0}^{p^m-1} q^{-(a+p^n x)} f(a+p^n x)(-q)^{a+p^n x} \\ &= \lim_{m\to\infty} \frac{(-1)^a}{[p^m]_{-q}} \sum_{x=0}^{p^{m-n}-1} f(a+p^n x)(-q)^{-p^n x} q^{p^n x}(-1)^x \\ &= \frac{[2]_q}{[2]_{q^{p^n}}} (-1)^a \lim_{m\to\infty} \frac{1}{[p^{m-n}]_{-q^{p^n}}} \sum_{x=0}^{p^{m-n}-1} f(a+p^n x)(-q^{p^n})^x \\ &= \frac{[2]_q}{[2]_{q^{p^n}}} (-1)^a \int_{\mathbb{Z}_p} q^{-p^n x} f(a+p^n x) d\mu_{-q^{p^n}}(x).\end{aligned} \tag{2.2}$$

By (2.2), we get

$$\tilde{\mu}_{f,-q}(a + p^n\mathbb{Z}_p) = \frac{[2]_q}{[2]_{q^{p^n}}} (-1)^a \int_{\mathbb{Z}_p} q^{-p^n x} f(a+p^n x) d\mu_{-q^{p^n}}(x). \tag{2.3}$$

Therefore, by (2.3), we obtain the following theorem.

Theorem 2.1. *For $f, g \in C(\mathbb{Z}_p)$, we have*

$$\tilde{\mu}_{\alpha f+\beta g,-q} = \alpha\tilde{\mu}_{f,-q} + \beta\tilde{\mu}_{g,-q}, \tag{2.4}$$

where α, β are constants.

From (2.2) and (2.4), we note that

$$\left|\tilde{\mu}_{f,-q}(a+p^n\mathbb{Z}_p)\right| \le M\|f_q\|_\infty, \tag{2.5}$$

where $\|f_q\|_\infty = \sup_{x\in\mathbb{Z}_p} |q^{-x}f(x)|$ and M is some positive constant.

Now, we recall the definition of the strongly fermionic p-adic q-measure on $\mathbb{Z}_p$. If μ_{-q} is satisfied the following equation:

$$\left|\mu_{-q}(a+p^n\mathbb{Z}_p) - \mu_{-q}(a+p^{n+1}\mathbb{Z}_p)\right| \le \delta_{n,q}, \tag{2.6}$$

where $\delta_{n,q} \to 0$ and $n \to \infty$ and $\delta_{n,q}$ is independent of a, then μ_{-q} is called the weakly fermionic p-adic q-measure on $\mathbb{Z}_p$.

If $\delta_{n,q}$ is replaced by Cp^{-n} (C is some constant), then μ_{-q} is called *strongly fermionic p-adic q-measure* on $\mathbb{Z}_p$.

Let $P(x) \in \mathbb{C}_p[x]$ be an arbitrary polynomial with $\sum a_i x^i$. Then we see that $\mu_{P,-q}$ is strongly fermionic p-adic q-measure on $\mathbb{Z}_p$. Without a loss of generality, it is enough to prove the statement for $P(x) = x^k$.

Let a be an integer with $0 \le a < p^n$. Then we get

$$\tilde{\mu}_{P,-q}(a+p^n\mathbb{Z}_p) = \frac{[2]_q}{[2]_{q^{p^n}}}(-q)^a \lim_{m\to\infty} \sum_{i=0}^{p^{m-n}-1} (a+ip^n)^k(-1)^i q^{p^n i}, \tag{2.7}$$

and

$$(a+ip^n)^k = \sum_{l=0}^{k} a^{k-l}\binom{k}{l}(ip^n)^l \equiv a^k \pmod{p^n}.$$

By (1.8) and (2.7), we easily get

$$\begin{aligned}\tilde{\mu}_{P,-q}(a+p^n\mathbb{Z}_p) &\equiv \frac{2[2]_q}{[2]^2_{q^{p^n}}}(-q)^a a^k H_0(-q^{p^n}) \pmod{p^n}\\ &\equiv \frac{2[2]_q}{[2]^2_{q^{p^n}}}(-q)^a P(a) H_0(-q^{p^n}) \pmod{p^n}.\end{aligned} \tag{2.8}$$

We can rewrite (2.7) as

$$\begin{aligned}&\tilde{\mu}_{p,-q}(a+p^n\mathbb{Z}_p)\\ =&\frac{[2]_q}{[2]_{q^{p^n}}}(-q)^a \lim_{m\to\infty} \sum_{i=0}^{p^{m-n}-1} \left\{a^k q^{p^n i}(-1)^i + a^{k-1}(p^n i)q^{p^n i}(-1)^i + \cdots + (p^n i)^k q^{p^n i}(-1)^i\right\}\\ =&\frac{[2]_q}{[2]_{q^{p^n}}}(-q)^a \left\{a^k\tilde{\varepsilon}_{0,q^{p^n}} + a^{k-1}p^n\tilde{\varepsilon}_{1,q^{p^n}} + \cdots + p^{nk}\tilde{\varepsilon}_{k,q^{p^n}}\right\}\end{aligned}$$

where

$$\begin{aligned}\tilde{\varepsilon}_{i,q} &= \int_{\mathbb{Z}_p} q^x x^i d\mu_i(x)\\ &= \frac{2}{[2]_q} H_i(-q^{-1}\end{aligned}$$

(see [3]).

Let x be an arbitrary in $\mathbb{Z}_p$ with $x \equiv x_n \pmod{p^n}$ and $x \equiv x_{n+1} \pmod{p^{n+1}}$, where x_n and x_{n+1} are positive integers such that $0 \leq x_n < p^n$ and $0 \leq x_{n+1} < p^{n+1}$. Thus, by (2.8)), we have

$$\left|\tilde{\mu}_{P,-q}(a+p^n\mathbb{Z}_p) - \tilde{\mu}_{P,-q}(a+p^{n+1}\mathbb{Z}_p)\right| \leq Cp^{-n}, \tag{2.9}$$

where C is a positive some constant and $n \gg 0$.

Let

$$f_{\tilde{\mu}_{P,-q}}(a) = \lim_{n\to\infty} \tilde{\mu}_{P,-q}(a+p^n\mathbb{Z}_p). \tag{2.10}$$

Then, (2.5), (2.7), and (2.8), we get

$$\begin{aligned} f_{\tilde{\mu}_{P,-q}}(a) &= \frac{[2]_q}{2}(-1)^a a^k \\ &= \frac{[2]_q}{2}(-1)^a P(a). \end{aligned} \tag{2.11}$$

Since $f_{\tilde{\mu}_{P,-q}}(x)$ is continuous on $\mathbb{Z}_p$, it follows for all $x \in \mathbb{Z}_p$

$$f_{\tilde{\mu}_{P,-q}}(x) = \frac{[2]_q}{2}(-1)^x P(x). \tag{2.12}$$

Let $g \in C(\mathbb{Z}_p)$. By (2.10), (2.11) and (2.12), we get

$$\begin{aligned} \int_{\mathbb{Z}_p} g(x)d\tilde{\mu}_{P,-q}(x) &= \lim_{n\to\infty}\sum_{i=0}^{p^n-1} g(i)\tilde{\mu}_{P,-q}(i+p^n\mathbb{Z}_p) \\ &= \frac{[2]_q}{2}\lim_{n\to\infty}\sum_{i=0}^{p^n-1} g(i)(-q)^i i^k \\ &= \int_{\mathbb{Z}_p} q^{-x}g(x)x^k d\mu_{-q}(x). \end{aligned} \tag{2.13}$$

Therefore, by (2.13), we obtain the following theorem.

Theorem 2.2. *Let $P(x) \in \mathbb{C}_p[x]$ be an arbitrary polynomial with $\sum a_i x^i$. Then $\tilde{\mu}_{P,-q}$ is a strongly fermionic weighted p-adic q-measure on $\mathbb{Z}_p$ and for all $x \in \mathbb{Z}_p$*

$$f_{\tilde{\mu}_{P,-q}} = (-1)^x \frac{[2]_q}{2} P(x). \tag{2.14}$$

Furthermore, for any $g \in C(\mathbb{Z}_p)$, we have

$$\int_{\mathbb{Z}_p} g(x)d\tilde{\mu}_{P,-q}(x) = \int_{\mathbb{Z}_p} q^{-x}g(x)P(x)d\mu_{-q}(x), \tag{2.15}$$

where the second integral is fermionic p-adic q-integral on $\mathbb{Z}_p$.

We adopt the technique of Kim in [4].

Let $f(x) = \sum_{n=0}^{\infty} a_n \binom{x}{n}$ be the Mahler expansion of a uniformly differentiable function of f, where $\binom{x}{n}$ stands for the binomial coefficient. In this case, $\lim_{n\to\infty} n|a_n|_p = 0$. Let $f_m(x) = \sum_{i=0}^{m} a_i \binom{x}{i} \in \mathbb{C}_p[x]$. Then

$$||f - f_m|| \leq \sup_{n\geq m} n|a_n|_p. \tag{2.16}$$

Writing $f = f_m + f - f_m$, we easily get

$$\begin{aligned}&\left|\tilde{\mu}_{f,-q}(a+p^n\mathbb{Z}_p) - \tilde{\mu}_{f,-q}(a+p^{n+1}\mathbb{Z}_p)\right| \\ &\le \max\{\left|\tilde{\mu}_{f_m,-q}(a+p^n\mathbb{Z}_p) - \tilde{\mu}_{f_m,-q}(a+p^{n+1}\mathbb{Z}_p)\right|, \\ &\qquad \left|\tilde{\mu}_{f-f_m,-q}(a+p^n\mathbb{Z}_p) - \tilde{\mu}_{f-f_m,-q}(a+p^{n+1}\mathbb{Z}_p)\right|\}.\end{aligned} \tag{2.17}$$

From Theorem 2.2, we note that

$$\left|\tilde{\mu}_{f-f_m,-q}(a+p^n\mathbb{Z}_p)\right| \le \|f-f_m\|_\infty \le C_1 p^{-n}, \tag{2.18}$$

where C_1 is some positive constant.

For $m \gg 0$, we have $\|f\|_\infty = \|f_m\|_\infty$.

So,

$$\left|\tilde{\mu}_{f_m,-q}(a+p^n\mathbb{Z}_p) - \tilde{\mu}_{f_m,-q}(a+p^{n+1}\mathbb{Z}_p)\right| \le C_2 p^{-n}, \tag{2.19}$$

where C_2 is also some positive constant.

By (2.18) and (2.19), we see that

$$\begin{aligned}&\left|f(a) - \tilde{\mu}_{f,-q}(a+p^n\mathbb{Z}_p)\right| \\ &\le \max\{\left|f(a)-f_m(a)\right|, \left|f_m(a) - \tilde{\mu}_{f_m,-q}(a+p^n\mathbb{Z}_p)\right|, \left|\tilde{\mu}_{f-f_m,-q}(a+p^n\mathbb{Z}_p)\right|\} \\ &\le \max\{\left|f(a)-f_m(a)\right|, \left|f_m(a) - \tilde{\mu}_{f_m,-q}(a+p^n\mathbb{Z}_p)\right|, \|f-f_m\|_\infty\}.\end{aligned} \tag{2.20}$$

If we fix $\epsilon > 0$ and fix m such that $\|f - f_m\| \le \epsilon$, then for $n \gg 0$, we have

$$\left|f(a) - \tilde{\mu}_{f,-q}(a+p^n\mathbb{Z}_p)\right| \le \epsilon. \tag{2.21}$$

Hence, we have

$$f_{\mu_{f,-q}}(a) = \lim_{n\to\infty} \tilde{\mu}_{f,-q}(a+p^n\mathbb{Z}_p) = \frac{[2]_q}{2}(-1)^a f(a). \tag{2.22}$$

Let m be the sufficiently large number such that $\|f - f_m\|_\infty \le p^{-n}$.

Then we get

$$\begin{aligned}\tilde{\mu}_{f,-q}(a+p^n\mathbb{Z}_p) &= \tilde{\mu}_{f_m,-q}(a+p^n\mathbb{Z}_p) + \tilde{\mu}_{f-f_m,-q}(a+p^n\mathbb{Z}_p) \\ &= \tilde{\mu}_{f_m,-q}(a+p^n\mathbb{Z}_p) \\ &= (-1)^a \frac{[2]_q}{[2]_{q^{p^n}}} f(a) \pmod{p^n}.\end{aligned} \tag{2.23}$$

For any $g \in C(\mathbb{Z}_p)$, we have

$$\int_{\mathbb{Z}_p} g(x)d\tilde{\mu}_{f,-q}(x) = \int_{\mathbb{Z}_p} q^{-x} f(x)g(x)d\mu_{-q}(x). \tag{2.24}$$

Assume that f is the function from $C(\mathbb{Z}_p)$ to $Lip(\mathbb{Z}_p)$. By the definition of $\tilde{\mu}_{-q}$, we easily see that $\tilde{\mu}_{-q}$ is a strongly p-adic q-measure on $\mathbb{Z}_p$ and for $n \gg 0$

$$\left|f_{\tilde{\mu}_{-q}}(a) - \tilde{\mu}_{-q}(a+p^n\mathbb{Z}_p)\right| \le C_3 p^{-n}, \tag{2.25}$$

where C_3 is some positive constant.

If $\tilde{\mu}_{1,-q}$ is associated strongly fermionic weighted p-adic q-measure on $\mathbb{Z}_p$, then we have

$$\left|\tilde{\mu}_{1,-q}(a+p^n\mathbb{Z}_p) - f_{\tilde{\mu}_{-q}}(a)\right| \le C_4 p^{-n}, \tag{2.26}$$

where $n \gg 0$ and C_4 is some positive constant.

From (2.26), we get

$$\begin{aligned} &\left|\tilde{\mu}_{-q}(a+p^n\mathbb{Z}_p)-\tilde{\mu}_{1,-q}(a+p^n\mathbb{Z}_p)\right| \\ &\leq \left|\tilde{\mu}_{-q}(a+p^n\mathbb{Z}_p)-f_{\tilde{\mu}_{-q}}(a)\right|+\left|f_{\tilde{\mu}_{-q}}(a)-\tilde{\mu}_{1,-q}(a+p^n\mathbb{Z}_p)\right| \leq K, \end{aligned} \tag{2.27}$$

where K is some positive constant.

Therefore, $\tilde{\mu}_{-q}-\tilde{\mu}_{1,-q}$ is a q-measure on $\mathbb{Z}_p$. Hence, we obtain the following theorem.

Theorem 2.3. *Let $\tilde{\mu}_{-q}$ be a strongly fermionic weighted p-adic q-measure on $\mathbb{Z}_p$, and assume that the fermionic weighted Radon-Nikodym derivative $f_{\tilde{\mu}_{-q}}$ on $\mathbb{Z}_p$ is continuous function on $\mathbb{Z}_p$. Suppose that $\tilde{\mu}_{1,-q}$ is the strongly fermionic weighted p-adic q-measure associated to $f_{\tilde{\mu}_{-q}}$. Then there exists a q-measure $\tilde{\mu}_{2,-q}$ on $\mathbb{Z}_p$ such that*

$$\tilde{\mu}_{-q}=\tilde{\mu}_{1,-q}+\tilde{\mu}_{2,-q}. \tag{2.30}$$

Acknowledgements

This research was supported by the Kyungpook National University Research Fund 2012.

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J. COMPUTATIONAL ANALYSIS AND APPLICATIONS, VOL. 15, NO.7, 1317-1331, 2013, COPYRIGHT 2013 EUDOXUS PRESS, LLC

Generalized Tikhonov regularization method for large-scale linear inverse problems*

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Abstract

In this paper we propose a regularization of general Tikhonov type for large-scale ill-posed problems. We introduce the projection method of iterative bidiagonalization and show that the regularization parameter can be chosen without prior knowledge of the noise variance by using the method of balancing principle. An algorithm implicate the efficient numerical realization of the new choice rule. Numerical experiments for severely ill-show benchmark inverse problems show that new method is effective compared with other criterions.

Key words: General Tikhonov regularization; Lanczos bidiagonalization; Iterative method; Balancing princple.

1 Introduction

This paper is concerned with the computation of an approximate solution of linear inverse problems. We focus on a common degradation model:

$$Ax = b, \tag{1.1}$$

where $x \in \mathbf{C}^n$, $A \in \mathbf{C}^{m\times n}$, in particular A is severely ill-conditioned and may be singular. An additive zero-mean Gaussian white noise $e \in \mathbf{C}^m$ of standard deviation δ_0, and we assume that the δ_0 is unknown. Thus the right-hand side b is obtained by

$$b = \widehat{b} + e, \tag{1.2}$$

*This research is supported by NSFC (61170311), Chinese Universities Specialized Research Fund for the Doctoral Program (20110185110020), Sichuan Province Sci. & Tech. Research Project (12ZC1802).

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and assume that the unavailable noise-free system

$$Ax = \widehat{b}. \tag{1.3}$$

Let $\widehat{x}$ denote the solution of (1.3), e.g., the least-squares solution of minimal Euclidean norm. We would like to determine an approximation of $\widehat{x}$ by computing a suitable approximate solution of minimal least-squares (LS) problem

$$\min_{x \in \mathbf{C}^n} \|Ax - b\|, \tag{1.4}$$

where $\|\cdot\|$ denotes the Euclidean vector norm. Due to the vector b is very sensitive to perturbations, the naive least-squares solution $x_{ls} = A^{\dagger}b$ (where $A^{\dagger}$denotes the pseudoinverse of A) is dominated by inaccuracies, therefore the LS problem generally does not yield meaningful approximation of $\widehat{x}$. It is well known, the replacement of the LS problem commonly is referred to as Tikhonov regularization, which is one of the most popular method. This method amounts to replacing the LS problem (1.3) by

$$\min_{x \in \mathbf{C}^n} \{\|Ax - b\|^2 + \mu\|Lx\|^2\}, \tag{1.5}$$

where the matrix $L \in \mathbf{C}^{l \times n}$ is a regularization operator, with $l \leq n$, and the scalar $\mu > 0$ is a regularization parameter. For future reference, let M^* denote the adjoint of the matrix M. We note that the normal equations associated with (1.5) are given by

$$(A^*A + \mu L^*L)x = A^*b, \tag{1.6}$$

whose solution is $x_\mu = (A^*A + \mu L^*L)^{-1}A^*b$, and the problem is how to select the parameter μ such that x_μ becomes as close as possible to the noise-free solution. We assume that

$$N(A) \cap N(L) = \{0\},$$

where $N(M)$ denotes the null space of the matrix M, which guarantees the uniqueness of the minimizer.

The choice of a suitable value of μ is an essential part of Tikhonov regularization. The value of μ determines how sensitive the solution x_μ of (1.6) is to the error e and how close x_μ is to the solution $\widehat{x}$. How the discrepancy principle to determine a suitable value of parameter μ for large-scale problems is discussed in [3], but the discrepancy principle must be employed only when the norm of e is known. Other choice rules are especially attractive in not requiring any precise knowledge of the noise level δ, e.g. quasi-optimality criterion [10], generalized cross-validation (GCV) [9], and L-curve criterion [5, 16]. The latter two have been very popular in the engineering community since they have been delivered encouraging results for many practical inverse problems. In the case of GCV, efficient implementation for Tikhonov regularization requires computing the SVD of the matrix A [17], which may be computationally impractical for large scale ill-posed problems. Then we take the very popular L-curve criterion for an instance. Theoretically, various nonconvergence results have been established for the L-curve criterion, and the existence of a corner is not ensured. In this paper we propose an augmented Tikhonov functional balancing principle for choosing the regularization parameter, then we combine this rule with quasi-optimality criterion to form a new parameter selection method.

There are many efficient methods available for the solution of large-scale Tikhonov minimization problems (1.5) with a general linear regularization operator. When the matrices A and L are of small to moderate size, one of most popular method to solve (1.6) is the generalized singular value decomposition (GSVD) method, see, e.g., [1], [2]. If we are concerned with the situation when A and L are computed their GSVD, for the mass matrix singular value decomposition (SVD) is likely just a waste. So looking for a low-cost method is very necessary and meaningful.

A popular approach to determine an approximation of $\widehat{x}$ for large-scale discrete ill-posed problems is to apply a few steps of an iterative method to (1.5). The new choice rule applies readily to Tikhonov regularization of a very general type. An iterative algorithm is based on Lanczos bidiagonalization and QR factorization, which is chosen for solve the general type. This makes the method suitable for the solution of large-scale Tikhonov minimization problems (1.5) with fairly general linear regularization operators L. The iterative method is easier to calculate regularization parameters base on general type. It is very important to determine a reliable stopping rule that can be partially chosen by combining Krylov subspace projection method with the convergency of regularization algorithm.

The rest of the paper is organized as follows. Section 2 reviews the iterative method which transform the large-scale minimization problem into a small size model. Section 3 discusses how to compute an regularization parameter, proposes an iterative algorithm for efficient numerical computation and determines a stopping rules. Section 4 presents numerical results for several benchmark inverse problems to illustrate relevant features of the proposed method, and a comparison with the quasi-optimally, L-curve criterion. Concluding remarks can be found in section 5.

2 The Lanczos and QR projection

In this section we describe an approach to regularization of the projected problem that arises from using Krylov subspace method, give enough details to make the costs apparent and show that the ideas are easy to program. Many projected problems have been proposed in [9]. We can solve large-scale, ill-posed inverse problems efficiently through combination the projected problem like the Lanczos bidiagonalization (LBD) with a direct method like the Tikhonov regularization. Good low-rank approximations can be directly obtained from the Lanczos bidiagonalization process which apply to the given matrix without computing any SVD, and this technique reduces the corresponding residual computational cost. The Lanczos bidiagonal process is introduced in details by Simon and Zha [12].

We want to evaluate an approximate solution of the Tikhonov minimization problem (1.5), by computing a partial Lancos bidiagonalization of the matrix A.

The methods compute sequences of projections of A onto judiciously chosen low-dimensional subspaces. We apply k steps of partial Lanczos bidiagonalization to the matrix A with initial unit vector $u_1 = b/\|b\|$. After the k step iterations, it has effectively computed three matrices: a lower-bidiagonal matrix $B_k \in \mathbf{C}^{(k+1)\times k}$, $U_k \equiv [u_1, \ldots, u_{k+1}]$ and $V_k \equiv [\upsilon_1, \cdots, \upsilon_k]$, with the relationship

$$b = \|b\|u_1 = U_{k+1}e_1\|b\|, \quad AV_k = U_{k+1}B_k, \tag{2.1}$$

where e_i denotes the ith unit vector, $U_k \in \mathbf{C}^{m\times(k+1)}$, $V_k \in \mathbf{C}^{n\times k}$, columns of U_k and V_k form an orthogonal basis, V_k spanned the k dimension subspace.

Now suppose we want to solve (1.5), the solution we seek in k dimension subspace is the form of $x^{(k)} = V_k y^{(k)}$ for some vector $y^{(k)}$ of length k. The corresponding residual is given by $r^{(k)} = b - Ax^{(k)}$ and observe that

$$r^{(k)} = \|b\|u_1 - AV_k y^{(k)} = U_{k+1}(\|b\|e_1 - B_k y^{(k)}).$$

Since U_{k+1} has orthogonal columns, computed the solution of the Tikhonov minimization problem (1.5) that we wish to solve

$$\min_{y^{(k)}\in\mathbf{C}^n} \{\|B_k y^{(k)} - \|b\|e_1\|^2 + \mu\|LV_k y^{(k)}\|^2\}. \tag{2.2}$$

In this minimization problem, though the matrix L is sparse matrix and the effort of evaluating the matrix-vector products is much smaller than matrix A and A^T, we still need to calculate the matrix-vector products LV_k. It is convenient to use the QR factorization of LV_k, introduce the factorizations

$$LV_k = Q_k R_k, \tag{2.3}$$

where $Q_k \in \mathbf{C}^{p\times k}$ has orthogonal columns and $R_k \in \mathbf{C}^{k\times k}$ is upper triangular. In applications of interest $k \ll l$, the factorization (2.3) can be computed quite rapidly. Through the projection transformation, and unitary invariance of the norm, the data fitting term and the penalty term have been changed. So the problem (2.2) will be translated into the reduced minimization problem

$$\min_{y^{(k)}\in\mathbf{C}^n} \{\|B_k y^{(k)} - \|b\|e_1\|^2 + \mu\|R_k y^{(k)}\|^2\}, \tag{2.4}$$

with the associated normal equations

$$(B_k^T B_k + \mu R_k^T R_k)y^{(k)} = R_k^T\|b\|e_1. \tag{2.5}$$

Therefore, we store $[B_k,\ \mu R_k]^T$ and use it when solving the least squares problems. Since typically the $k-$dimension subspace is quite small, this Tikhovov minimization problem can be solved efficiently by (2.5), also this method makes the evaluation of the parameter selection method cheaper than the initial evaluation. When the number of bidiagonalization steps k is increasing, the QR factorization of LV_k has to be updated, because of the k is quite small, the QR factorizations can be updated at negligible cost. It is worth noting that only the upper triangular matrices $R_k,\ k = 1, 2, \cdots$, are required, but not the associate matrices Q_k with orthogonal columns. After a suitable parameter values is calculated, the third part will introduce parameter selection method, then we choose a method working out the minimum solution $y^{(k)}$ of (2.4) which is easy to solve, the corresponding approximate solution $x^{(k)}$ of (1.5) is given by

$$x^{(k)} = V_k y^{(k)}, \text{ and } \|x^{(k)}\| = \|y^{(k)}\|.$$

Since the projection process only used k steps of the Lanczos bidiagonalization, we must choose an integer k properly. It is worth noting that the integer k is assumed to be small, so that the approximate solution $y_\mu^{(k)}$ for μ-values of interest provide meaningful approximations of the corresponding solution x_u of (1.6). There may be many approaches for selecting a suitable number of bidiagonalization steps. In generally, it was choose at will, but in this paper, we set the smallest integer for which

$$\min_k \{\sigma_k < \epsilon\sigma_1,\ \ 30\}. \tag{2.6}$$

A typical value of ϵ is $\epsilon = \sqrt{\text{machine precision}}$, where σ_k are the singular values of B_k given by its SVD.

3 Determining the regularization parameter

3.1 Parameter selection method

Firstly, we give the definition of the value function $F(\mu)$ as follow

$$F(\mu) = \inf_x \{\|Ax - b\|^2 + \mu\|Lx\|^2\}. \tag{3.1}$$

The value function $F(\mu)$ is monotonically increasing and concave. Thus it is continuous everywhere and differentiable except perhaps on a countable set (see [7] for the theoretical studies details). In this section we discuss the computation of μ based on the balancing principle so that the solution of (2.4) meet $y_k = y_k^{(\mu)}$.

We introduce the augmented Tikhonov (a-Tikh) functional $\mathcal{J}(x, \lambda, \tau)$ which is derived from the hierarchical Bayesian inference [4]. The functional is defined by

$$\mathcal{J}(x, \lambda, \tau) = \tau\|Ax - b\|^2 + \lambda\|Lx\|^2 + \beta_0\lambda - \alpha_0 \ln\lambda + \beta_1\tau - \alpha_1 \ln\tau, \tag{3.2}$$

where $\alpha_0 \approx \frac{m'}{2}$ ($m' = \text{rank}(L)$), $\alpha_1 \approx \frac{n}{2}$, and the parameter pairs $(\alpha_0,\ \beta_0)$, $(\alpha_1,\ \beta_1)$ are related to shape parameters of Gamma distributions for the scalar unknowns λ and τ, respectively, which afford a priori statistical knowledge of the fidelity and the penalty [4, 13]. Let $\mu = \lambda/\tau$. Then the necessary optimality condition of the a-Tikh functional (3.2) is given by

$$\begin{cases} x_\mu^\delta = \arg\min_x\{\|Ax-b\|^2+\mu\|Lx\|^2\}, \\ \lambda^* = \frac{\alpha_0}{\|Lx_\mu^\delta\|^2+\beta_0}, \\ \tau^* = \frac{\alpha_1}{\|Ax_\mu^\delta-b\|^2+\beta_1}. \end{cases}$$

Hence the regularization parameter μ^* satisfies

$$\frac{\alpha_1}{\alpha_0}\mu^*(\|Lx_\mu^\delta\|^2+\beta_0) = \|Ax_\mu^\delta-b\|^2+\beta_1. \tag{3.3}$$

The fix-point method can be regarded as a realization of a parameter choice rule which was devised in [8]. We assume $F(\mu)$ is positive for all $\mu > 0$, which holds for all commonly models. A rule finds a $\mu > 0$ by minimizing

$$\Phi_\gamma(\mu) = \frac{(F(\mu)+\mu\beta_0+\beta_1)^{1+\gamma}}{\mu}, \tag{3.4}$$

for proper $\gamma > 0$, i.e. $\gamma = \frac{\alpha_1}{\alpha_0}$. The rule $\Phi(\mu)$ follows from the equation (3.3) and the derivation method of $\Phi(\mu)$ is similar to the rule in [7]. If $\mu^* > 0$ is a local minimizer of $\Phi(\mu)$, then $\mu^* = \lambda^*/\tau^*$ holds for all minimizers x_μ of (1.5), when F is differentiable at μ.

Next we use a-Tikh functional based on the iterative decompose method as described in the previous section. Respectively, $y_\mu^\delta,\ \lambda^*,\ \tau^*$, were expressed as follows,

$$\begin{cases} y_\mu^\delta = \arg\min_y\{\|B_k y-\|b\|e_1\|^2+\mu\|R_k y\|^2\}, \\ \lambda^* = \frac{\alpha_0}{\|R_k y_\mu^\delta\|^2+\beta_0}, \\ \tau^* = \frac{\alpha_1}{\|B_k y_\mu^\delta-\|b\|e_1\|^2+\beta_1}. \end{cases} \tag{3.5}$$

Equation (3.5) and the numerical experiments in [4] indicate that the quantity

$$\delta^2 = \tau^{*-1} = (\|B_k y^{(k)}-\|b\|e_1\|^2+\beta_1)\alpha_1^{-1},$$

which estimates the accurate noise level δ_0^2. However, for $\alpha_0 \sim \delta_0^{-d}$ with $0 < d < 2$, that is to say $\alpha_0 \sim \tau^d,\ 0 < d < 1,\ \alpha_0$ is positive and it would been required by the convergence. In this where α_0 is replaced by $\alpha_0\tau^d$, we rewrite the estimate of λ^*:

$$\lambda^* = \frac{\alpha_0\tau^{*d}}{\|R_p y_\mu^\delta\|^2+\beta_0}. \tag{3.6}$$

which help the algorithm as follows faster convergence to the optimal solution.

Now, we consider the following alternating iterative algorithm, through combining the equation (3.5) with Tikhonov's quasi-optimality principle to solve the the projected problem (2.4). The algorithm constructs a finite parameter sequence of $\{\mu_i\}$, which convergence to the minimizer of criterion Φ_γ.

Algorithm 1. *Alternating iterative algorithm*

1. Choose $\mu_0,\ k_{\max}$, the parameter pairs $(\alpha_0,\ \alpha_1)$ and $(\beta_0,\ \beta_1)$.
2. Apply k LBD steps to A with starting vector b and form the matrix B_p.
3. Apply QR decomposition to LV_k and form the matrix R_p.
4. For $i = 0,\ 1,\ \cdots,\ I_{\max}$.
5. Solve for y_{i+1} by the Tikhonov regularization method
$$y_{i+1} \in \arg\min_y \{\|B_k y - \|b\|e_1\|^2 + \mu_k \|R_k y\|^2\}.$$
Set $x_{i+1} = V_k y_{i+1}$.
6. Update the parameter λ_{i+1} and τ_{i+1} by
$$\tau_{i+1} = \frac{\alpha_1}{\|B_p y_{i+1} - \|b\|e_1\|^2 + \beta_1}, \qquad \lambda_{i+1} = \frac{\alpha_0 \tau_{i+1}^d}{\|R_p y_{i+1}\|^2 + \beta_0},$$
set $\mu_{i+1} = \lambda_{i+1}\tau_{i+1}^{-1}$,
7. Check the stopping criterion, until
$$i = \arg\min_i \|x_{i+1} - x_i\|,$$
do $\mu^* = \mu_i$.
8. Compute the regularized solution $x_{\mu^*}^k$.

For large-scale problems, we use a projection method to change it into a small- or medium-scale problem. We would point out that we do not specify the solver for the regularization problem in step 5 deliberately. Therefore, the linear system may be solved directly, or solved by other methods, i.e. the conjugate gradient method. Our numerical experiments indicate that an accurate approximate solution suffices.

3.2 Stopping criteria analysis

The stopping rules are easy to find. We could choose the criteria base on the changes or convergence of either the regularization parameter μ or the solution x. We can stop the iteration when $|\mu_i - \mu_0| < \epsilon_1|\mu_0|$, where ϵ_1 is a small tolerance parameter. We note that because the μ_0 is often random. A disadvantage of the stopping criterion is that the approximate solution have the greater error relative to the true solution. To circumvent this trouble, we use another stopping criterion. The following lemma which provides a surprising and important observation on the monotonicity of the sequence $\mu_{i+1} = \lambda_{i+1}\tau_{i+1}^{-1}$ which are generated by alternating iterative algorithm. The monotonicity is the key about the demonstration of the convergence of the total algorithm.

Lemma 3.1. *For any initial guess μ_0, the sequence x_i is generated by the iterative algorithm and converges to a critical point x^* of (3.1). Moreover, the sequence μ_k is monotonically convergent, showing that there exists some μ^*, such that*

$$\lim_{i\to\infty} \mu_i = \mu^*, \tag{3.7}$$

from which it follows that

$$\lim_{i\to\infty} x_i^\delta = x^*. \tag{3.8}$$

Proof. See [15 Lemma 4.1]. □

For each μ_i the corresponding regularized solution is now denoted by y_i^δ, then $x_i^\delta = V_k y_i^\delta$. For a parameter choice algorithm, we have to choose a certain i as the stopping criteria. This is done by the quasi-optimality principle, the discredited version we use in this paper could be found in [6]:

Definition 3.1 (quasi-optimality). *For x_i^δ and μ_i as in (3.5) the regularization parameter μ_{i_*} defined by the quasi-optimality principle is obtained as*

$$i_* = \arg\min_{i\geq 0} \|x_i^\delta - x_{i+1}^\delta\|. \tag{3.9}$$

Notice that because the sequence x_i is convergent, then $\|x_i - x_{i+1}\|$ is monotone decreasing, especially in section 4.1 some examples illustrate the convergence property of $\|x_i - x_{i+1}\|$ and show that the sequence μ_i increase very quickly. So the solution x_{i_*} approximate equals the solution x^*. The solution x_{i_*} is the iterative optimal solutions for any given a max iterations. In other words, it is stopped if the relative change of the iterates solution (x) at the low point. However this method needs calculate the all iteration solutions, so in order to reduce iterative time, we could set a critical value as the minimum jif maximum iterations is very large, or we set a small maximum iterations.

4 Numerical results

In this section, we illustrate the efficiency of the Algorithms 1 when applied parameter selection method to typical large-scale linear ill-posed problems. For this purpose the numerical results can be divided into two parts, Section 4.1 we choose three benchmark linear inverse problems, e.g. baart, shaw, gravity, which are considered as the test problems, from Hansen's package Regularization Tools [14]. Section 4.2 we consider the restoration problem of a grayscale image as the test problem.

In each case we generate triples A, $\widehat{x}$, $\widehat{b}$, so that $A\widehat{x} = \widehat{b}$. The size of A is taken to be 256×256 and then simulated distinct noisy vector b, $b = \widehat{b} + e$, where e was generated by the Matlab randn function with the seed value set to zero. The vector e is scaled to yield a specified noise level $\xi = \|e\|/\|\widehat{b}\|$. The noise level ξ, i.e., $\xi = 5 \times 10^{-3}$, is considered in section 4.1. In algorithm 1, the initial guess μ_0

is taken to be 1×10^{-6}, and we get access to the i_* when $\|x_i^\delta - x_{i+1}^\delta\|$ falls below $10^{-4}\|x_{i+1}^\delta\|$. The parameter d is set to $d = \frac{1}{4}$. The choice of the parameter pairs (α_0, α_1), (β_0, β_1) are based on the value of k. In the numerical examples, for the regularization of small dimension inverse problem, we choice $\alpha_0 = \frac{k}{2}$ and $\alpha_1 = \frac{k}{2}$. The tridiagonal regularization operator L is a scaled approximation of the second derivative operator.

The relative error (ReErr) is used to measure the quality of the regularized solutions of different algorithms. It is defined as follows:

$$\text{ReErr} = \frac{\|x - \widehat{x}\|}{\|\widehat{x}\|}.$$

The accuracy of the solution x_μ^δ is measured by ReERR. In follows, δ and δ_{at} stand for the norms of true noise level and estimated noise level by Algorithm 1.

4.1 Test problems from Hansen's package

Comparisons are made for the regularized solutions of the Algorithm 1 chosen by different parameter selection method. In this example, numerical results are given to compare the quasi-optimal (q-o) method, L-curve (L-c) method against the optimal (opt) choice of the regularization parameter on several test problems. To illustrate the performance of algorithm on the above test problems, we run 10 realizations and then compute average values of regularization parameters, average relative errors. The optimal regularized solution produces the minimum relative error, the parameter values are are summarized in parentheses, and comparison of ReErr for three parameter selection methods on the projection problem in Table 1. First we observe that the estimated residual noise δ_{at} agree very well with the exact

Table 1: Numerical results for three problems from Hansen's MATLAB package.

	(δ) δ_{at}	(μ_{at}) ReErr	(L-c) ReErr	(q-o) ReErr	(opt) ReErr
baart	(1.45e-2)	(6.98e-3)	(2.64)	(6.16e-5)	(1.13e-6)
	1.56e-2	1.65e-1	4.49e-1	1.79e-1	1.05e-1
shaw	(1.86e-1)	(6.59e-4)	(4.28e-6)	(3.36e-3)	(2.34e-4)
	1.84e-1	1.47e-1	3.37e-1	1.68e-1	1.49e-1
gravity	(3.74e-1)	(3.10e-3)	(3.36e-3)	(1.00e-10)	(3.36e-3)
	3.71e-1	1.49e-1	1.50e-1	7.1534	1.50e-1

one δ. Second observation is that the balancing principle gives an error fairly close to the optimal one. This illustrates clearly the benefit of using iterative method for large-scale inverse problem. The results of the comparison for three problems are displayed in Fig.1- Fig.3, where the figures display the reconstructed solutions and exact solution. In each of figures the third line show the sequence $\{\mu_i\}$ is monotonic

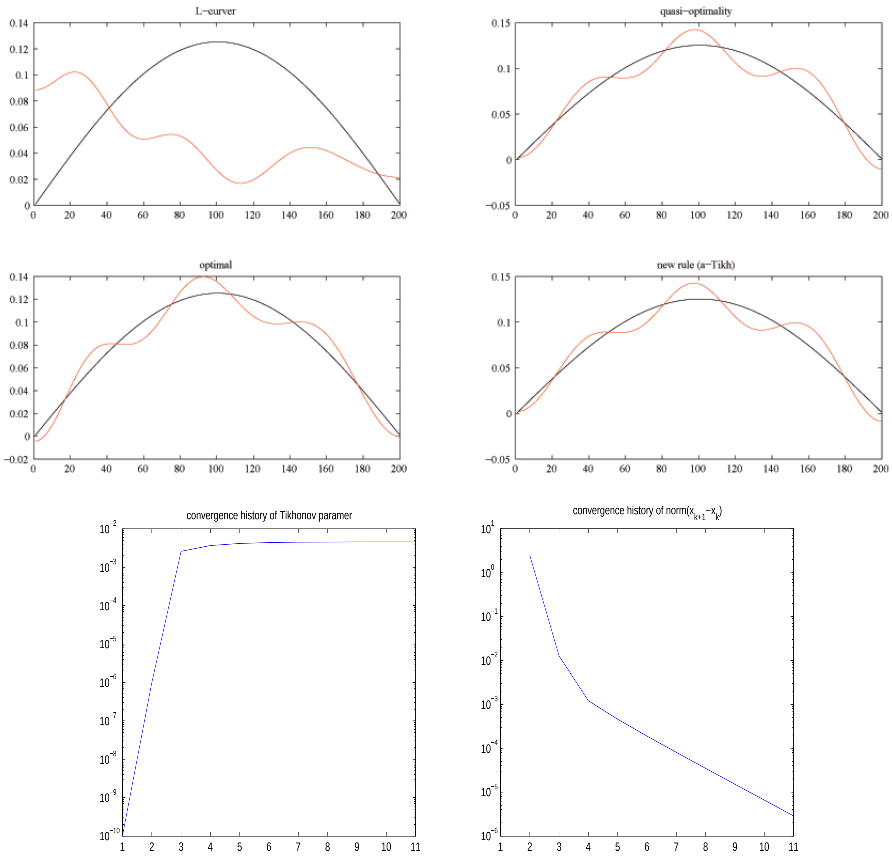
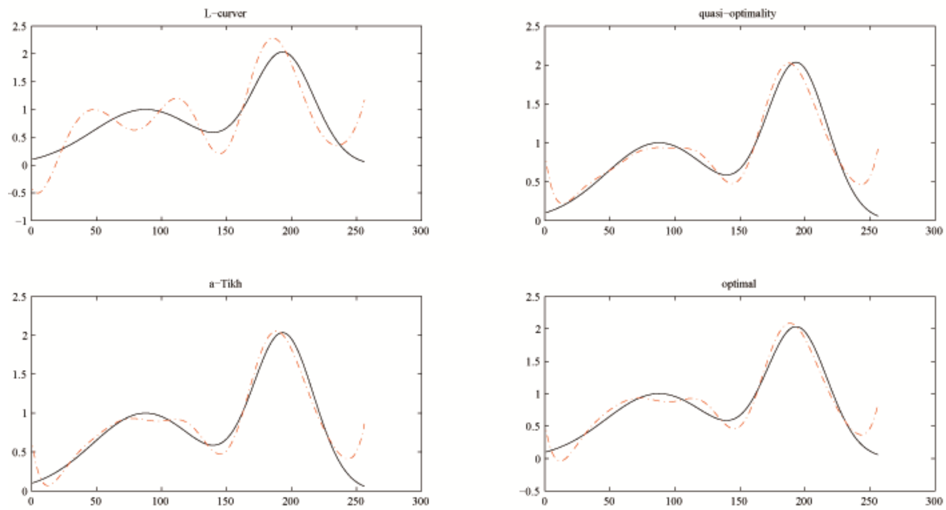


Figure 1: *General Tikhonov for Baart problem. The first four graphs show the approximate solution with three parameter selected methods and the true solution(solid line). Bottom: the convergence analysis of the parameter and the norm of difference of neighbouring approximate solution.*



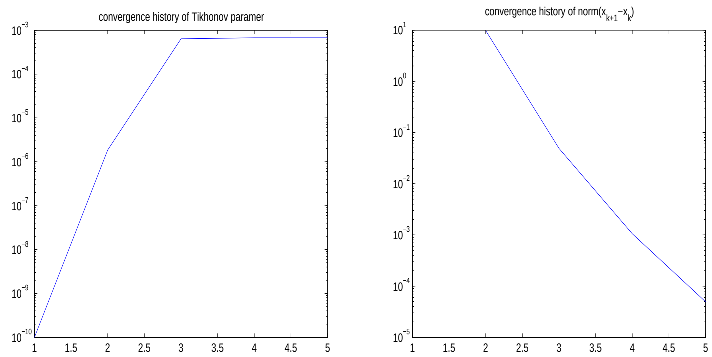


Figure 2: *General Tikhonov for Shaw problem. The first four graphs show the approximate solution (red dashed line) with three parameter selected methods and the true solution (solid line). Bottom: the convergence analysis of the parameter and the norm of difference of adjacent to approximate solution.*

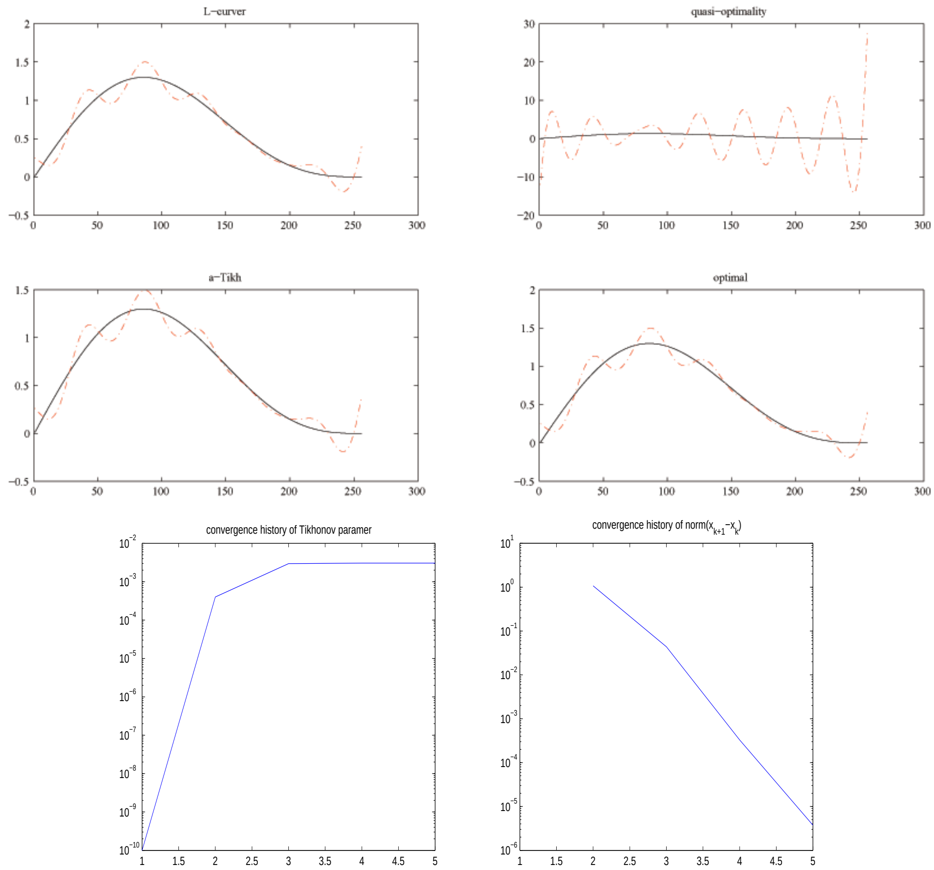


Figure 3: *General Tikhonov for Gravity problem. The first four graphs show the approximate solution (red dashed line) with three parameter selected methods and the true figurename solution (solid line). Bottom: the convergence analysis of the parameter and the norm of difference of adjacent to approximate solution.*

increasing and the relative change of the regularization solutions which are solved by the Algorithm 1 is monotone decreasing. Other quantities are shown in the third line, the sequences $\{\mu_i\}$ are convergent and the convergent rates are very quickly, such as in Fig.1, at about $k = 3$ the μ_i begin to flat. The stopping criterion for Algorithm 1 may be based on this quantities, however we choose some combination of the quantities as the stopping criteria. Combined the Fig.1 and Fig.2 with Table 1, the ReErr of the computed approximate solutions with L-curve parameter choice method, are larger than the other two methods. In Fig.3 the computed approximate solutions with quasi-optimal method is deviating from the optimal solution, therefore the ReErr is the largest of three princples. So we summarize that in three problems the solutions for our method is more close to the true solution.

4.2 Example for grayscale

To test our algorithm on a large-scale problem we consider a denoising problem of a greyscale image cameraman that is represented by an array of 256×256 pixels. The pixels are stored columnwise in a vector in $\mathcal{R}^{65536}$. A block Toeplitz with Toephlitz blocks blurring matrix $A \in \mathcal{R}^{65536\times 65536}$ is determined with Gaussian point spread function and the width sigma= 4.0. Three different relative noise values are generated with $\xi = 5 \times 10^{-3},\ 5 \times 10^{-3},\ 5 \times 10^{-3}$. As we can see from the figures, the computed solutions yield images that resemble the true image relatively well. The stopping criterion is important which determined the time cost. The conclusion in this case is that the alternating iterations i of the Algorithm 1 is very small. By comparing with L-curve, quasi-optimal criterion when they achieved the optimal solutions when the ReErr are identical, respectively the iterations are $i = 3,\ i = 2,\ i = 2$ for different perturbation levels. However the quasi-optimal and L-curve have to calculate all approximate solutions, and then choice the best one.

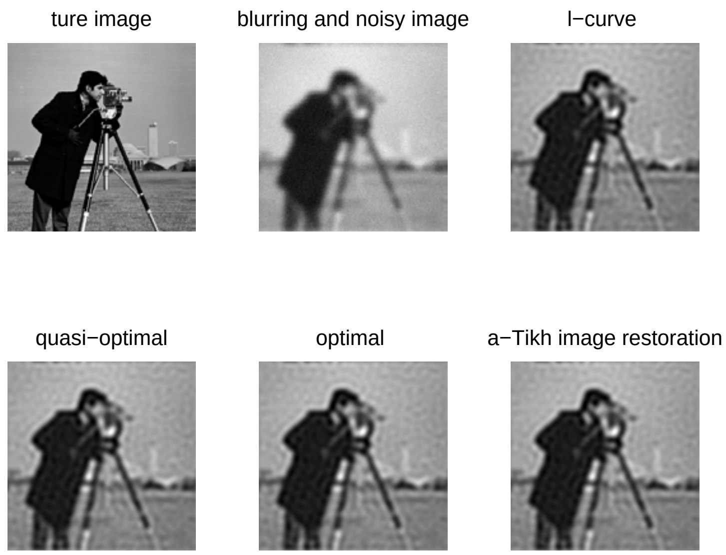


Figure 4: *General Tikhonov for greyscale image. Image restoration with relative noise level* 5×10^{-2}.

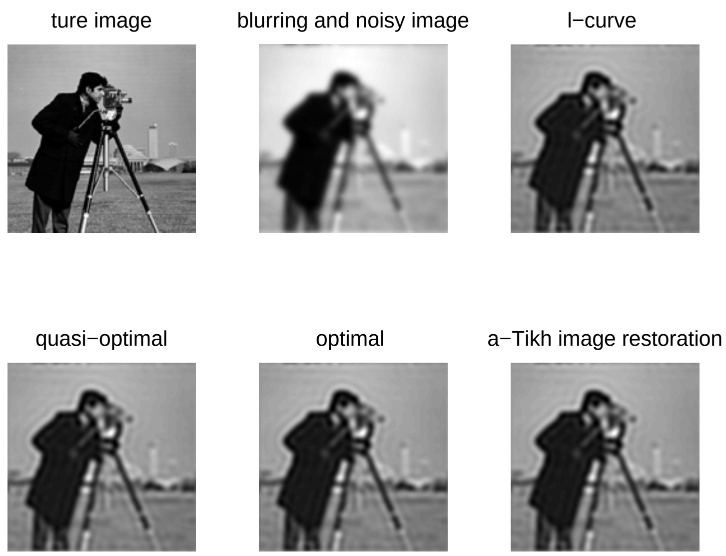


Figure 5: *General Tikhonov for greyscale image. Image restoration with relative noise level* 5×10^{-3}.

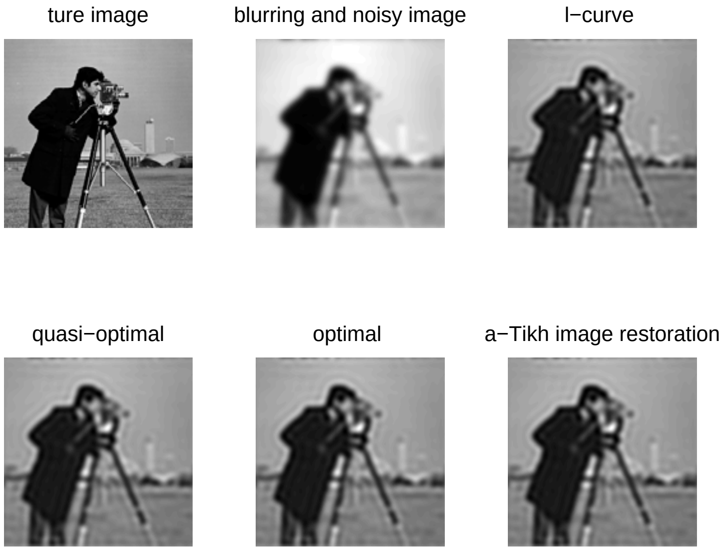


Figure 6: *General Tikhonov for greyscale image. Image restoration with relative noise level* 5×10^{-4}.

5 Conclusion

In this work we have presented a method for solving the general Tikhonov regularization on large-scale ill-posed problems. We have shown that determining regularizing parameters based on the k-dimensional subspace, our selection method is convenient. The examples indicate that the combination of a-Tikh parameter choice method and the iterative projection method is perfected. And our computing method involves less computational expense for solving large-scale ill-posed problems.

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J. COMPUTATIONAL ANALYSIS AND APPLICATIONS, VOL. 15, NO.7, 1332-1343, 2013, COPYRIGHT 2013 EUDOXUS PRESS, LLC

Local and global well-posedness of stochastic Zakharov-Kuznetsov equation

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Abstract. We consider the Cauchy problem for stochastic Zakharov-Kuznetsov equation forced by a random term of additive white noise type. We obtain a local existence and uniqueness result for the solution of this problem. Our proposed technique is based on employing Banach contraction principle method, fixed point theory, Fourier analysis and some basic inequalities. We also get global existence of solution in the function space $Z_s(T)$. Detailed computations and implemented examples are explicitly provided.

Keywords: Stochastic; Well-Posedness; Zakharov-Kuznetsov.

1 Introduction

This paper is devoted to establish local and global well-posedness to stochastic Zakharov-Kuznetsov equation (SZK) forced by a random term of additive white noise type i.e.,

$$\begin{cases} du + (u\frac{\partial u}{\partial x} + \frac{\partial^3 u}{\partial x^3} + \frac{\partial^3 u}{\partial x \partial y^2})dt = \Phi dW, \quad (x,y,t) \in \mathbb{R}^2 \times \mathbb{R}_+ \\ u(x,y,0) = u_0(x,y) \quad \text{for all} \quad (x,y) \in \mathbb{R}^2. \end{cases} \tag{1.1}$$

Where u is a stochastic process on $\mathbb{R}^2 \times \mathbb{R}_+$, $W(t)$ is a cylindrical Wiener process on $L^2(\mathbb{R}^2)$ and Φ is a linear bounded operator not depend on u i.e., the noise ΦdW is additive. The notion of well-posedness will be the usual one in the context of nonlinear dispersive equations, that is, it includes existence, uniqueness, persistence property, and continuous dependence upon the data. Equation (1.1) can be considered as a 2-dimensional generalization of the stochastic KdV equation and arises when modelling the propagation of weakly nonlinear ion-acoustic waves in noisy plasma[1,2,3]. Recently, many researchers pay more attention to study of random waves, which are important subjects of stochastic partial differential equation (SPDE). Wadati [4] first answered the interesting question, How does external noise affect the motion of solitons? and studied the diffusion of soliton of the KdV equation under Gaussian noise, which satisfies a diffusion equation in transformed coordinates. Wadati and Akutsu also studied the behaviors of solitons under the Gaussian white noise of the stochastic KdV equations with and without damping [5]. In addition, a nonlinear partial differential equation which describes wave propagations in random media was presented by Wadati [4]. Debussche and Printems [6,7], de Bouard and Debussche [8,9], Konotop and Vazquez [10], Printems [11], Ghany [12] and others also researched stochastic KdV-type equations. By local well-posedness (LWP) of a stochastic PDE we mean pathwise LWP almost surely. That is, for almost every fixed $\omega \in \Omega$, the corresponding PDE is LWP. Similarly, global well-posedness (GWP) of a stochastic PDE will be defined as pathwise GWP almost surely. Linares and Pastor [13] studied the initial value problems (IVPs) associated with both the ZK and modified ZK equations. They improved the results in [14,15] by showing that both IVPs are locally well-posed for initial data in $H^s(\mathbb{R}^2)$, $s > 0.75$. Moreover, by using the techniques introduced in Birnir at al. [16,17], they proved that the IVP associated with the modified ZK equation is ill-posed, in the sense that the flow-map data-solution is not uniformly continuous, for data in $H^s(\mathbb{R}^2)$, $s \leqslant 0$. It should be noted that the method employed in [13,14] to show local well-posedness, was the one developed by Kenig, Ponce, and Vega [18] (when dealing with the generalized KdV equation), which combines smoothing effects, Strichartz-type estimates, and a maximal function estimate together with the Banach contraction principle. This paper is organized as follows: In Section 2, we introduce some notations and some function spaces along with their embeddings and state deterministic linear estimates from [19,20]. In Section 3, we state two Theorems, as main result of our paper, that guarantees and establishes local and global well-posedness for stochastic Zakharov-Kuznetsov equation forced by a random term of additive white noise type. In Section 4, we prove our main results by establishing the type nonlinear estimate on the second iteration for the integral formulation of the mild solution of equation (1.1).

2 Notations and Preliminaries

Suppose that $\mathcal{S}(\mathbb{R}^d)$ and $\mathcal{S}'(\mathbb{R}^d)$ denote the Schwartz space and its completion with respect to the family of seminorms

$$\|f\|_{k,\alpha} := \sup_{x\in\mathbb{R}^d}\{(1+\|x\|^k_{\mathbb{R}^d})|\partial^\alpha f(x)|\}, \quad \alpha\in\mathbb{N}_0^d, \quad k\in\mathbb{N}_0.$$

For a Banach space X and $s \in \mathbb{R}$ we denote by $H^s(\mathbb{R}^d;X)$ the space of all functions $f \in \mathcal{S}'(\mathbb{R}^d;X)$ such that

$$\|f\|_{H^s(\mathbb{R}^d;X)} := \left(\int_{\mathbb{R}^d}(1+\|\zeta\|^2_{\mathbb{R}^d})^{s/2}\|\hat{f}(\zeta)\|^2_X d\zeta\right)^{1/2} < \infty$$

where $\hat{.}$ denote the Fourier transform. In general case equation (1.1) can be considered on a stochastic basis $(\Omega,\mathcal{F},\mathbb{P};\{\mathcal{F}_t\}_{t\geqslant 0};\{W(t)\}_{t\geqslant 0})$, where $(\Omega,\mathcal{F},\mathbb{P})$ a probability space, $\{\mathcal{F}_t\}_{t\geqslant 0}$ a filtration on Ω and $\{W(t)\}_{t\geqslant 0}$ a cylindrical Wiener process adapted to $\{\mathcal{F}_t\}_{t\geqslant 0}$. The mild solution of equation (1.1) is given in the form

$$u(t) = U(t)u_0 + \int_0^t U(t-s)uu_x ds + \int_0^t U(t-s)\Phi dW(s) \tag{2.1}$$

where $\{U(t)\}_{t\geqslant 0}$ is the unitary group of operators generated by the deterministic Zakharov-Kuznetsov equation, more precisely the solution of the linear equation

$$v_t + v_{xxx} + v_{xyy} = 0, \quad (x,y,t) \in \mathbb{R}^2 \times \mathbb{R}_+ \tag{2.2}$$

with $v(x,y,0) = v_0(x,y)$ for all $(x,y) \in \mathbb{R}^2$ is given by

$$v(x,y,t) = \widehat{U(t)v_0}(\zeta,\eta) = e^{it\Phi}\hat{v}_0(\zeta,\eta) \tag{2.3}$$

where the phase function Φ is given by $\Phi(\zeta,\eta) = \zeta(\zeta^2+\eta)$. The solution of the linear equation

$$du_L + (\frac{\partial^3 u_L}{\partial x^3} + \frac{\partial^3 u_L}{\partial x \partial y^2})dt = \Phi dW \tag{2.4}$$

with $u_L(x,y,0) = 0$ for all $(x,y) \in \mathbb{R}^2$ is given by

$$u_L = \int_0^t U(t-s)\Phi dW(s) \tag{2.5}$$

Suppose that $L_2^{0,s} := L_2^0(L^2(\mathbb{R}^d); H^s(\mathbb{R}^d))$ denote the space of Hilbert-Schmidt operators from $L^2(\mathbb{R}^d)$ into $H^s(\mathbb{R}^d)$. Its norm is given by

$$\|\Phi\|_{L_2^{0,s}} := \sum_{i\geqslant 1} \|\Phi e_i\|^2_{H^s(\mathbb{R}^d)}$$

where $\{e_i\}_{i\geqslant 1}$ is any orthonormal basis of $L^2(\mathbb{R}^d)$. For simplicity we will use the following shorter notations: $L^p([0,T]; L^q(\mathbb{R}^d)) := L_t^p(L_x^q)$ and $L^q(\mathbb{R}^d; L^p([0,T])) := L_x^q(L_t^p)$. For a fixed $\omega \in \Omega$ we define

$$Z_s(T) = \{u \in L^2_\omega(C_t(H^s_{x,y})) \cap L^2_\omega(L^2_{x,y}(L^\infty_t)), D^s\partial_x u \in L^2_\omega(L^\infty_{x,y}(L^2_t)), \partial_x u \in L^2_\omega(L^4_t(L^\infty_{x,y}))\} \tag{2.6}$$

where the Riesz's operator D^s [21] is defined by

$$\widehat{D^s u}(\zeta,\eta) = (\zeta^2+\eta^2)^s \hat{u}(\zeta,\eta), \quad s \in \mathbb{R} \tag{2.7}$$

3 Main Results

In this section we give the precise statement of our results, more precisely, we give two theorems below. Theorem 1 gives the sufficient conditions for obtaining local will posedness of equation (1.1). Theorem 2 concerning the linearized stochastic Zakharov-Kuznetsov equation (2.4). As usual in the context of nonlinear estimation, Theorem 2 is essential for proving Theorem 1. Eventually, one can find that the results of Theorem 1 are true for arbitrary large T , this gives the global well-posedness of equation (1.1).

Theorem 1.

Assume that $u_0 \in L^2_\omega(H^1_{x,y}) \cap L^4_\omega(L^2_{x,y})$ is $\mathcal{F}_0-$ measurable and $\Phi \in L_2^{0,1}$, then there exists a unique solution of equation (1.1) in $Z_s(T_0)$ almost surely for any T_0 and any s with $0.75 < s < 1$.

By virtue the arguments of fixed point theory and the following theorem we can easily prove the above theorem.

Theorem 2.
Assume that $\Phi \in L_2^{0,\bar{s}}$ for some $\bar{s} > 0.75$ then u_L is almost surely in $Z_s(T)$ for any $T > 0$ and any s such that $0.75 < s < \bar{s}$. Moreover there exists a constant $C(s, \bar{s}, T)$ such that

$$\mathbb{E}[\|u_L\|^2_{Z_s(T)}] \leqslant C(s, \bar{s}, T)\|\Phi\|^2_{L_2^{0,\bar{s}}} \tag{3.1}$$

4 Computations and Proofs

The proof of Theorem 1 will require four key Propositions concerning the above mentioned spaces. In this section we present these Propositions.

Proposition 3.
For any $s \leqslant \bar{s}$ we have $u_L \in L^2_\omega(L^\infty_t(H^s_{x,y})$ and

$$\mathbb{E}[\sup_{0\leqslant t\leqslant T} \|u_L\|^2_{H^s_{x,y}}] \leqslant C(T)\|\Phi\|^2_{L_2^{0,s}} \tag{4.1}$$

Proof. We use Itô formula on the functional $\|.\|^2_{H^s_{x,y}}$ [16] and deduce

$$\|u_L\|^2_{H^s_{x,y}} = 2\int_0^t (J_s u_L, J_s \Phi dW(s))_{L^2_{x,y}} + \int_0^t \mathrm{Tr}(J_s^2 \Phi\Phi^*)ds$$

where the Bessel's operator J_s is defined by

$$\widehat{J_s u}(\zeta, \eta) = (1 + \zeta^2 + \eta^2)^{s/2}\widehat{u}(\zeta, \eta) \tag{4.2}$$

and has the property [21]

$$\|J_s.\|_{L^2_{x,y}} = \|.\|^2_{H^s_{x,y}} \tag{4.3}$$

Now, we write $Tr(J_s^2\Phi\Phi^*) = \|\Phi\|_{L_2^{0,s}}$ and hence applying a martingale inequality[20]

$$\begin{aligned}\sup_t \int_0^t (J_s u_L, J_s \Phi dW(s))_{L^2_{x,y}} &\leqslant 3\mathbb{E}[(\int_0^t \|\Phi^* u_L\|^2_{H^s_{x,y}} ds)^{0.5}] \\ &\leqslant \frac{1}{4}\mathbb{E}[\sup_t \|u_L\|^2_{H^s_{x,y}}] + C(T)\|\Phi\|_{L_2^{0,s}} \end{aligned} \tag{4.4}$$

implies the required result.

The proof of the above proposition implies directly the following

Corollary.

$$u_L \in L^2_\omega(C_t(H^s_{x,y})) \tag{4.5}$$

The above Proposition and its corollary give a draws attention regularity property of the solution u_L of the linear problem, that is, they decide that u_L is a square integrable random variable with values in $L^\infty_t(H^s_{x,y})$ especially in $C_t(H^s_{x,y})$ for any $s \leqslant \bar{s}$.

Now, we will give a simple priori estimate of u_L by giving the following result:

Proposition 4.
$u_L \in L^2_\omega(L^2_{x,y}(L^\infty_t)$ and

$$\mathbb{E}[\int_{\mathbb{R}^2} \sup_{0\leqslant t\leqslant T} |u_L|^2 dxdy] \leqslant C(\bar{s},T)\|\Phi\|^2_{L_2^{0,\bar{s}}} \tag{4.6}$$

Proof. Let $\{e_i\}_{i\geqslant 1}$ be an orthonormal basis for $L^2(\mathbb{R}^2)$ and $\{h_k\}_{k\geqslant 1}$ a partition of unity on $\mathbb{R}^2_+$ such that:
a) $h_k(\zeta,\eta) = h_1(\frac{\zeta}{2^{k-1}}, \frac{\eta}{2^{k}-1}), \quad (\zeta,\eta)\in\mathbb{R}^2_+, \quad k\geqslant 1;$
b) $\mathrm{supp} h_k \subseteq [2^{k-1}, 2^{k+1}]^2, \quad k \geqslant 1;$
c) $\mathrm{supp} h_0 \subseteq [-1,1]^2$.
We also consider $\widetilde{h}_k \in C^\infty(\mathbb{R}^2)$ with $\mathrm{supp} h_k \subseteq [2^{k-2}, 2^{k+2}]$ such that $\widetilde{h}_k \geqslant 0$ and $\widetilde{h}_k = 1$ on $\mathrm{supp} h_k$. For $k\in\mathbb{N}$, we define the group $\{U_k(t)\}_{t\in\mathbb{R}}$ by

$$\widehat{U_k(t)f}(\zeta,\eta) = h_k(|\zeta|,|\eta|)\widehat{U(t)f}(\zeta,\eta) = e^{it\phi}h_k(|\zeta|,|\eta|)\widehat{f}(\zeta,\eta) \tag{4.7}$$

and the operator Φ_k by

$$\widehat{\Phi_k e_i}(\zeta,\eta) = \widetilde{h}_k(|\zeta|,|\eta|)\widehat{\Phi e_i}(\zeta,\eta) \tag{4.8}$$

Since, $U_k(t)\Phi = U_k(t)\Phi_k$ implies $U(t)\Phi = \sum_{k\geqslant 1} U_k(t)\Phi_k$. Then, by using Minkowski's integral inequality we will get

$$\mathbb{E}[\int_{\mathbb{R}^2}\sup_t|\int_0^t U(t-s)\Phi dW(s)|^2dxdy]^{0.5} \leqslant \sum_{k\geqslant 1}\mathbb{E}[\int_{\mathbb{R}^2}\sup_t|\int_0^t U_k(t-s)\Phi_k dW(s)|^2dxdy]^{0.5}$$

$$\leqslant C(T,\bar{s})\sum_{k\geqslant 1}2^{sk}\|\Phi_k\|_{L_2^{0,0}} \leqslant C(T,\bar{s})(\sum_{k\geqslant 1}2^{2(s-\bar{s})k})^{0.5}(\sum_{k\geqslant 1}2^{2\bar{s}k}\|\Phi_k\|^2_{L_2^{0,0}})^{0.5} \leqslant C(T,s,\bar{s})\|\Phi\|_{L_2^{0,\bar{s}}}$$

where, $0.75 < s < \bar{s}$.

Remark.
From [16] We have used

$$\mathbb{E}[\int_{\mathbb{R}^2}\sup_t|\int_0^t U_k(t-s)\Phi_k dW(s)|^2dxdy] \leqslant C(T,\bar{s})2^{2sk}\|\Phi_k\|^2_{L_2^{0,0}}$$

and for $0.75 < s < \bar{s}$ we were used

$$\sum_k 2^{2\bar{s}k}\|\Phi_k\|^2_{L_2^{0,0}} = \sum_{k,i} 2^{2\bar{s}k}\|\Phi_k e_i\|^2_{L_{x,y}}$$

and

$$\sum_k 2^{2\bar{s}k}\|\Phi_k u_L\|^2_{L_{x,y}} \leqslant C(\bar{s})\|\Phi u_L\|^2_{H^{\bar{s}}_{x,y}}$$

Its well known that, the Riesz's operator D^s is a powerful tool for checking the regularity of the solutions of nonlinear partial differential equations. Proposition 5 will clarify the success of the solution u_L under this checking.

Proposition 5.
Suppose $0 < \delta < \inf\{\bar{s}, 2\}$, then $D^{\bar{s}-\delta}\partial_x u_L \in L^2_\omega((L^2_{x,y}(L^2_t))$ and

$$\mathbb{E}[\sup_{x,y\in\mathbb{R}} \int_0^T |D^{\bar{s}-\delta}\partial_x u_L|^2 dt] \leqslant C(\delta, T)\|\Phi\|^2_{L_2^{0,\bar{s}}} \tag{4.9}$$

Proof. Let $q = \frac{4}{\delta}$. By virtue of the stochastic integral properties[20]:

$$\mathbb{E}|\int_0^t D^{1+\bar{s}}U(t-\tau)\Phi dW(\tau)|^2 dt \leqslant \int_0^t \sum_{i\geqslant 1} |D^{1+\bar{s}}U(t-\tau)\Phi e_i|^2 d\tau$$

So, we can easily find that:

$$\begin{aligned}
\|D^{1+\bar{s}}u_L\|^q_{L^\infty_{x,y}(L^q_\omega(L^2_t))} &= \sup_{x,y\in\mathbb{R}} \mathbb{E}[(\int_0^T |\int_0^t D^{1+\bar{s}}U(t-\tau)\Phi dW(\tau)|^2 dt)^{q/2}] \\
&\leqslant C \sup_{x,y\in\mathbb{R}} \int_0^T \mathbb{E}[|\int_0^t D^{1+\bar{s}}U(t-\tau)\Phi dW(\tau)|^2]^{q/2} dt \\
&\leqslant C \sup_{x,y\in\mathbb{R}} \int_0^T (\int_0^t \sum_{i\geqslant 1} |D^{1+\bar{s}}U(t-\tau)\Phi e_i|^2 d\tau)^{q/2} dt \\
&\leqslant C \int_0^T (\sum_{i\geqslant 1} \sup_{x,y\in\mathbb{R}} \int_0^t |D^{1+\bar{s}}U(t-\tau)\Phi e_i|^2 d\tau)^{q/2} dt
\end{aligned}$$

As pointed in [21, Lemma 2.1], we have

$$\sup_{x,y\in\mathbb{R}} \int_0^t |D^{1+\bar{s}}U(t-\tau)\Phi e_i|^2 d\tau \leqslant C\|D^{\bar{s}}\Phi e_i\|^2_{L^2_{x,y}} \leqslant C\|\Phi e_i\|^2_{H^{\bar{s}}_{x,y}} \tag{4.10}$$

hence

$$\|D^{1+\bar{s}}u_L\|^q_{L^\infty_{x,y}(L^q_\omega(L^2_t))} \leqslant C\int_0^T (\sum_{i\geqslant 1} \|\Phi e_i\|_{H^{\bar{s}}_{x,y}})^{q/2} dt \leqslant C(T)\|\Phi\|_{L_2^{0,\bar{s}}} \tag{4.11}$$

Similarly we can derive

$$\|D^{\bar{s}}u_L\|^q_{L^2_{x,y}(L^q_\omega(L^2_t))} \leqslant C\|\Phi\|^2_{L_2^{0,\bar{s}}} \tag{4.12}$$

Inequality (4.11) and [9, proposition A.1] implies

$$D^{1+\bar{s}-\frac{\delta}{2}}u_L \in L^q_{x,y}(L^q_\omega(L^2_t)) \tag{4.13}$$

and

$$\|D^{1+\bar{s}-\frac{\delta}{2}}u_L\|_{L^q_\omega(L^q_{x,y}(L^2_t))} = \|D^{1+\bar{s}-\frac{\delta}{2}}u_L\|_{L^q_{x,y}(L^q_\omega(L^2_t))} \leqslant C\|\Phi\|^2_{L_2^{0,\bar{s}}} \tag{4.14}$$

Also we have

$$\begin{aligned}\|u_L\|^q_{L^q_\omega(L^q_{x,y}(L^2_t))} &= \int_{\mathbb{R}^2}\mathbb{E}[(\int_0^T|\int_0^t U(t-\tau)\Phi dW(\tau)|^2dt)^{q/2}]dxdy\\ &\leqslant C\int_{\mathbb{R}^2}\int_0^T\mathbb{E}(|\int_0^t U(t-\tau)\Phi dW(\tau)|^2)^{q/2}dtdxdy\\ &\leqslant C\int_{\mathbb{R}^2}\int_0^T(\int_0^t\sum_{i\geqslant1}|U(t-\tau)\Phi e_i|^2d\tau)^{q/2}dtdxdy\end{aligned}$$

So,

$$\|u_L\|_{L^q_\omega(L^q_{x,y}(L^2_t))} \leqslant C\|\Phi\|_{L_2^{0,\bar{s}}} \tag{4.15}$$

Hence,

$$\|u_L\|^q_{L^q_\omega(L^q_{x,y}(L^2_t))} \leqslant C\int_{\mathbb{R}^2}(\int_0^T\sum_{i\geqslant1}|U(t)\Phi e_i|^2d\tau)^{q/2}dxdy$$

Applying Minkowski's intgral inequality gives,

$$\|u_L\|^2_{L^q_\omega(L^q_{x,y}(L^2_t))} \leqslant C\sum_{i\geqslant1}(\int_{\mathbb{R}^2}(\int_0^T|U(t)\Phi e_i|^2d\tau)^{q/2}dxdy)^{q/2}$$

So,

$$\|u_L\|_{L^q_\omega(L^q_{x,y}(L^2_t))} \leqslant C\sum_{i\geqslant1}\|U(t)\Phi e_i\|^2_{L^\infty_t(L^q_{x,y})} \leqslant C\|\Phi\|^2_{L_2^{0,\bar{s}}} \tag{4.16}$$

Obviously, equations (4.14) and (4.15) implies that $D^{1+\bar{s}-\delta}u_L \in L^q_\omega(L^\infty_{x,y}(L^2_t))$ and

$$\|D^{1+\bar{s}-\delta}u_L\|_{L^q_\omega(L^\infty_{x,y}(L^2_t))} \leqslant C\|\Phi\|^2_{L_2^{0,\bar{s}}} \tag{4.17}$$

Recalling the definition of the Hilbert transform [21]

$$\widehat{\mathrm{H}f}(\zeta,\eta) = (\frac{\zeta}{|\zeta|}+\frac{\eta}{|\eta|})\hat{f}(\zeta,\eta) \tag{4.18}$$

implies,

$$D^{\bar{s}-\delta}\partial_x u_L = \int_0^t D^{\bar{s}-\delta}\partial_x U(t-\tau)\Phi dW(\tau)$$
$$= \int_0^t D^{1+\bar{s}-\delta}\partial_x U(t-\tau)H\Phi dW(\tau)$$

Then,

$$\|D^{\bar{s}-\delta}\partial_x u_L\|_{L^q_\omega(L^\infty_{x,y}(L^2_t))} \leqslant C\|H\Phi\|^2_{L_2^{0,\bar{s}}} \leqslant C\|\Phi\|^2_{L_2^{0,\bar{s}}} \tag{4.19}$$

Now we can present the last regularity property of the solution u_L by giving the following result:

Proposition 6.
$\partial_x u_L \in L^2_\omega(L^4_t(L^\infty_{x,y})$ and

$$\mathbb{E}[(\int_0^T \sup_{x,y\in\mathbb{R}} |\partial_x u_L|^4 dt)^{0.5}] \leqslant \|\Phi\|^2_{L_2^{0,\bar{s}}} \tag{4.20}$$

Proof. Let $\epsilon = \bar{s} - 0.75$ and $q = 4(1+1/\epsilon)$. Noting that $D^{1+\epsilon}u_L \in L^4_t(L^\infty_{x,y}(L^q_\omega))$ we have,

$$\|D^{1+\epsilon}u_L\|_{L^4_t(L^\infty_{x,y}(L^q_\omega))} = \int_0^T \sup_{x,y\in\mathbb{R}} \mathbb{E}[|\int_0^t D^{\bar{s}+1/4}U(t-\tau)\Phi dW(\tau)|^q]^{4/q}$$
$$\leqslant C\int_0^T \sup_{x,y\in\mathbb{R}} [\sum_{i\geqslant 1}\int_0^t |D^{\bar{s}+1/4}U(t-\tau)\Phi e_i d\tau|^2]^{4/2}dt$$
$$\leqslant C(T)[\sum_{i\geqslant 1}(\int_0^T \sup_{x,y\in\mathbb{R}} |D^{\bar{s}+1/4}U(t-\tau)\Phi e_i|^4 d\tau)^{\frac{1}{2}}]^2$$

Applying [21, Theorem 2.4] with $\alpha = 2, \theta = 1, \beta = 1/2$ we get

$$\int_0^T \sup_{x,y\in\mathbb{R}} |D^{\bar{s}+1/4}U(t-\tau)\Phi e_i|^4 d\tau \leqslant C\|D^{\bar{s}}\Phi e_i\|^4_{L^2_{x,y}}$$

So,

$$\|D^{1+\epsilon}u_L\|_{L^4_t(L^\infty_{x,y}(L^q_\omega))} \leqslant C\|\Phi\|_{L_2^{0,\bar{s}}}$$

Therefore,

$$\|u_L\|_{L^4_t(L^2_{x,y}(L^q_\omega))} \leqslant C\|\Phi\|_{L_2^{0,0}} \leqslant C\|\Phi\|_{L_2^{0,\bar{s}}}$$

By virtue of the above inequalities and [9, proposition A.1] we obtain for all $t \in [0,T]$ that

$$\|D^{1+\epsilon/2}u_L\|_{L^q_{x,y}(L^q_\omega))} \leqslant C\|u_L\|^{2/q}_{L^2_{x,y}(L^q_\omega))}\|D^{1+\epsilon}u_L\|^{1-2/q}_{L^\infty_{x,y}(L^q_\omega)}$$

Since $q = 4(1 + 1/\epsilon) \geqslant 4$, so

$$\|D^{1+\epsilon/2}u_L\|_{L^4_\omega(L^4_t(L^q_{x,y}))} \leqslant C\|D^{1+\epsilon/2}u_L\|_{L^4_t(L^q_{x,y}(L^q_\omega))} \leqslant$$
$$\leqslant C\|u_L\|_{L^4_t(L^2_{x,y}(L^q_\omega))}\|D^{1+\epsilon}u_L\|^{1-2/q}_{L^4_t(L^\infty_{x,y}(L^q_\omega))} \leqslant C\|\Phi\|_{L_2^{0,\bar{s}}}$$

Using Fuibini's theorem, we have

$$\begin{aligned}\|u_L\|_{L^4_\omega(L^4_t(L^q_{x,y}))} &\leqslant C\|u_L\|_{L^4_t(L^4_\omega(H^s_{x,y}))}\\ &\leqslant C(\int_0^T \mathbb{E}[\|\int_0^t U(t-\tau)\Phi dW(\tau)\|^4_{H^{\bar{s}}_{x,y}}]dt)^{1/4}\\ &\leqslant C(\int_0^T [\int_0^t \sum_{i\geqslant 1}\|U(t-\tau)\Phi e_i\|^2_{H^{\bar{s}}_{x,y}}d\tau]dt)^{1/4}\end{aligned}$$

So,

$$\|u_L\|_{L^4_\omega(L^4_t(L^q_{x,y}))} \leqslant C\|\Phi\|_{L_2^{0,\bar{s}}}$$

Since $q\epsilon/2 > 1$, Then

$$\|\partial_x u_L\|_{L^4_\omega(L^4_t(L^\infty_{x,y}))} \leqslant C(T)\|\Phi\|_{L_2^{0,\bar{s}}} \tag{4.21}$$

Now, Theorem 2 is a direct result from the global results of the above propositions. To prove Theorem 1 i.e., to solve the stochastic Zakharov-Kuznetsov equation forced by a random term of additive white noise (1.1). We will use a fixed point argument in $Z_s(T)$ for some $T > 0$ and $s \in (0.75, 1)$, then a priori estimate will give us the global solution in $H^1_{x,y}$. From Theorem 2, we have $u_L \in Z_s(T_0), T_0 > 0$ for almost all $\omega \in \Omega$.

Proposition 7.[21] For any $s > 0.75$ and any $T > 0$ there exists $C(T,s)$ nondecreasing with respect to T such that:

$$\|\int_0^T U(t-\tau)(u\partial_x v)d\tau\|_{Z_s(T)} \leqslant C(T,s)\|u\|_{Z_s(T)}\|v\|_{Z_s(T)} \tag{4.22}$$

for any $u, v \in Z_s(T)$ and

$$\|U(t)u_0\|_{Z_s(T)} \leqslant C(T,s)\|u\|_{H^s_{x,y}} \quad \text{for all} \quad u_0 \in H^s_{x,y} \tag{4.23}$$

Proof of Theorem 1. Firstly, we introduce the mapping $\mathcal{J}$ defined by

$$\mathcal{J}u(t) = U(t)u_0 + \int_0^t U(t-\tau)(u\partial_x u)d\tau + u_L(t) \tag{4.24}$$

Let $0.75 < s < 1$, since $\Phi \in L_2^{0,1}$ so by Theorem 2 and Proposition 5 we have $u_0 \in H^s(\mathbb{R}^2)$, $\mathcal{J}$ maps $Z_s(T)$ into itself. Moreover, let R_0 satisfies:

$$R_0 \geqslant C(T_0,s)\|u_0\|_{H^s(\mathbb{R}^2)} + \|u_L\|_{Z_s(T)}$$

and choose T such that:

$$C(T_0, s)T^{\frac{1}{2}}R_0 \leqslant 1$$

then, $\mathcal{J}$ maps the ball of center 0 and radius $2R_0$ in $Z_s(T)$ into itself and

$$\|\mathcal{J}u - \mathcal{J}v\|_{Z_s(T)} \leqslant \frac{1}{2}\|u - v\|_{Z_s(T)} \tag{4.25}$$

for any $u, v \in Z_s(T)$ with norm less than 2 R_0 . By virtue of fixed point theorem, $\mathcal{J}$ has a unique fixed point, denote by u , in this ball. It is obvious that this solution u for Equation (1.1) belongs to the function space $Z_s(T)$.

5 Concluding Remarks

This paper is devoted to establish some methods like Banach contraction principle and successive approximations method for handling stochastic nonlinear partial differential equations and for proving local and global well-posedness results for their solutions in selected function spaces. In fact, we restricted our efforts in stochastic Zakharov-Kuznetsov equation, but we believe that, similar ideas can be applied to other stochastic nonlinear partial differential equations in mathematical physics, such as the generalized KdV, KdV-Burgers, Modified KdV-Burgers and Swada-Kotera equations. Also we remark that, if we assume that $u_0 \in L^2_\omega(H^{\bar{s}}_{x,y}) \cap L^4_\omega(L^2_{x,y})$ with $0.75 \leqslant \bar{s} < 1$ and u_0 is $\mathcal{F}_0-$ measurable , then we cannot construct a solution on a fixed interval, even a finite one of the form $[0, T_0]$. Moreover, by using a standard truncation argument we can extend our results under the assumption that $u_0 \in H^1(\mathbb{R}^2)$ almost surely.

Acknowledge. Authors Thanks the reviewers' for their notes which improved the quality of the paper.

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Volume 15, Number 8 December 2013
ISSN:1521-1398 PRINT,1572-9206 ONLINE

Journal of

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EUDOXUS PRESS,LLC

Journal of Computational Analysis and Applications
ISSNno.'s:1521-1398 PRINT,1572-9206 ONLINE

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A modified nonlinear Uzawa algorithm for solving symmetric saddle point problems *

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Abstract

In this paper, a modified nonlinear Uzawa algorithm for solving symmetric saddle point problems is proposed, and also the convergence rate is analyzed. The results of numerical experiments are presented when we apply the algorithm to Stokes equations discretized by mixed finite elements.

Keywords: Convergence rate; Modified nonlinear Uzawa algorithm; Saddle point problems; Schur complement

AMS classification: 65F10

1 Introduction

Let H_1 and H_2 be finite-dimensional Hilbert spaces with inner product denoted by $(\cdot,\cdot)$. In this paper, we propose a modified nonlinear Uzawa algorithm for solving systems of linear equations with the following two-by-two block structure:

$$\mathcal{A}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} A & B^T \\ B & -C \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} f \\ g \end{pmatrix}, \tag{1}$$

where$f \in H_1, g \in H_2$ are given, and $x \in H_1, y \in H_2$ are unknown. Here $A : H_1 \to H_1$ is assumed to be linear, symmetric and positive definite operator, $B : H_1 \to H_2$ is a linear map and $B^T : H_2 \to H_1$ is its adjoint. In addition, $C : H_2 \to H_2$ is linear symmetric and positive semidefinite. Such system is usually referred to as saddle point problem, which is typically resulted from mixed or hybrid finite element approximations of second-order elliptic problems, or the Stokes equation, including computational fluid dynamics as well as constrained optimization problems [1, 2, 6-11,14].

On the solution methods for saddle point systems there is a very good reference [2].

In [1], Bramble et al, considered the linear system (1) with $C = 0$ and assumed that the following LBB condition [13] holds, i.e.,

$$(BA^{-1}B^Tv, v) \equiv \sup_{u\in H_1} \frac{(v, Bu)^2}{(Au, u)} \geq c_0\|v\|^2, \ \ \forall v \in H_2, \tag{2}$$

*This research was supported by Doctoral Research Project of NCWU (2001119) and by NSFC of Tianyuan Mathematics Youth Fund (11226337, 11126323).

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for some positive number c_0. A nonlinear Uzawa algorithm is **first** proposed by defining the nonlinear approximate inverse of A as a map $\phi : H_1 \to H_1$, i.e., for any $\varphi \in H_1$, $\phi(\varphi)$ is an approximation to the solution ξ of $A\xi = \varphi$.

In [3], Cao considered the linear system (1), and assumed that the following stabilized condition [7, 8] holds, i.e.,

$$((BA^{-1}B^T + C)v, v) \geq c_0 \|v\|^2, \ \ \forall v \in H_2, \tag{3}$$

for some positive number c_0. Cao proposed another nonlinear Uzawa algorithm by defining the nonlinear approximate inverse of approximate Schur complement $(BQ_A^{-1}B^T + C)$ as a map $\psi : H_2 \to H_2$, i.e., for any $\varphi \in H_2$, $\psi(\varphi)$ is an approximation to the solution ξ of $(BQ_A^{-1}B^T + C)\xi = \varphi$, where Q_A is a symmetric positive definite operator.

In [4], Lin and Cao proposed another nonlinear Uzawa algorithm by defining the nonlinear approximate inverse of A and the Schur complement $(BA^{-1}B^T + C)$. In [5], Lin and Wei proposed a modified nonlinear Uzawa algorithm and modified the Cao's results. In this paper, we present another modified nonlinear Uzawa algorithm for solving the system (1). At the same time, its convergence is analyzed.

The inexact Uzawa algorithms [1,3,4,6,14] are of interest because they are simple, efficient and have minimal numerical computer memory requirements. this could be important in large-scale scientific applications implemented for today's computing architectures. Therefore, the inexact Uzawa methods are widely used in the engineer community.

The paper is organized as follows. In section 2, we review the Uzawa type algorithms mentioned in section 1 and their convergence results. In section 3, we give our modified nonlinear Uzawa algorithm (MNUAS) and analyze convergence results. In section 4, the MNUAS algorithm is applied to solve system (1), which is resulted from the discretization of Stokes equations by mixed finite element method and the results of the numerical experiments are presented. Finally, the conclusions are drawn.

2 The Uzawa algorithms and convergence

First, some notions are given. Let Q be a symmetric and positive definite matrix, we define a inner product

$$\langle v, u \rangle_Q = (Qv, u) = \left(Q^{\frac{1}{2}}v, Q^{\frac{1}{2}}u\right), \ \ \forall v, u \in H_2,$$

and denote the Euclidean norm by $\|\cdot\|$. So

$$\|v\|_Q = \langle v, v \rangle_Q^{\frac{1}{2}} \equiv \left(Q^{\frac{1}{2}}v, Q^{\frac{1}{2}}v\right)^{\frac{1}{2}} \equiv \left\|Q^{\frac{1}{2}}v\right\|_2.$$

Denote residue of x and y as

$$e_i^x = x - x_i, \ \ e_i^y = y - y_i.$$

The Nonlinear Uzawa algorithm (which is related to the approximate inverse of the matrix A, and is called as **NUA algorithm**) for solving system (1) is as follows ([1,3,4]).

Algorithm 1 (**NUA algorithm**) ([1, 3]) For $x_0 \in H_1$ and $y_0 \in H_2$ given, the iterative sequence $\{(x_i, y_i)\}$ is defined, Q_B is a symmetric positive definite operator, for $i = 0, 1, ...,$ by

$$x_{i+1} = x_i + \phi(f - Ax_i - B^T y_i), \tag{4}$$

$$y_{i+1} = y_i + Q_B^{-1}(Bx_{i+1} - Cy_i - g). \tag{5}$$

It is assumed that

$$\|\phi(v) - A^{-1}v\|_A \le \delta \|v\|_{A^{-1}}, \forall v \in H_1, \tag{6}$$

for some positive $\delta < 1$. In [1], the authors also pointed out that (6) is a reasonable assumption which is satisfied by the approximate inverse associated with the Preconditioned Conjugate Gradient algorithm (PCG algorithm) [12].

It is assumed that the following inequality

$$(1-\gamma)(Q_B w, w) \le ((BA^{-1}B^T + C)w, w) \le (Q_B w, w), \forall w \in H_2 \tag{7}$$

holds for some γ in the interval $[0, 1)$. In practice, preconditioners satisfy (7) with γ bounded away from one.

The result on the convergence of the **NUA algorithm** is given as follows [1,3].

Theorem 1 *Assume that (6) and (7) hold. Let $\{(x, y)\}$ be the solution pair of (1), and $\{(x_i, y_i)\}$ be defined by the Algorithm 1. Then, x_i and y_i converge to x and y, respectively, if*

$$\delta < \frac{1-\gamma}{3-\gamma}. \tag{8}$$

In this case, the following two inequalities hold:

$$\frac{\delta}{1+\delta}(Ae_i^x, e_i^x) + (Q_B e_i^y, e_i^y) \le \rho^{2i}\left(\frac{\delta}{1+\delta}(Ae_0^x, e_0^x) + (Q_B e_0^y, e_0^y)\right), \tag{9}$$

and

$$(Ae_i^x, e_i^x) \le (1+\delta)(1+2\delta)\rho^{2i-2}\left(\frac{\delta}{1+\delta}(Ae_0^x, e_0^x) + (Q_B e_0^y, e_0^y)\right), \tag{10}$$

where

$$\rho = \frac{\gamma + 2\delta + \sqrt{(\gamma+2\delta)^2 + 4\delta(1-\gamma)}}{2}. \tag{11}$$

The following Algorithm 2 is the Nonlinear Uzawa method, which is relate to the approximate inverse of the approximate Schur complement matrix $BQ_A^{-1}B^T + C$. We call it as **NUS algorithm**.

Algorithm2 (**NUS algorithm**) ([3]) For $x_0 \in H_1$ and $y_0 \in H_2$ given, Q_A is a symmetric positive definite, the iterative sequence $\{(x_i, y_i)\}$ is defined, for $i = 0, 1, ...$, by

$$x_{i+1} = x_i + Q_A^{-1}(f - Ax_i - B^T y_i), \tag{12}$$

$$y_{i+1} = y_i + \psi(Bx_{i+1} - Cy_i - g), \tag{13}$$

where $\psi(w)$ is an approximation to the solution ξ of the system

$$(BQ_A^{-1}B^T + C)\xi = w.$$

It is assumed that

$$(1-\omega)(Q_A v, v) \le (Av, v) \le (Q_A v, v), \forall v \in H_1, v \ne 0. \tag{14}$$

holds for some ω in the interval $[0, 1)$, and the approximate Schur complement matrix satisfies

$$\|\psi(w) - (BQ_A^{-1}B^T + C)^{-1}w\|_{(BQ_A^{-1}B^T+C)} \le \varepsilon \|w\|_{(BQ_A^{-1}B^T+C)^{-1}}, \forall w \in H_2. \tag{15}$$

for some positive $\varepsilon < 1$. Analogous to (6) in [1], (15) is a reasonable assumption [3], which is satisfied by the approximate inverse associated with the Conjugate Gradient algorithm (CG algorithm).

In [3], Cao gave the following convergence result.

Theorem 2 *Assume that (14) and (15) hold. Let $\{(x, y)\}$ be the solution pair of (1), and $\{(x_i, y_i)\}$ be defined by the Algorithm 2. Then, x_i and y_i converge to x and y, respectively, if*

$$\omega < \frac{1}{2} \text{ and } \varepsilon < 1 - 2\omega. \tag{16}$$

In this case, the following two inequalities hold:

$$\begin{aligned} &\omega(1+\varepsilon)(Q_A e_i^x, e_i^x) + ((BQ_A^{-1}B^T + C)e_i^y, e_i^y) \\ \leq &\rho^{2i}(\omega(1+\varepsilon)(Q_A e_0^x, e_0^x) + ((BQ_A^{-1}B^T + C)e_0^y, e_0^y)), \end{aligned} \tag{17}$$

and

$$\begin{aligned} (Q_A e_i^x, e_i^x) \leq &\left(1 + \frac{\omega}{1+\varepsilon}\right)\rho^{2i-2} \times \\ &(\omega(1+\varepsilon)(Q_A e_0^x, e_0^x) + (BQ_A^{-1}B^T + C)e_0^y, e_0^y)), \end{aligned} \tag{18}$$

where

$$\rho = \frac{\omega + \varepsilon + \sqrt{(\omega+\varepsilon)^2 + 4\omega}}{2}. \tag{19}$$

The following Algorithm 3 is another **Nonlinear Uzawa** method, which is relate to the approximate inverse of the matrix A and the approximate inverse of the **Schur complemen**t matrix $BA^{-1}B^T + C$. We call it as **NUAS algorithm**.

Algorithm 3 (**NUAS algorithm**) ([4]) For $x_0 \in H_1$ and $y_0 \in H_2$ given, the iterative sequence $\{(x_i, y_i)\}$ is defined, for $i = 0, 1, ...,$ by

$$x_{i+1} = x_i + \phi(f - Ax_i - B^T y_i), \tag{20}$$
$$y_{i+1} = y_i + \psi(Bx_{i+1} - Cy_i - g), \tag{21}$$

where $\phi(v)$ is an approximation to the solution φ of the system

$$A\varphi = v,$$

and $\psi(w)$ is an approximation to the solution ξ of the system

$$(BA^{-1}B^T + C)\xi = \omega.$$

Let

$$S = BA^{-1}B^T + C.$$

It is assumed that

$$\|\phi(v) - A^{-1}v\|_A \leq \delta\|v\|_{A^{-1}}, \forall v \in H_1, \tag{22}$$
$$\|\psi(w) - S^{-1}w\|_S \leq \varepsilon\|w\|_{S^{-1}}, \forall w \in H_2, \tag{23}$$

hold for some positive $\delta < 1$ and $\varepsilon < 1$, respectively.

The result on the convergence of the **NUAS algorithm** is given as follow [4].

Theorem 3 *Assume that (22) and (23) hold, Let $\{(x, y)\}$ be the solution pair of (1), and $\{(x_i, y_i)\}$ be defined by the Algorithm 3. Then, x_i and y_i converge to x and y, respectively, if*

$$0 < \delta < \frac{1}{3} \text{ and } 0 < \varepsilon < \frac{1 - 3\delta}{1 + \delta}. \tag{24}$$

In this case, the following two inequalities hold:

$$
\begin{aligned}
&\delta(1+\varepsilon)(Ae_i^x, e_i^x) + (1+\delta)(Se_i^y, e_i^y)\\
\le&\rho^{2i}(\delta(1+\varepsilon)(Ae_0^x, e_0^x) + (1+\delta)(Se_0^y, e_0^y)),
\end{aligned} \tag{25}
$$

and

$$
\begin{aligned}
(Ae_i^x, e_i^x) \le \left(1+\delta+\frac{\delta}{(1+\varepsilon)}\right)\rho^{2i-2}\times&\\
(\delta(1+\varepsilon)(Ae_0^x, e_0^x) + (1+\delta)(Se_0^y, e_0^y)),&
\end{aligned} \tag{26}
$$

where

$$
\rho = \frac{\varepsilon+2\delta+\varepsilon\delta+\sqrt{(\varepsilon+2\delta+\varepsilon\delta)^2+4\delta}}{2}. \tag{27}
$$

In [5], Lin and Wei modified the Algorithm 2, and gave the following **Modified NUS algorithm** (called **MNUS algorithm**).

Algorithm 4 (**MNUS algorithm**) For $x_0 \in H_1$ and $y_0 \in H_2$ given, the iterative sequence $\{(x_i, y_i))\}$ is defined, for $i = 0, 1, ...$, by

$$
\bar{x}_{i+1} = x_i + Q_A^{-1}(f - Ax_i - B^T y_i), \tag{28}
$$

$$
y_{i+1} = y_i + \psi(B\bar{x}_{i+1} - Cy_i - g), \tag{29}
$$

$$
x_{i+1} = \bar{x}_{i+1} - Q_A^{-1}B^T(y_{i+1} - y_i). \tag{30}
$$

The result on the convergence of the **MNUS algorithm** is given as follow [5].

Theorem 4 *Assume that (14) and (15) hold. Let $\{(x, y)\}$ be the solution pair of (1), and $\{(x_i, y_i)\}$ be defined by the Algorithm 4. Then, x_i and y_i converge to x and y, respectively, if*

$$
\omega < \frac{1}{2} \text{ and } \varepsilon < 1 - 2\omega. \tag{31}
$$

In this case, the following two inequalities hold:

$$
\begin{aligned}
&\omega(1+\varepsilon)(Q_Ae_i^x, e_i^x) + \varepsilon((BQ_A^{-1}B^T + C)e_i^y, e_i^y)\\
\le&\rho^{2i}(\omega(1+\varepsilon)(Q_Ae_0^x, e_0^x) + \varepsilon((BQ_A^{-1}B^T + C)e_0^y, e_0^y)),
\end{aligned} \tag{32}
$$

and

$$
\begin{aligned}
(Q_Ae_i^x, e_i^x) \le\varepsilon\left(1+\frac{\omega(2+\varepsilon)^2}{(1+\varepsilon)}\right)\rho^{2i-2}\times&\\
(\omega(1+\varepsilon)(Q_Ae_0^x, e_0^x) + \varepsilon((BQ_A^{-1}B^T + C)e_0^y, e_0^y)),&
\end{aligned} \tag{33}
$$

where

$$
\rho = \frac{\varepsilon+2\omega+\varepsilon\omega+\sqrt{(\varepsilon+2\omega+\varepsilon\omega)^2-4\varepsilon\omega}}{2}. \tag{34}
$$

In [5], Lin and Wei compare the convergence rate between **NUS algorithm** and **MNUS algorithm**, and also gave the conclusion that **MNUS algorithm** is better than **NUS algorithm**.

In fact, inequalities (22) and (23) contain exact inverse, and too many iterations may be need in order to evaluate $A^{-1}u$. A practical nonlinear Uzawa algorithm was proposed in [4]. But the authors only consider using Q_A replace A in inequality (23), the inequality (22) also contain A. Here, we replace A^{-1} with Q_A^{-1} in both inequalities (22) and (23) to result in another result for the algorithm 3.

First, we give two assumptions and some lemmas. For the symmetric and positive definite matrix Q_A, $\phi(v)$ is an approximation to the solution φ of the system

$$Q_A\varphi = v, \tag{35}$$

and $\psi(w)$ is an approximation to the solution ξ of the system

$$(BQ_A^{-1}B^T + C)\xi = w. \tag{36}$$

Let

$$S_a = BQ_A^{-1}B^T + C.$$

It is assumed that

$$\|\phi(v) - Q_A^{-1}v\|_{Q_A} \le \delta\|v\|_{Q_A^{-1}}, \forall v \in H_1, \tag{37}$$

$$\|\psi(w) - S_a^{-1}w\|_{S_a} \le \varepsilon\|w\|_{S_a^{-1}}, \forall w \in H_2, \tag{38}$$

hold for some positive $\delta < 1$ and $\varepsilon < 1$, respectively, and also the inequality (14) holds. Inequalities (37) and (38) are also two reasonable assumptions which are satisfied by the approximate inverse associated with the CG algorithm.

Lemma 1 For any $v \in H_1$, we have the following inequality.

$$\|Bv\|_{S_a^{-1}} \le \|v\|_{Q_A}. \tag{39}$$

Proof.

$$\begin{aligned}
(S_a^{-1}Bv, Bv) \equiv \|Bv\|^2_{S_a^{-1}} &= \sup_{\omega\in H_2} \frac{(S_a^{-1}Bv,\omega)^2}{(S_a^{-1}\omega,\omega)} \\
&= \sup_{\omega\in H_2} \frac{(Bv,\omega)^2}{(S_a\omega,\omega)} = \sup_{\omega\in H_2} \frac{(Q_A^{\frac{1}{2}}v, Q_A^{-\frac{1}{2}}B^T\omega)^2}{(S_a\omega,\omega)} \\
&\le \sup_{\omega\in H_2} \frac{(Q_Av,v)(BQ_A^{-1}B^T\omega,\omega)}{(S_a\omega,\omega)} \\
&\le (Q_Av,v) \equiv \|v\|^2_{Q_A}.
\end{aligned}$$

The proof of the lemma 1 is completed. □

Lemma 2 For a symmetric positive definite matrix Q, $\|A\|_Q = \|Q^{\frac{1}{2}}AQ^{\frac{1}{2}}\|_2$. Proof. By the definition of the matrix norm [15], $\|A\|_Q = \max\limits_{x\neq 0}\frac{\|Ax\|_Q}{\|x\|_Q}$, then

$$\begin{aligned}
\|A\|_Q &= \max_{x\neq 0} \frac{\left(Q^{\frac{1}{2}}Ax, Q^{\frac{1}{2}}Ax\right)^{\frac{1}{2}}}{\left(Q^{\frac{1}{2}}x, Q^{\frac{1}{2}}x\right)^{\frac{1}{2}}} \\
&= \max_{y\neq 0,\ y=Q^{\frac{1}{2}}x} \frac{\left(Q^{\frac{1}{2}}AQ^{-\frac{1}{2}}y, Q^{\frac{1}{2}}AQ^{-\frac{1}{2}}y\right)^{\frac{1}{2}}}{(y,y)^{\frac{1}{2}}} \\
&= \max_{y\neq 0} \frac{\|Q^{\frac{1}{2}}AQ^{-\frac{1}{2}}y\|_2}{\|y\|_2} = \|Q^{\frac{1}{2}}AQ^{\frac{1}{2}}\|_2.
\end{aligned}$$

Therefore, Lemma 2 holds. □

Lemma 3 Assume inequality (14) holds, I is a unity matrix with appropriate dimension, we have the following inequality

$$\|I - Q_A^{-1}A\|_{Q_A} \leq \omega, \tag{40}$$

$$\|Av + B^T w\|_{Q_A^{-1}} \leq \|v\|_{Q_A} + \|w\|_{S_a}. \tag{41}$$

Proof. From inequality (14), we know that

$$((I - Q_A^{-1}A)v, v) \leq \omega(v, v),$$

so $\rho(I - Q_A^{-1}A) \leq \omega$, where ρ is the spectral radius of the corresponding operator.

By the Lemma 2, we have

$$\begin{aligned}
||I - Q_A^{-1}A||_{Q_A} &= ||Q_A^{\frac{1}{2}}(I - Q_A^{-1}A)Q_A^{-\frac{1}{2}}||_2 \\
&= ||I - Q_A^{-\frac{1}{2}}AQ_A^{-\frac{1}{2}}||_2 \\
&= \rho(I - Q_A^{-\frac{1}{2}}AQ_A^{-\frac{1}{2}}) \\
&= \rho(I - Q_A^{-1}A) \leq \omega
\end{aligned}$$

i.e., $||I - Q_A^{-1}A||_{Q_A} \leq \omega$. It follows from the triangular inequality that:

$$\begin{aligned}
||Av + B^T w||_{Q_A^{-1}} &\leq ||Av||_{Q_A^{-1}} + ||B^T w||_{Q_A^{-1}} \\
&\leq ||Q_A v||_{Q_A^{-1}} + ||w||_{BQ_A^{-1}B^T} \\
&\leq ||v||_{Q_A} + ||w||_{S_a}.
\end{aligned}$$

The proof of the Lemma 3 is completed. □

Theorem 5 *Assume that (37),(38) and (14) hold, Let $\{(x, y)\}$ be the solution pair of (1), and $\{(x_i, y_i)\}$ be defined by the Algorithm 3. Then, x_i and y_i converge to x and y, respectively, if*

$$0 < \omega < \frac{1}{2}\ ,\ 0 < \delta < \frac{1 - 2\omega}{3} \ \text{ and } \ 0 < \varepsilon < \frac{1 - 3\delta - 2\omega}{1 + \delta}. \tag{42}$$

In this case, the following two inequalities hold:

$$\begin{aligned}
&(\varepsilon + 1)(\delta + \omega)(Q_A e_i^x, e_i^x) + (1 + \delta)(S_a e_i^y, e_i^y) \\
\leq &\rho^{2i}(\varepsilon + 1)(\delta + \omega)(Q_A e_0^x, e_0^x) + (1 + \delta)(S_a e_0^y, e_0^y),
\end{aligned} \tag{43}$$

and

$$\begin{aligned}
(Q_A e_i^x, e_i^x) \leq &\left(1 + \delta + \frac{\delta + \omega}{\varepsilon + 1}\right)\rho^{2i-2}\times \\
&((\varepsilon + 1)(\delta + \omega)(Q_A e_0^x, e_0^x) + (1 + \delta)(S_a e_0^y, e_0^y)),
\end{aligned} \tag{44}$$

where

$$\rho = \frac{\varepsilon + 2\delta + \varepsilon\delta + \omega + \sqrt{(\varepsilon + 2\delta + \varepsilon\delta + \omega)^2 + 4(\delta + \omega)}}{2}. \tag{45}$$

Proof. From Algorithm 3, then we have the following equations

$$e^x_{i+1} = e^x_i - \phi(Ae^x_i + B^T e^y_i), \tag{46}$$
$$e^y_{i+1} = e^y_i - \psi(Ce^y_i - Be^x_{i+1}). \tag{47}$$

Eq. (46) gives

$$\begin{aligned} e^x_{i+1} =&(Q_A^{-1} - \phi)(Ae^x_i + B^T e^y_i) + e^x_i - Q_A^{-1}(Ae^x_i + B^T e^y_i) \\ =&(Q_A^{-1} - \phi)(Ae^x_i + B^T e^y_i) + (I - Q_A^{-1}A)e^x_i - Q_A^{-1}B^T e^y_i. \end{aligned}$$

Substituting e^x_{i+1} in Eq. (47) by the above equation, we have

$$Ce^y_i - Be^x_{i+1} = S_a e^y_i - B(I - Q_A^{-1}A)e^x_i - B(Q_A^{-1} - \phi)(Ae^x_i + B^T e^y_i). \tag{48}$$

$$\begin{aligned} e^y_{i+1} =&(S_a^{-1} - \psi)(Ce^y_i - Be^x_{i+1}) + e^y_i - S_a^{-1}(Ce^y_i - Be^x_{i+1}) \\ =&(S_a^{-1} - \psi)(Ce^y_i - Be^x_{i+1}) \\ &+S_a^{-1}B((I - Q_A^{-1}A)e^x_i + (Q_A^{-1} - \phi)(Ae^x_i + B^T e^y_i)). \end{aligned} \tag{49}$$

It follows from the triangular inequality that:

$$\begin{aligned} ||e^x_{i+1}||_{Q_A} =&||(Q_A^{-1} - \phi)(Ae^x_i + B^T e^y_i) + (I - Q_A^{-1}A)e^x_i - Q_A^{-1}B^T e^y_i||_{Q_A} \\ \le&||(Q_A^{-1} - \phi)(Ae^x_i + B^T e^y_i)||_{Q_A} + ||(I - Q_A^{-1}A)e^x_i||_{Q_A} + ||Q_A^{-1}B^T e^y_i||_{Q_A} \\ \le&\delta||Ae^x_i + B^T e^y_i||_{Q_A^{-1}} + ||I - Q_A^{-1}A||_{Q_A}||e^x_i||_{Q_A} + ||e^y_i||_{S_a} \ \ \text{(by(38))} \\ \le&\delta(||e^x_i||_{Q_A} + ||e^y_i||_{S_a}) + \omega||e^x_i||_{Q_A} + ||e^y_i||_{S_a} \ \ \text{(by Lemma 3)} \\ =&(\delta + \omega)||e^x_i||_{Q_A} + (1 + \delta)||e^y_i||_{S_a}. \end{aligned} \tag{50}$$

Using triangular inequality, from Eq. (49) and Lemma 1 we have

$$\begin{aligned} ||e^y_{i+1}||_{S_a} =&||(S_a^{-1} - \psi)(Ce^y_i - Be^x_{i+1}) + e^y_i - S_a^{-1}(Ce^y_i - Be^x_{i+1})||_{S_a} \\ \le&||(S_a^{-1} - \psi)(Ce^y_i - Be^x_{i+1})||_{S_a} \\ &+ ||S_a^{-1}B((I - Q_A^{-1}A)e^x_i + (Q_A^{-1} - \phi)(Ae^x_i + B^T e^y_i))||_{S_a} \ \ \text{(by(49))} \\ \le&\varepsilon||Ce^y_i - Be^x_{i+1}||_{S_a^{-1}} + ||(I - Q_A^{-1}A)e^x_i + (Q_A^{-1} - \phi)(Ae^x_i + B^T e^y_i)||_{Q_A} \\ \le&\varepsilon||e^y_i||_{S_a} + (\varepsilon + 1)(||(I - Q_A^{-1}A)e^x_i||_{Q_A} + ||(Q_A^{-1} - \phi)(Ae^x_i + B^T e^y_i)||_{Q_A}) \\ \le&\varepsilon||e^y_i||_{S_a} + (\varepsilon + 1)((\delta + \omega)||e^x_i||_{Q_A} + \delta||e^y_i||_{S_a}) \ \ \text{(by(37) and Lemma 3)} \\ =&(\varepsilon + 1)(\delta + \omega)||e^x_i||_{Q_A} + (\varepsilon + \delta + \varepsilon\delta)||e^y_i||_{S_a}. \end{aligned} \tag{51}$$

It follow from (50) and (51) that

$$\begin{pmatrix} \|e^x_i\|_{Q_A} \\ \|e^y_i\|_{S_a} \end{pmatrix} \le M^i \begin{pmatrix} \|e^x_0\|_{Q_A} \\ \|e^y_0\|_{S_a} \end{pmatrix}, \tag{52}$$

where M is given by

$$M = \begin{pmatrix} \delta + \omega & 1 + \delta \\ (\varepsilon + 1)(\delta + \omega) & \varepsilon + \delta + \varepsilon\delta \end{pmatrix}.$$

Obviously, M is symmetric with respect to the following inner product of the two-dimensional Euclidean space

$$\begin{aligned} \left[\begin{pmatrix} x_1 \\ y_1 \end{pmatrix}, \begin{pmatrix} x_2 \\ y_2 \end{pmatrix}\right] \equiv& \left(\begin{pmatrix} (\varepsilon + 1)(\delta + \omega) & 0 \\ 0 & 1 + \delta \end{pmatrix}\begin{pmatrix} x_1 \\ y_1 \end{pmatrix}, \begin{pmatrix} x_2 \\ y_2 \end{pmatrix}\right) \\ =& (\varepsilon + 1)(\delta + \omega)x_1 x_2 + (1 + \delta)\, y_1 y_2. \end{aligned}$$

Thus, from (52), we have

$$
\begin{aligned}
&(\varepsilon+1)(\delta+\omega)(Q_A e_i^x, e_i^x) + (1+\delta)(S_a e_i^y, e_i^y)\\
=&\left[\begin{pmatrix} \|e_i^x\|_{Q_A} \\ \|e_i^y\|_{S_a} \end{pmatrix}, \begin{pmatrix} \|e_i^x\|_{Q_A} \\ \|e_i^y\|_{S_a} \end{pmatrix}\right]\\
\leq&\left[M^i \begin{pmatrix} \|e_0^x\|_{Q_A} \\ \|e_0^y\|_{S_a} \end{pmatrix}, M^i \begin{pmatrix} \|e_0^x\|_{Q_A} \\ \|e_0^y\|_{S_a} \end{pmatrix}\right]\\
\leq&\rho^{2i}((\varepsilon+1)(\delta+\omega)(Q_A e_0^x, e_0^x) + (1+\delta)(S_a e_0^y, e_0^y)),
\end{aligned}
$$

where ρ is the spectral radius of M. The eigenvalues of M are the roots of

$$\lambda^2 - (2\delta + \varepsilon + \varepsilon\delta + \omega)\lambda - (\omega + \delta) = 0.$$

From above equation, we know that $\lambda \in R$ and $2\delta + \varepsilon + \varepsilon\delta + \omega > 0$. Obviously, the spectral radius ρ of M is equal to its positive eigenvalue which is given by (45).

It is easy to see if (42) is satisfied, then $\rho < 1$. This completes the proof of (43).

To prove (44), we apply the following elementary inequality

$$(a+b)^2 \leq (1+\eta)a^2 + (1+\eta^{-1})b^2$$

to (50), and get for any $\eta > 0$,

$$||e_i^x||_{Q_A}^2 \leq (1+\eta)(\delta+\omega)^2||e_{i-1}^x||_{Q_A}^2 + (1+\eta^{-1})(1+\delta)^2||e_{i-1}^y||_{S_a}^2.$$

Inequality (44) follow from taking $\eta = \frac{(1+\varepsilon)(1+\delta)}{\delta+\omega}$ and applying (43). This completes the proof of the theorem. □

Remark 1. When $\omega = 0$, Theorem 5 is the theorem 3, it is the result of [4]. In the experiment, we compute the incomplete Cholesky factorization of A, i.e., $A = LL^T - R$, where L is the incomplete Cholesky factor. Let $Q_A = LL^T$, which can insure $\omega < \frac{1}{2}$ in (14).

3 A new Nonlinear Uzawa method and convergence results

In this section, we propose a new **Nonlinear Uzawa method** by using the modified idea of [5,9] to modified NUAS method. We call this algorithm as **MNUAS algorithm**.

Algorithm 5(MNUAS algorithm) For $x_0 \in H_1$ and $y_0 \in H_2$ given, the iterative sequence $\{(x_i, y_i)\}$ is defined, for $i = 0, 1, ...$, by

$$\bar{x}_{i+1} = x_i + \phi(f - Ax_i - B^T y_i), \tag{53}$$

$$y_{i+1} = y_i + \psi(B\bar{x}_{i+1} - Cy_i - g), \tag{54}$$

$$x_{i+1} = \bar{x}_{i+1} - Q_A^{-1}B^T(y_{i+1} - y_i), \tag{55}$$

where $\phi(v)$ is an approximation to the solution φ of the system

$$Q_A\varphi = v,$$

for the symmetric positive definite operator Q_A and $\psi(w)$ is an approximation to the solution ξ of the system

$$(BQ_A^{-1}B^T + C)\xi = w.$$

It is also assumed that (14), (37) and (38) hold. We will give the main results of the paper in the following series.

Theorem 6 *Assume that (14), (37) and (38) hold, Let $\{(x, y)\}$ be the solution pair of (1), and $\{(x_i, y_i)\}$ be defined by the Algorithm 5. Then, x_i and y_i converge to x and y, respectively, if*

$$0 < \omega < \frac{1}{2}\ ,\ 0 < \delta < \frac{1-2\omega}{3} \ \text{ and } \ 0 < \varepsilon < \frac{1-3\delta-2\omega}{1+\delta}. \tag{56}$$

In this case, the following two inequalities hold:

$$\begin{aligned}&(\delta+\omega)(1+\varepsilon)(Q_A e_i^x, e_i^x) + (\varepsilon+2\delta+\varepsilon\delta)(S_a e_i^y, e_i^y)\\ \le&\rho^{2i}((\delta+\omega)(1+\varepsilon)(Q_A e_0^x, e_0^x) + (\varepsilon+2\delta+\varepsilon\delta)(S_a e_0^y, e_0^y)),\end{aligned} \tag{57}$$

and

$$\begin{aligned}(Q_A e_i^x, e_i^x) \le &\left(\varepsilon+2\delta+\varepsilon\delta+\frac{(\varepsilon+2)^2(\delta+\omega)}{1+\varepsilon}\right)\rho^{2i-2}\times\\ &((\delta+\omega)(1+\varepsilon)(Q_A e_0^x, e_0^x) + (\varepsilon+2\delta+\varepsilon\delta)(S_a e_0^y, e_0^y)),\end{aligned} \tag{58}$$

where

$$\rho = \frac{3\delta+\varepsilon+2\varepsilon\delta+2\omega+\varepsilon\omega+\sqrt{(3\delta+\varepsilon+2\varepsilon\delta+2\omega+\varepsilon\omega)^2-4\varepsilon(\delta+\omega)}}{2}. \tag{59}$$

Proof. Let $\bar{e}_{i+1}^x = x - \bar{x}_{i+1}$. From (53)-(55), then we have the following equations:

$$\bar{e}_{i+1}^x = e_i^x - \phi\left(Ae_i^x + B^T e_i^y\right), \tag{60}$$

$$e_{i+1}^y = e_i^y - \psi\left(Ce_i^y - B\bar{e}_{i+1}^x\right), \tag{61}$$

$$e_{i+1}^x = \bar{e}_{i+1}^x + Q_A^{-1}B^T(e_i^y - e_{i+1}^y). \tag{62}$$

In fact, by the proof of **Theorem 5**, we know that

$$||e_{i+1}^y||_{S_a} \le (\varepsilon+1)(\delta+\omega)||e_i^x||_{Q_A} + (\varepsilon+\delta+\varepsilon\delta)||e_i^y||_{S_a}. \tag{63}$$

From (60) and (62), it can be concluded that

$$\begin{aligned}e_{i+1}^x &= (Q_A^{-1}-\phi)(Ae_i^x + B^T e_i^y) - Q_A^{-1}(Ae_i^x + B^T e_i^y) + e_i^x + Q_A^{-1}B^T(e_i^y - e_{i+1}^y)\\ &= (Q_A^{-1}-\phi)(Ae_i^x + B^T e_i^y) + (I - Q_A^{-1}A)e_i^x - Q_A^{-1}B^T e_{i+1}^y.\end{aligned} \tag{64}$$

Using triangular inequality, from Eq.(64) we have

$$\begin{aligned}||e_{i+1}^x||_{Q_A} &= ||(Q_A^{-1}-\phi)(Ae_i^x + B^T e_i^y)||_{Q_A} + ||(I - Q_A^{-1}A)e_i^x||_{Q_A} + ||Q_A^{-1}B^T e_{i+1}^y||_{Q_A}\\ &\le \delta||Ae_i^x + B^T e_i^y||_{Q_A^{-1}} + ||I - Q_A A||_{Q_A}||e_i^x||_{Q_A} + ||e_{i+1}^y||_{S_a} \quad (\text{by}(37))\\ &\le (\delta+\omega)||e_i^x||_{Q_A} + \delta||e_i^y||_{S_a} + ||e_{i+1}^y||_{S_a} \quad (\text{by Lemma 3})\\ &\le (\varepsilon+2)(\delta+\omega)||e_i^x||_{Q_A} + (\varepsilon+2\delta+\varepsilon\delta)||e_i^y||_{S_a}. \ (\text{by}(63))\end{aligned} \tag{65}$$

It follows from (63) and (65) that

$$\begin{pmatrix} \|e_i^x\|_{Q_A} \\ \|e_i^y\|_{S_a} \end{pmatrix} \le N^i \begin{pmatrix} \|e_0^x\|_{Q_A} \\ \|e_0^y\|_{S_a} \end{pmatrix}, \tag{66}$$

where N is given by

$$N=\begin{pmatrix} (\varepsilon+2)(\delta+\omega) & \varepsilon+2\delta+\varepsilon\delta \\ (\varepsilon+1)(\delta+\omega) & \varepsilon+\delta+\varepsilon\delta \end{pmatrix}.$$

Obviously, N is symmetric with respect to the following inner product of the two-dimensional Euclidean space

$$\begin{aligned}\left[\begin{pmatrix} x_1 \\ y_1 \end{pmatrix},\begin{pmatrix} x_2 \\ y_2 \end{pmatrix}\right] &\equiv \left(\begin{pmatrix} (\varepsilon+1)(\delta+\omega) & 0 \\ 0 & \varepsilon+2\delta+\varepsilon\delta \end{pmatrix}\begin{pmatrix} x_1 \\ y_1 \end{pmatrix},\begin{pmatrix} x_2 \\ y_2 \end{pmatrix}\right) \\ &= (\varepsilon+1)(\delta+\omega)x_1x_2+(\varepsilon+2\delta+\varepsilon\delta)\,y_1y_2.\end{aligned}$$

Thus, from (66), we have

$$\begin{aligned}&(\varepsilon+1)(\delta+\omega)(Q_Ae_i^x,e_i^x)+(\varepsilon+2\delta+\varepsilon\delta)(S_ae_i^y,e_i^y) \\ =&\left[\begin{pmatrix} \|e_i^x\|_{Q_A} \\ \|e_i^y\|_{S_a} \end{pmatrix},\begin{pmatrix} \|e_i^x\|_{Q_A} \\ \|e_i^y\|_{S_a} \end{pmatrix}\right] \\ \le&\left[N^i\begin{pmatrix} \|e_0^x\|_{Q_A} \\ \|e_0^y\|_{S_a} \end{pmatrix},N^i\begin{pmatrix} \|e_0^x\|_{Q_A} \\ \|e_0^y\|_{S_a} \end{pmatrix}\right] \\ \le&\rho^{2i}((\varepsilon+1)(\delta+\omega)(Q_Ae_0^x,e_0^x)+(\varepsilon+2\delta+\varepsilon\delta)(S_ae_0^y,e_0^y)),\end{aligned}$$

where ρ is the spectral radius of N. The eigenvalues of N are the roots of

$$\lambda^2-(3\delta+\varepsilon+2\varepsilon\delta+2\omega+\varepsilon\omega)\lambda+4\varepsilon(\delta+\omega)=0.$$

From above equation, we know that $\lambda\in R$ and $\lambda>0$. Obviously, the spectral radius ρ of N is equal to its max positive eigenvalue which is given by (59). It is easy to see if (56) is satisfied, then $\rho<1$. This completes the proof of (57).

To prove (58), we apply the following elementary inequality

$$(a+b)^2\le(1+\eta)\,a^2+\left(1+\eta^{-1}\right)b^2$$

to (65), and get for any $\eta>0$,

$$||e_i^x||_{Q_A}^2\le(1+\eta)[(\varepsilon+2)(\delta+\omega)]^2||e_{i-1}^x||_{Q_A}^2+(1+\eta^{-1})(\varepsilon+2\delta+\varepsilon\delta)^2||e_{i-1}^y||_{S_a}^2.$$

Inequality (58) follow from taking $\eta=\frac{(\varepsilon+1)(\varepsilon+2\delta+\varepsilon\delta)}{(\varepsilon+2)^2(\delta+\omega)}$ and applying (57). This completes the proof of the theorem. □

Corollary 1. In Algorithm 5, assume that $Q_A=A$ hold, let $\{(x,y)\}$ be the solution pair of (1), and $\{(x_i,y_i)\}$ be defined by the Algorithm 5. Then, x_i and y_i converge to x and y, respectively, if

$$0<\delta<\frac{1}{3}\ \text{ and }\ 0<\varepsilon<\frac{1-3\delta}{1+\delta}. \tag{67}$$

In this case, the following two inequalities hold:

$$\begin{aligned}&\delta(\varepsilon+1)(Q_Ae_i^x,e_i^x)+(\varepsilon+2\delta+\varepsilon\delta)(S_ae_i^y,e_i^y) \\ \le&\rho^{2i}\delta(\varepsilon+1)(Q_Ae_0^x,e_0^x)+(\varepsilon+2\delta+\varepsilon\delta)(S_ae_0^y,e_0^y),\end{aligned} \tag{68}$$

and

$$\begin{aligned}(Q_Ae_i^x,e_i^x)\le&\left(\varepsilon+2\delta+\varepsilon\delta+\frac{(\varepsilon+2)^2\delta}{1+\varepsilon}\right)\rho^{2i-2}\times \\ &(\delta(1+\varepsilon)(Q_Ae_0^x,e_0^x)+(\varepsilon+2\delta+\varepsilon\delta)(S_ae_0^y,e_0^y)),\end{aligned} \tag{69}$$

where

$$\rho = \frac{3\delta + \varepsilon + 2\varepsilon\delta + \sqrt{(3\delta + \varepsilon + 2\varepsilon\delta)^2 - 4\varepsilon\delta}}{2}. \tag{70}$$

Proof. For $Q_A = A$, hence $\omega = 0$. In Theorem 6, let $w = 0$, we can complete this corollary. □

In the rest of this section, we analyze the convergence factors between MNUAS Algorithm and NUAS Algorithm under the same condition with different nonlinear approximate assumption.

1. Under the conditions (22) and (23), denote by ρ_{MN} and ρ_N the convergence factors of **MNUAS** and **NUAS**, respectively;

2. under the conditions (37) and (38), denote by ρ_{IMN} and ρ_{IN} the convergence factors of **MNUAS** and **NUAS**, respectively.

Case 1: From Theorem 3 in [4], the convergence factor of **NUAS** is

$$\rho_N = \frac{\varepsilon + 2\delta + \varepsilon\delta + \sqrt{(\varepsilon + 2\delta + \varepsilon\delta)^2 + 4\delta}}{2}.$$

when $\delta \gg \varepsilon$, the convergence factorρ_N is approximately equal to $\delta + \sqrt{\delta^2 + \delta}$. Thus

$$\rho_N \approx \delta + \sqrt{\delta^2 + \delta}.$$

From the above corollary, the convergence factor of **MNUAS** is

$$\rho_{MN} = \frac{3\delta + \varepsilon + 2\varepsilon\delta + \sqrt{(3\delta + \varepsilon + 2\varepsilon\delta)^2 - 4\varepsilon\delta}}{2}.$$

When $\delta \gg \varepsilon$, the convergence factorρ_{MN} is approximately equal to 3δ. Thus

$$\rho_{MN} \approx 3\delta.$$

If $0 < \delta < \frac{1}{3}$, we have $\rho_{MN} \approx 3\delta < \delta + \sqrt{\delta^2 + \delta} \approx \rho_N$.

Case 2: Theorem 5 gives the convergence factor of **NUAS** is

$$\rho_{IN} = \frac{\varepsilon + 2\delta + \varepsilon\delta + \omega + \sqrt{(\varepsilon + 2\delta + \varepsilon\delta + \omega)^2 + 4(\delta + \omega)}}{2}.$$

when $\omega \gg \delta,\ \varepsilon$, the convergence factor ρ_{IN} is approximately equal to $\frac{\omega + \sqrt{\omega^2 + 4\omega}}{2}$. Thus

$$\rho_{IN} \approx \frac{\omega + \sqrt{\omega^2 + 4\omega}}{2}.$$

Theorem 6 gives the convergence factor of **MNUAS** is

$$\rho_{IMN} = \frac{3\delta + \varepsilon + 2\varepsilon\delta + 2\omega + \varepsilon\omega + \sqrt{(3\delta + \varepsilon + 2\varepsilon\delta + 2\omega + \varepsilon\omega)^2 - 4\varepsilon(\delta + \omega)}}{2}.$$

When $\omega \gg \delta,\ \varepsilon$, the convergence factor ρ_{IMN} is approximately equal to 2ω. Thus

$$\rho_{IMN} \approx 2\omega.$$

If $0 < \omega < \frac{1}{2}$, we have $\rho_{IMN} \approx 2\omega < \frac{\omega + \sqrt{\omega^2 + 4\omega}}{2} \approx \rho_{IN}$. From the above comparison of the convergence factors, we expect that the **MNUAS** may be better than **NUAS**, if the nonlinear approximation is appropriate.

In the next section, numerical experiments confirm our analysis of the results on the convergence of the nonlinear Uzawa methods.

4 Numerical experiments

The problem under consideration is the classic incompressible steady state Stokes problem

$$\begin{cases} -\nu\Delta u + \nabla p = f, \text{in } \Omega, \\ \text{div} u = 0, \text{in } \Omega, \end{cases} \tag{71}$$

here ν is the viscosity. Many discretization schemes for this problem will lead to saddle point problems of the form (1) see for instance [2]. We generate the test problem (leaky lid-driven cavity) with the IFISS software written by Howard Elman, Alison Ramage and David Silvester [11]. The mixed finite element used is the bilinear-constant velocity-pressure $Q_1 - P_0$ pair with global stabilization or local stabilization [10]. The finite element subdivision is based on $n \times n$ uniform grids of square elements. Using the IFISS software to discretize (71), then the coefficient matrix $\mathcal{A}$ of the linear system, which is equivalent to (1), is the following

$$\mathcal{A} = \begin{pmatrix} A & B^T \\ B & -\beta C \end{pmatrix},$$

where β is a stabilizing parameter [10]. A remark on the local stabilization was given by Cao in [3] to state that D. Silvester pointed out that β should be 0.25 in the local stabilized case. Consequently, Cao took $\beta = 1$ and 0.25 in the numerical experiments for global and local stabilizations, respectively. Similarly, in our numerical experiments we also take $\beta = 1$ and 0.25 for global and local stabilizations, respectively.

Now we describe the algorithms and notions for the test problem. In our experiments, we take $\nu = 1$. NUS denotes the NUS algorithm only using the nonlinear approximation to the inverse of Schur complement $(BQ_A^{-1}B^T + C)^{-1}$. MNUS denotes the Modified algorithm of NUS. NUAS denotes the NUAS algorithm using both the nonlinear approximations to Q_A^{-1} and $(BQ_A^{-1}B^T + C)^{-1}$ at the same time. MNUAS denotes the Modified algorithm of NUAS. In the next experiments, we only compare those four algorithm.

By using the sparsity of A, we may compute the incomplete Cholesky factorization of A, i.e., $A = LL^T - R$, where L is the incomplete Cholesky factor [12]. In the incomplete Cholesky factorization, we consider the case in which the drop tolerance is tol=0.01 and the case with no fill-in [12]. In NUS and MNUS, $Q_A = LL^T$ and ψ is defined by two steps of CG applied to approximate the action of inverse of the approximate Schur complement. In NUAS and MNUAS, ϕ is defined by two steps of CG applied to approximate the action of Q_A^{-1} and ψ is defined by two steps of CG applied to approximate the action of the inverse of the approximate Schur complement $BQ_A^{-1}B^T + C$ with $Q_A = LL^T$.

All computations are performed in Matlab 7.0. The stop criterion for the iteration is

$$\frac{\|r_k\|}{\|r_0\|} \leq 10^{-6},$$

where r_0 is the initial residual vector and r_k is the kth residual vector of (1).

In Table 1-4, we show the iteration numbers and the CPU time (in seconds) of the four algorithms in four mesh grids for global and local stabilizations, respectively. We can see from these tables that all these algorithms are convergent, but NUS and NUAS converge rather slowly. In contrast, MNUS and MNUAS converge rapidly. It means that the modified algorithms have better convergerce rate. With the mesh grids refined, among all these algorithms, MNUAS need the smallest number of steps in the four algorithm, and most case the CPU time is the smallest. Finally, we give, in Figures 1-4, convergence plots for these algorithms. These figures show that the modified algorithms converge more quickly, and MNUAS is better than NUAS.

Table 1: Iteration number and CPU time for global stabilization (no fill-in)

Grid	NUS	MNUS	NUAS	MNUAS
8×8	20(0.0310)	19(0.0310)	20(0.0320)	16(0.0310)
16×16	58(0.2180)	36(0.1560)	35(0.1720)	22(0.1410)
32×32	174(2.7970)	118(2.2500)	95(1.8440)	54(1.2340)
64×64	>500(33.8590)	343(28.1400)	311(25.9070)	188(18.5320)

Table 2: Iteration number and CPU time for global stabilization (tol=0.01)

Grid	NUS	MNUS	NUAS	MNUAS
8×8	22(0.0150)	21(0.0310)	23(0.0470)	21(0.0320)
16×16	24(0.0940)	18(0.0930)	24(0.1250)	17(0.0940)
32×32	59(0.9840)	36(0.7190)	45(0.9380)	24(0.5930)
64×64	184(12.0780)	108(8.8900)	141(12.0160)	75(7.4070)

Table 3: Iteration number and CPU time for local stabilization (no fill-in)

Grid	NUS	MNUS	NUAS	MNUAS
8×8	25(0.0310)	9(0.0160)	16(0.0470)	12(0.0310)
16×16	61(0.2180)	24(0.1250)	38(0.1880)	20(0.1090)
32×32	176(2.5160)	70(1.2660)	119(2.2030)	61(1.3280)
64×64	485(27.7190)	309(21.7180)	419(29.8600)	171(15.0470)

Table 4: Iteration number and CPU time for local stabilization (tol=0.01)

Grid	NUS	MNUS	NUAS	MNUAS
8×8	12(0.0160)	9(0.0150)	12(0.0160)	9(0.0160)
16×16	27(0.0940)	9(0.0470)	19(0.0940)	12(0.0620)
32×32	62(0.9530)	23(0.4380)	48(0.9370)	24(0.5620)
64×64	163(9.8750)	70(5.4060)	126(9.6250)	66(6.0940)

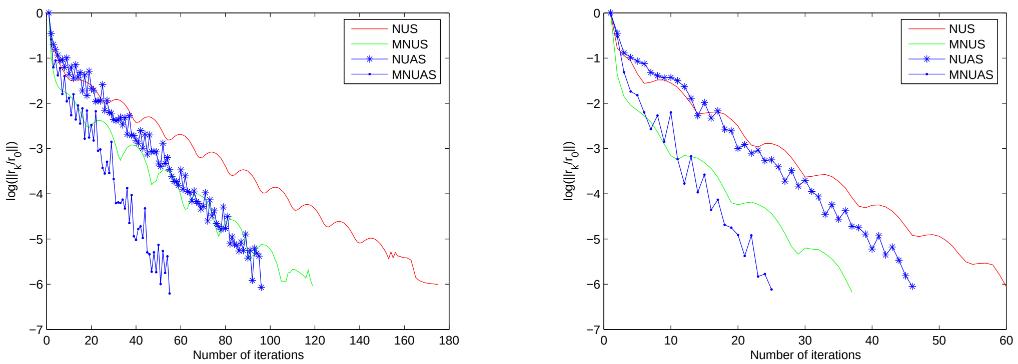

Figure 1: *Global stabilization.* 32×32 *mesh. Left: no fill-in; right: tol* $= 0.01$

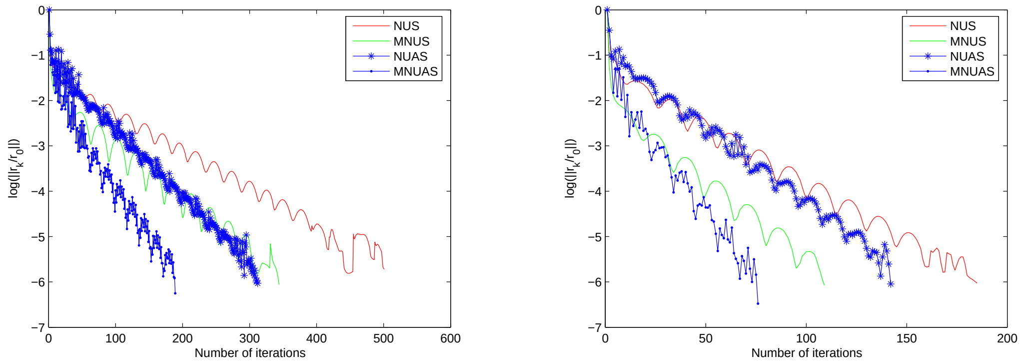

Figure 2: *Global stabilization.* 64×64 *mesh. Left: no fill-in; right:* $tol = 0.01$

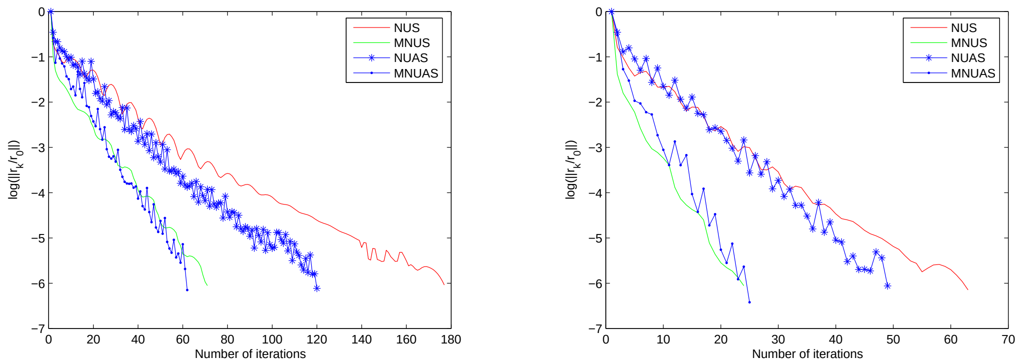

Figure 3: *Local stabilization.* 32×32 *mesh. Left: no fill-in; right:* $tol = 0.01$

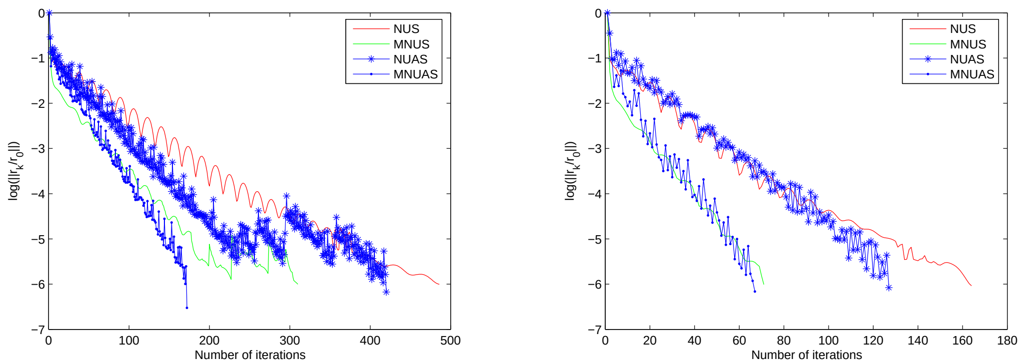

Figure 4: *Local stabilization.* 64×64 *mesh. Left: no fill-in; right:* $tol = 0.01$

5 Conclusion

In this paper, we present a Modified Nonlinear Uzawa algorithm (MNUAS), which is modified the NUAS algorithm contained two nonlinear approximate inverses, for solving symmetric saddle point problems. At the same time, its convergence result is given, and the convergence factors are compared. Numerical experiments show that MNUAS algorithm needs less iterations and CPU time than the NUAS algorithm when applied to the Stokes equation by mixed finite element discretization.

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J. COMPUTATIONAL ANALYSIS AND APPLICATIONS, VOL. 15, NO.8, 1372-1380, 2013, COPYRIGHT 2013 EUDOXUS PRESS, LLC

A boundary value problem of fractional differential equations with anti-periodic type integral boundary conditions

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Abstract

In this paper, we discuss the existence of solutions for fractional differential equations of order $q \in (2, 3]$ with anti-periodic type integral boundary conditions. Our results are based on Leray-Schauder nonlinear alternative and some standard tools of fixed point theory.

Key words and phrases: Fractional differential equations; antiperiodic; integral boundary conditions; existence; nonlinear alternative of Leray Schauder type; fixed point theorems.
AMS (MOS) Subject Classifications: 34A08, 34B10, 34B15.

1 Introduction

Boundary value problems for nonlinear differential equations arise in a variety of areas of applied mathematics, physics and variational problems of control theory. The existing literature mainly deals with second-order boundary value problems and there are a few papers on third/higher order problems ([1]-[3]).

In the last few years, much work has been completed on boundary value problems of fractional differential equations. For examples and details, we refer the reader to the books ([4]-[9]) and papers ([10]-[23]). In [19], the authors studied a boundary value problem of nonlinear fractional differential equations of order $q \in (1, 2]$ with non-separated integral boundary conditions. In this paper, we study the existence of solutions for a boundary value problem of fractional differential equations of order $q \in (2, 3]$ with anti-periodic type integral boundary conditions. Precisely, we consider the following problem

$$\begin{cases} {}^cD^q x(t) = f(t, x(t)), \quad t \in [0, T], \ T > 0, \quad 2 < q \leq 3, \\ x^{(j)}(0) - \lambda_j x^{(j)}(T) = \mu_j \displaystyle\int_0^T g_j(s, x(s))ds, \ j = 0, 1, 2 \end{cases} \tag{1}$$

where ${}^cD^q$ denotes the Caputo derivative of fractional order q, $x^{(j)}(\cdot)$ denotes jth derivative of $x(\cdot)$ with $x^{(0)}(\cdot) = x(\cdot)$, $f, g_j : [0, T] \times \mathbb{R} \to \mathbb{R}$ are given continuous functions and $\lambda_j, \ \mu_j \in \mathbb{R}$ $(\lambda_j \neq 1)$.

We remark that problem (1) reduces to a fractional boundary value problem with anti-periodic type boundary conditions for $\lambda_j = -1, \mu_j = 0, j = 0, 1, 2$ [12].

The main aim of the present work is to establish some existence results for problem (1) by means of Leray-Schauder nonlinear alternative, Banach contraction mapping principle and Krasnoselskii fixed point theorem.

2 Preliminary result

Let us recall some basic definitions of fractional calculus [4, 6].

Definition 2.1 *For $(n-1)-$times absolutely continuous function $g:[0,\infty)\to\mathbb{R}$, the Caputo derivative of fractional order q is defined as*

$$ {}^cD^q g(t)=\frac{1}{\Gamma(n-q)}\int_0^t (t-s)^{n-q-1}g^{(n)}(s)ds,\quad n-1<q<n, n=[q]+1, $$

where $[q]$ denotes the integer part of the real number q.

Definition 2.2 *The Riemann-Liouville fractional integral of order q is defined as*

$$ I^q g(t)=\frac{1}{\Gamma(q)}\int_0^t \frac{g(s)}{(t-s)^{1-q}}ds,\quad q>0, $$

provided the integral exists.

In the sequel, the following lemma plays a pivotal role.

Lemma 2.3 *For a given $y\in C([0,T],\mathbb{R})$ and $2<q\le 3$, the unique solution of the equation ${}^cD^q x(t)=y(t),\ t\in[0,T]$ subject to the boundary conditions of (1) is given by*

$$ \begin{aligned} x(t) \;&= \int_0^t \frac{(t-s)^{q-1}}{\Gamma(q)}y(s)ds-\lambda_0\xi_1\int_0^T \frac{(T-s)^{q-1}}{\Gamma(q)}y(s)ds \\ &+\lambda_1\eta_2\int_0^T \frac{(T-s)^{q-2}}{\Gamma(q-1)}y(s)ds+\lambda_2\eta_1\int_0^T \frac{(T-s)^{q-3}}{\Gamma(q-2)}y(s)ds \\ &-\mu_0\xi_1\int_0^T g_0(s,x(s))ds+\mu_1\eta_2\int_0^T g_1(s,x(s))ds \\ &+\mu_2\eta_1\int_0^T g_2(s,x(s))ds, \end{aligned} \tag{2} $$

where

$$ \eta_1=\xi_3\Big[-\lambda_0(\lambda_1+1)T^2+2\lambda_1(\lambda_0-1)tT-(\lambda_0-1)(\lambda_1-1)t^2\Big], $$
$$ \eta_2=\xi_2[\lambda_0 T-(\lambda_0-1)t], $$
$$ \xi_1=\frac{1}{\lambda_0-1},\quad \xi_2=\frac{1}{(\lambda_0-1)(\lambda_1-1)},\quad \xi_3=\frac{1}{2(\lambda_0-1)(\lambda_1-1)(\lambda_2-1)}. $$

Proof. For $2<q\le 3$, it is well known [6] that the solution of fractional differential equation ${}^cD^q x(t)=y(t)$ can be written as

$$ x(t)=\int_0^t \frac{(t-s)^{q-1}}{\Gamma(q)}y(s)ds-c_0-c_1t-c_2t^2,\ t\in[0,T], \tag{3} $$

where $c_0,c_1,c_2\in\mathbb{R}$ are arbitrary constants. Applying the boundary conditions of (1), we get

$$ \begin{cases} (\lambda_0-1)c_0+\lambda_0Tc_1+\lambda_0T^2c_2 &= \mu_0\displaystyle\int_0^T g_0(t,x(s))ds+\lambda_0\int_0^T \frac{(T-s)^{q-1}}{\Gamma(q)}y(s)ds, \\ (\lambda_1-1)c_1+2\lambda_1Tc_2 &= \mu_1\displaystyle\int_0^T g_1(s,x(s))ds+\lambda_1\int_0^T \frac{(T-s)^{q-2}}{\Gamma(q-1)}y(s)ds \\ 2(\lambda_2-1)c_3 &= \mu_2\displaystyle\int_0^T g_2(s,x(s))ds+\lambda_2\int_0^T \frac{(T-s)^{q-3}}{\Gamma(q-2)}y(s)ds. \end{cases} \tag{4} $$

Solving the system (4), we find the values of c_0,c_1 and c_2. Substituting these values in (3), we obtain (2).

3 Main results

Let $\mathcal{C} = \mathrm{C}([0,T],\mathbb{R})$ denotes the Banach space of all continuous functions from $[0,T] \to \mathbb{R}$ endowed with the usual sup-norm ($\|x\| = \sup_{t\in[0,T]} |x(t)|$).

By Lemma 2.3, the problem (1) can be transformed to a fixed point problem as $x = F(x)$, where $F : \mathcal{C} \to \mathcal{C}$ is given by

$$\begin{aligned}(Fx)(t) &= \int_0^t \frac{(t-s)^{q-1}}{\Gamma(q)} f(s,x(s))ds - \lambda_0\xi_1 \int_0^T \frac{(T-s)^{q-1}}{\Gamma(q)} f(s,x(s))ds \\ &+\lambda_1\eta_2 \int_0^T \frac{(T-s)^{q-2}}{\Gamma(q-1)} f(s,x(s))ds + \lambda_2\eta_1 \int_0^T \frac{(T-s)^{q-3}}{\Gamma(q-2)} f(s,x(s))ds \\ &-\mu_0\xi_1 \int_0^T g_0(s,x(s))ds + \mu_1\eta_2 \int_0^T g_1(s,x(s))ds \\ &+\mu_2\eta_1 \int_0^T g_2(s,x(s))ds, \quad t\in[0,T].\end{aligned} \tag{5}$$

For the sake of computational convenience, we introduce

$$\Lambda_1 = \frac{T^q}{\Gamma(q+1)} \left\{1 + |\lambda_0\xi_1| + |\lambda_1\eta_2| q T^{-1} + |\lambda_2\eta_1| q(q-1)T^{-2}\right\}. \tag{6}$$

Our first existence result is based on Leray-Schauder nonlinear alternative.

Theorem 3.1 *(Nonlinear alternative for single valued maps)[24]. Let E be a Banach space, C a closed, convex subset of E, U an open subset of C and $0 \in U$. Suppose that $F : \overline{U} \to C$ is a continuous, compact (that is, $F(\overline{U})$ is a relatively compact subset of C) map. Then either*

(i) F has a fixed point in $\overline{U}$, or

(ii) there is a $u \in \partial U$ (the boundary of U in C) and $\lambda \in (0,1)$ with $u = \lambda F(u)$.

Theorem 3.2 *Assume that $f, g_j : [0,1]\times\mathbb{R} \to \mathbb{R}$ are continuous functions and the following conditions hold:*

(A_1) *there exist a function $p \in L^1([0,1],\mathbb{R}^+)$, and $\psi : \mathbb{R}^+ \to \mathbb{R}^+$ nondecreasing such that $|f(t,x)| \leq p(t)\psi(\|x\|)$ for each $(t,x) \in [0,T]\times\mathbb{R}$;*

(A_2) *there exist continuous nondecreasing functions $\psi_j : [0,\infty) \to (0,\infty)$ and functions $p_j \in L^1([0,T],\mathbb{R}^+)$ such that*

$$|g_j(t,x)| \leq p_j(t)\psi_j(\|x\|),\ j = 0,1,2, \text{ for each } (t,x) \in [0,T]\times\mathbb{R};$$

(A_3) *there exists a number $M > 0$ such that*

$$\frac{M}{\psi(M)\Omega_1\|p\|_{L^1} + \psi_0(M)|\mu_0\xi_1|\|p_0\|_{L^1} + \psi_1(M)|\mu_1\eta_2|\|p_1\|_{L^1} + \psi_2(M)|\mu_2\eta_1|\|p_2\|_{L^1}} > 1,$$

where

$$\Omega_1 = \frac{T^{q-1}}{\Gamma(q)} \left\{1 + |\lambda_0\xi_1| + |\lambda_1\eta_2|(q-1)T^{-1} + |\lambda_2\eta_1| q(q-1)(q-2)T^{-2}\right\}.$$

Then the boundary value problem (1) has at least one solution on $[0,1]$.

Proof. Consider the operator $F : \mathcal{C} \to \mathcal{C}$ defined by (5). It is easy to prove that F is continuous. Next, we show that F maps bounded sets into bounded sets in $C([0,T],\mathbb{R})$. For a positive number ρ, let $B_\rho = \{x \in C([0,T],\mathbb{R}) : \|x\| \le \rho\}$ be a bounded set in $C([0,T],\mathbb{R})$. Then, for each $x \in B_\rho$,we have

$$\begin{aligned}
|(Fx)(t)| &\le \int_0^t \frac{(t-s)^{q-1}}{\Gamma(q)}|f(s,x(s))|ds + |\lambda_0\xi_1|\int_0^T \frac{(T-s)^{q-1}}{\Gamma(q)}|f(s,x(s))|ds \\
&\quad +|\lambda_1\eta_2|\int_0^T \frac{(T-s)^{q-2}}{\Gamma(q-1)}|f(s,x(s))|ds + |\lambda_2\eta_1|\psi(\|x\|)\int_0^T \frac{(T-s)^{q-3}}{\Gamma(q-2)}p(s)|ds \\
&\quad +|\mu_0\xi_1|\int_0^T |g_0(s,x(s))|ds + |\mu_1\eta_2|\int_0^T |g_1(s,x(s))|ds \\
&\quad +|\mu_2\eta_1|\int_0^T |g_2(s,x(s))|ds \\
&\le \psi(\|x\|)\left\{\frac{T^{q-1}}{\Gamma(q)} + |\lambda_0\xi_1|\frac{T^{q-1}}{\Gamma(q)} + |\lambda_1\eta_2|\frac{T^{q-2}}{\Gamma(q-1)} + |\lambda_2\eta_1|\frac{T^{q-3}}{\Gamma(q-2)}\right\}\int_0^T p(s)ds \\
&\quad +\psi_0(\|x\|)|\mu_0\xi_1|\int_0^T p_0(s)ds + \psi_1(\|x\|)|\mu_1\eta_2|\int_0^T p_1(s)ds \\
&\quad +\psi_2(\|x\|)|\mu_2\eta_1|\int_0^T p_2(s)ds \\
&\le \psi(\|x\|)\Omega_1\|p\|_{L^1} + \psi_0(\|x\|)|\mu_0\xi_1|\|p_0\|_{L^1} + \psi_1(\|x\|)|\mu_1\eta_2|\|p_1\|_{L^1} \\
&\quad +\psi_2(\|x\|)|\mu_2\eta_1|\|p_2\|_{L^1}.
\end{aligned}$$

Thus,

$$\|Fx\| \le \psi(\rho)\Omega_1\|p\|_{L^1} + \psi_0(\rho)|\mu_0\xi_1|\|p_0\|_{L^1} + \psi_1(\rho)|\mu_1\eta_2|\|p_1\|_{L^1} + \psi_2(\rho)|\mu_2\eta_1|\|p_2\|_{L^1}.$$

Now we show that F maps bounded sets into equicontinuous sets of $C([0,T],\mathbb{R})$. Let $t', t'' \in [0,T]$ with $t' < t''$ and $x \in B_\rho$, where B_ρ is a bounded set of $C([0,T],\mathbb{R})$. Then we have

$$\begin{aligned}
&|(Fx)(t'') - (Fx)(t')| \\
\le\ & \left|\psi(\|x\|)\int_0^{t'}\left[\frac{(t''-s)^{q-1} - (t'-s)^{q-1}}{\Gamma(q)}\right]p(s)ds + \int_{t'}^{t''}\frac{(t''-s)^{q-1}}{\Gamma(q)}f(s,x(s))ds\right| \\
&+|(1-\lambda_0)\lambda_1\xi_2||t''-t'|\psi(\|x\|)\int_0^T \frac{(T-s)^{q-2}}{\Gamma(q-1)}p(s)ds + |\lambda_2\xi_3|\Big[2|(1-\lambda_0)\lambda_1|T|t''-t'| \\
&+|(1-\lambda_0)(1-\lambda_1)||t''^2 - t'^2|\Big]\psi(\|x\|)\int_0^T \frac{(T-s)^{q-3}}{\Gamma(q-2)}p(s)ds \\
&+|(1-\lambda_0)|\mu_1\lambda_1\xi_2||t''-t'|\psi_1(\|x\|)\int_0^T p_1(s)|ds \\
&+|\lambda_2\xi_3\mu_2|\Big[2|(1-\lambda_0)\lambda_1 T|t''-t'| + |(1-\lambda_0)(1-\lambda_1)||t''^2 - t'^2|\Big]\psi_2(\|x\|)\int_0^T p_2(s)ds.
\end{aligned}$$

Obviously the right hand side of the above inequality tends to zero independently of $x \in B_\rho$ as $t''-t' \to 0$. Therefore it follows by the Ascoli-Arzelá theorem that $F : C([0,T],\mathbb{R}) \to C([0,T],\mathbb{R})$ is completely continuous.

The result will follow from the Leray-Schauder nonlinear alternative (Theorem 3.1) once we have proved the boundedness of the set of all solutions to equations $x = \lambda Fx$ for $\lambda \in [0,1]$.

Let x be a solution. Then, for $t \in [0,T]$, and using the computations in proving that F is bounded, we have

$$|x(t)| \le \psi(\|x\|)\left\{\frac{T^{q-1}}{\Gamma(q)} + |\lambda_1\xi_1|\frac{T^{q-1}}{\Gamma(q)} + |\lambda_2\eta_2|\frac{T^{q-2}}{\Gamma(q-1)} + |\lambda_3\eta_1|\frac{T^{q-3}}{\Gamma(q-2)}\right\}\int_0^T p(s)ds$$

$$
\begin{aligned}
&+\psi_0(\|x\|)|\mu_1\xi_1|\int_0^T p_0(s)ds+\psi_1(\|x\|)|\mu_2\eta_2|\int_0^T p_1(s)ds\\
&+\psi_2(\|x\|)|\mu_3\eta_1|\int_0^T p_2(s)ds\\
\leq\ &\psi(\|x\|)\Omega_1\|p\|_{L^1}+\psi_0(\|x\|)|\mu_1\xi_1|\|p_0\|_{L^1}+\psi_1(\|x\|)|\mu_2\eta_2|\|p_1\|_{L^1}\\
&+\psi_2(\|x\|)|\mu_3\eta_1|\|p_2\|_{L^1}.
\end{aligned}
$$

Consequently, we have

$$
\frac{\|x\|}{\psi(\|x\|)\Omega_1\|p\|_{L^1}+\psi_0(\|x\|)|\mu_0\xi_1|\|p_0\|_{L^1}+\psi_1(\|x\|)|\mu_1\eta_2|\|p_1\|_{L^1}+\psi_2(\|x\|)|\mu_2\eta_1|\|p_2\|_{L^1}}\leq 1.
$$

In view of (A_3), there exists M such that $\|x\|\neq M$. Let us set

$$
U=\{x\in C([0,T],\mathbb{R}):\|x\|<M+1\}.
$$

Note that the operator $F:\overline{U}\to C([0,T],\mathbb{R})$ is continuous and completely continuous. From the choice of U, there is no $x\in\partial U$ such that $x=\lambda Fx$ for some $\lambda\in(0,1)$. Consequently, by the Leray-Schauder alternative (Theorem 3.1), we deduce that F has a fixed point $x\in\overline{U}$ which is a solution of the problem (1). □

Our next result is based on the celebrated fixed point theorem due to Banach.

Theorem 3.3 *Assume that $f,g_j:[0,T]\times\mathbb{R}\to\mathbb{R}$ are continuous functions satisfying the conditions:*

(A_4) $|f(t,x)-f(t,y)|\leq L|x-y|,\forall t\in[0,T],\ L>0,\ x,y\in\mathbb{R}$;

(A_5) $|g_j(t,x)-g_j(t,y)|\leq\mathcal{L}_j|x-y|,\forall t\in[0,T],\ \mathcal{L}_j>0,\ j=0,1,2,\ x,y\in\mathbb{R}$.

Then the boundary value problem (1) has a unique solution if

$$
L\Lambda_1+\Big\{\mathcal{L}_0|\mu_0\xi_1|+\mathcal{L}_1|\mu_1\eta_2|+\mathcal{L}_2|\mu_2\eta_1|\Big\}T<1,
$$

where Λ_1 is given by (6).

Proof. Let us fix $\sup_{t\in[0,T]}|f(t,0)|=M$, $\sup_{t\in[0,T]}|g_j(t,0)|=\mathcal{M}_j,\ j=0,1,2$ and choose

$$
r\geq\frac{M\Lambda_1+\Big\{\mathcal{M}_0|\mu_0\xi_1|+\mathcal{M}_1|\mu_1\eta_2|+\mathcal{M}_2|\mu_2\eta_1|\Big\}T}{1-\Big(L\Lambda_1+\Big\{\mathcal{L}_0|\mu_0\xi_1|+\mathcal{L}_1|\mu_1\eta_2|+\mathcal{L}_2|\mu_2\eta_1|\Big\}T\Big)}.
$$

Then we show that $FB_r\subset B_r$, where $B_r=\{x\in\mathcal{C}:\|x\|\leq r\}$. For $x\in B_r$, we have

$$
\begin{aligned}
|(Fx)(t)|\ \leq\ &\sup_{t\in[0,T]}\Bigg\{\int_0^t\frac{(t-s)^{q-1}}{\Gamma(q)}|f(s,x(s))|ds+|\lambda_0\xi_1|\int_0^T\frac{(T-s)^{q-1}}{\Gamma(q)}|f(s,x(s))|ds\\
&+|\lambda_1\eta_2|\int_0^T\frac{(T-s)^{q-2}}{\Gamma(q-1)}|f(s,x(s))|ds+|\lambda_2\eta_1|\int_0^T\frac{(T-s)^{q-3}}{\Gamma(q-2)}|f(s,x(s))|ds\\
&+|\mu_0\xi_1|\int_0^T|g_0(s,x(s))|ds+|\mu_1\eta_2|\int_0^T|g_1(s,x(s))|ds\\
&+|\mu_2\eta_1|\int_0^T|g_2(s,x(s))|ds\Bigg\}\\
\leq\ &\sup_{t\in[0,T]}\Bigg\{\int_0^t\frac{(t-s)^{q-1}}{\Gamma(q)}\Big[|f(s,x(s))-f(s,0)|+|f(s,0)|\Big]ds
\end{aligned}
$$

$$
\begin{aligned}
&+|\lambda_0\xi_1|\int_0^T \frac{(T-s)^{q-1}}{\Gamma(q)}\Big[|f(s,x(s))-f(s,0)|+|f(s,0)ds|\Big]ds\\
&+|\lambda_1\eta_2|\int_0^T \frac{(T-s)^{q-2}}{\Gamma(q-1)}\Big[|f(s,x(s))-f(s,0)|+|f(s,0)ds|\Big]ds\\
&+|\lambda_2\eta_1|\int_0^T \frac{(T-s)^{q-3}}{\Gamma(q-2)}\Big[|f(s,x(s))-f(s,0)|+|f(s,0)|\Big]ds\\
&+|\mu_0\xi_1|\int_0^T \Big[|g_0(s,x(s))-g_0(s,0)|+|g_0(s,0)|\Big]ds\\
&+|\mu_1\eta_2|\int_0^T \Big[|g_1(s,x(s))-g_1(s,0)|+|g_1(s,0)|\Big]ds\\
&+|\mu_2\eta_1|\int_0^T \Big[|g_2(s,x(s))-g_2(s,0)|+|g_2(s,0)|\Big]ds\Bigg\}\\
\le\ & (Lr+M)\frac{T^q}{\Gamma(q+1)}\left\{1+|\lambda_1\xi_1|+|\lambda_2\eta_2|qT^{-1}+|\lambda_3\eta_1|q(q-1)T^{-2}\right\}\\
&+(\mathcal{L}_0 r+\mathcal{M}_0)|\mu_0\xi_1|T+(\mathcal{L}_1 r+\mathcal{M}_1)|\mu_1\eta_2|T+(\mathcal{L}_2 r+\mathcal{M}_2)|\mu_2\eta_1|T\\
=\ & (L\Lambda_1+\mathcal{L}_0|\mu_0\xi_1|T+\mathcal{L}_1|\mu_1\eta_2|T+\mathcal{L}_2|\mu_2\eta_1|T)r\\
&+(M\Lambda_1+\mathcal{M}_0|\mu_0\xi_1|T+\mathcal{M}_1|\mu_1\eta_2|T+\mathcal{M}_2|\mu_2\eta_1|T)\le r.
\end{aligned}
$$

Now, for $x, y \in \mathcal{C}$ and for each $t \in [0,T]$, we obtain

$$
\begin{aligned}
\|(Fx)(t)-(Fy)(t)\| \le\ & \sup_{t\in[0,T]}\left\{\int_0^t \frac{(t-s)^2}{2}|f(s,x(s))-f(s,y(s))|ds\right.\\
&+|\lambda_0\xi_1|\int_0^T \frac{(T-s)^{q-1}}{\Gamma(q)}|f(s,x(s))-f(s,y(s))|ds\\
&|\lambda_1\eta_2|\int_0^T \frac{(T-s)^{q-2}}{\Gamma(q-1)}|f(s,x(s))-f(s,y(s))|ds\\
&+|\lambda_2\eta_1|\int_0^T \frac{(T-s)^{q-3}}{\Gamma(q-2)}|f(s,x(s))-f(s,y(s))|ds\\
&+|\mu_0\xi_1|\int_0^T |g_0(s,x(s))-g_0(s,y(s))|ds\\
&+|\mu_1\eta_2|\int_0^T |g_1(s,x(s))-g_1(s,y(s))|ds\\
&\left.+|\mu_2\eta_1|\int_0^T |g_2(s,x(s))-g_2(s,y(s))|ds\right\}\\
\le\ & \|x-y\|\frac{LT^q}{\Gamma(q+1)}\left\{1+|\lambda_1\xi_1|+|\lambda_2\eta_2|qT^{-1}+|\lambda_3\eta_1|q(q-1)T^{-2}\right\}\\
&+\mathcal{L}_0\|x-y\||\mu_0\xi_1|+\mathcal{L}_1\|x-y\||\mu_1\eta_2|T+\mathcal{L}_2|\mu_2\eta_1|T\|x-y\|\\
=\ & (L\Lambda_1+\mathcal{L}_0|\mu_0\xi_1|T+\mathcal{L}_1|\mu_1\eta_2|T+\mathcal{L}_2|\mu_2\eta_1|T)\|x-y\|.
\end{aligned}
$$

As $L\Lambda_1+(\mathcal{L}_0|\mu_0\xi_1|+\mathcal{L}_1|\mu_1\eta_2|+\mathcal{L}_2|\mu_2\eta_1|)T<1$, therefore F is a contraction. Thus, the conclusion of the theorem follows by the contraction mapping principle (Banach fixed point theorem). □

Our final existence result is based on Krasnoselskii's fixed point theorem [25].

Lemma 3.4 *(Krasnoselskii's fixed point theorem) [25]. Let M be a closed bounded, convex and nonempty subset of a Banach space X. Let A, B be the operators such that (i) $Ax+By \in M$ whenever $x, y \in M$; (ii) A is compact and continuous and (iii) B is a contraction mapping. Then there exists $z \in M$ such that $z = Az + Bz$.*

Theorem 3.5 *Let $f, g_j : [0,T]\times\mathbb{R} \to \mathbb{R}$ be continuous functions satisfying the assumptions $(A_4)-(A_5)$. In addition we suppose that*

(A_6) $|f(t,x)| \le \nu(t), \quad \forall (t,x) \in [0,T]\times\mathbb{R},$ *and* $\nu \in C([0,T],\mathbb{R}^+)$;

(A_7) $|g_j(t,x)| \le \nu_j(t),\ j=0,1,2,\ \forall (t,x) \in [0,T]\times\mathbb{R},$ *and* $\nu_j \in C([0,T],\mathbb{R}^+)$.

If

$$\frac{L(\Lambda_1\Gamma(q+1)-T^q)}{\Gamma(q+1)} + \Big(\mathcal{L}_0|\mu_0\xi_1| + \mathcal{L}_1|\mu_1\eta_2| + \mathcal{L}_2|\mu_2\eta_1|\Big)T < 1, \tag{7}$$

then problem (1) has at least one solution on $[0,T]$.

Proof. Letting $\sup_{t\in[0,T]}|\nu(t)| = \|\nu\|$, $\sup_{t\in[0,T]}|\nu_j(t)| = \|\nu_j\|,\ j=0,1,2,$ we fix

$$\overline{r} \ge \Lambda_1\|\nu\| + (|\mu_0\xi_1|\|\nu_0\| + |\mu_1\eta_2|\|\nu_1\| + |\mu_2\eta_1|\|\nu_2\|)T$$

and consider $B_{\overline{r}} = \{x \in \mathcal{C} : \|x\| \le \overline{r}\}$. We define the operators $\mathcal{P}$ and $\mathcal{Q}$ on $B_{\overline{r}}$ as

$$\begin{aligned}
(\mathcal{P}x)(t) &= \int_0^t \frac{(t-s)^{q-1}}{\Gamma(q)} f(s,x(s))ds,\\
(\mathcal{Q}x)(t) &= -\lambda_0\xi_1\int_0^T \frac{(T-s)^{q-1}}{\Gamma(q)} f(s,x(s))ds + \lambda_1\eta_2\int_0^T \frac{(T-s)^{q-2}}{\Gamma(q-1)} f(s,x(s))ds\\
&\quad +\lambda_2\eta_1\int_0^T \frac{(T-s)^{q-3}}{\Gamma(q-2)} f(s,x(s))ds - \mu_0\xi_1\int_0^T g_0(s,x(s))ds\\
&\quad +\mu_1\eta_2\int_0^T g_1(s,x(s))ds + \mu_2\eta_1\int_0^T g_2(s,x(s))ds, \quad t\in[0,T].
\end{aligned}$$

For $x, y \in B_{\overline{r}}$, we find that

$$\|\mathcal{P}x + \mathcal{Q}y\| \le \Lambda_1\|\nu\| + \Big(|\mu_0\xi_1|\|\nu_0\| + |\mu_1\eta_2|\|\nu_1\| + |\mu_2\eta_1|\|\nu_2\|\Big)T \le \overline{r}.$$

Thus, $\mathcal{P}x + \mathcal{Q}y \in B_{\overline{r}}$. It follows from the assumption (A_4) together with (7) that $\mathcal{Q}$ is a contraction mapping. Continuity of f implies that the operator $\mathcal{P}$ is continuous. Also, $\mathcal{P}$ is uniformly bounded on $B_{\overline{r}}$ as

$$\|\mathcal{P}x\| \le \frac{T^q}{\Gamma(q+1)}\|\mu\|.$$

Now we prove the compactness of the operator $\mathcal{P}$.

We define $\sup_{(t,x)\in[0,T]\times B_{\overline{r}}}|f(t,x)| = f_s < \infty$, and consequently, for $t_1, t_2 \in [0,T]$ with $t_2 < t_1$, we have

$$|(\mathcal{P}x)(t_2) - (\mathcal{P}x)(t_1)| \le \frac{f_s}{\Gamma(q)}\left|\int_0^{t_1}[(t_2-s)^{q-1} - (t_1-s)^{q-1}]ds + \int_{t_1}^{t_2}(t_2-s)^{q-1}ds\right|,$$

which is independent of x. Thus, $\mathcal{P}$ is equicontinuous. So $\mathcal{P}$ is relatively compact on $B_{\overline{r}}$. Hence, by the Arzelá-Ascoli Theorem, $\mathcal{P}$ is compact on $B_{\overline{r}}$. Thus all the assumptions of Lemma 3.4 are satisfied. So the conclusion of Lemma 3.4 implies that the boundary value problem (1) has at least one solution on $[0,T]$. $\square$

Example 3.6 *Consider the following boundary value problem*

$$\begin{cases} {}^cD^{5/2}x(t) = L(\cos t + \tan^{-1} x(t)), \quad t \in [0,1], \\ x(0) + x(1) = \displaystyle\int_0^1 \frac{x(s)}{(1+s)^2} ds, \\ x'(0) + x'(1) = \displaystyle\frac{1}{2}\int_0^1 \Big(\frac{e^s x(s)}{1+2e^s} + \frac{1}{2}\Big) ds, \\ x''(0) + x''(1) = \displaystyle\frac{1}{3}\int_0^1 \Big(\frac{x(s)}{1+e^s} + \frac{3}{4}\Big) ds, \end{cases} \tag{8}$$

where

$$f(t,x) = L(\cos t + \tan^{-1} x(t)),\ g_0(t,x) = \frac{x(t)}{(1+t)^2},\ g_1(t,x) = \frac{e^t x(t)}{1+2e^t} + \frac{1}{2},\ g_2(t,x) = \frac{x(t)}{1+e^t} + \frac{3}{4}$$

(L to be fixed later), and $\lambda_1 = \lambda_2 = \lambda_3 = -1,\ \mu_1 = 1,\ \mu_2 = \frac{1}{2},\ \mu_3 = \frac{1}{3}$.

Clearly, $\xi_1 = -1/2,\ \xi_2 = 1/4,\ \xi_3 = -1/16,\ \eta_1 = 1/16,\ \eta_2 = 1/4,$

$$|f(t,x) - f(t,y)| \le L|x-y|,\ |g_0(t,x) - g_0(t,y)| \le |x-y|,\ |g_1(t,x) - g_1(t,y)| \le \frac{1}{3}|x-y|,$$
$$|g_2(t,x) - g_2(t,y)| \le \frac{1}{2}|x-y|, \mathcal{L}_0 = 1,\ \mathcal{L}_1 = \frac{1}{3}, \mathcal{L}_2 = \frac{1}{2}.$$

$$\Lambda_1 = \frac{T^q}{\Gamma(q+1)}\left\{1 + |\lambda_1\xi_1| + |\lambda_2\eta_2|qT^{-1} + |\lambda_3\eta_1|q(q-1)T^{-2}\right\} = \frac{151}{120\sqrt{\pi}},$$

and

$$L\Lambda_1 + \Big\{\mathcal{L}_0|\mu_1\xi_1| + \mathcal{L}_1|\mu_2\eta_2| + \mathcal{L}_2|\mu_3\eta_1|\Big\} < 1$$

implies that $L < \frac{215\sqrt{\pi}}{604}$. Thus, all the conditions of Theorem 3.3 are satisfied. So there exists at least one solution of the problem (8) on $[0,1]$.

Remark 3.7 *The existence results for a third-order nonlinear boundary vale problem of ordinary differential equations with anti-periodic type integral boundary conditions follow as a special case if we take* $q = 3$ *in the results of this paper. We emphasize that these results are new.*

Acknowledgment. The research of B. Ahmad was partially supported by Deanship of Scientific Research (DSR), King Abdulaziz University, Jeddah, Saudi Arabia.

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Coupled common fixed point theorems for weakly increasing mappings with two variables*

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Abstract

In this paper, we introduce a notion of weakly increasing mappings with two variables. Several coupled common fixed point theorems for weakly increasing mappings in ordered metric spaces are established. Then, by using a scalarization method, we obtain two coupled common fixed point theorems in ordered cone metric spaces, which extend some earlier results about weakly increasing mappings with one variable.

Keywords: Weakly increasing mapping; fixed point; coupled common fixed point; ordered metric space; ordered cone metric space.

Mathematics Subject Classification: 47H10, 54H25.

1 Introduction

Recently, there is a large literature about fixed point theorems in cone metric spaces and ordered cone metric spaces, and such problems have attracted more and more authors. We refer the reader to [1–12] and references therein for some of recent developments on such topics. Especially, Altun et al. [3] introduced the notion of weakly increasing mapping, and obtained the following result:

Theorem 1.1. *Let $(X, \sqsubseteq, d)$ be an ordered complete cone metric space, and (f, g) be a weakly increasing pair of self-maps on X w.r.t. $\sqsubseteq$. Suppose that the following conditions hold:*

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*The work was supported by the NSF of China (11101192), the Key Project of Chinese Ministry of Education (211090), the NSF of Jiangxi Province of China (20114BAB211002), and the Young Talents Programme of Jiangxi Normal University.

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(i) there exist $\alpha, \beta, \gamma \geq 0$ such that $\alpha + 2\beta + 2\gamma < 1$ and

$$d(fx, gy) \preceq \alpha d(x, y) + \beta[d(x, fx) + d(y, gy)] + \gamma[d(x, gy) + d(y, fx)]$$

for all comparable $x, y \in X$;

(ii) f or g is continuous, or

(ii′) if an nondecreasing sequence $\{x_n\}$ converges to x in X, then $x_n \sqsubseteq x$ for all $n \in \mathbb{N}$.

Then f and g have a common fixed point $x^* \in X$.

In fact, Theorem 1.1 can be seen as an “ordered” variant of a result of Abbas and Rhoades [2]. Very recently, Kadelburg et al. [8] generalized Theorem 1.1 and obtained the following theorem:

Theorem 1.2. *Let $(X, \sqsubseteq, d)$ be an ordered complete cone metric space, and (f, g) be a weakly increasing pair of self-maps on X w.r.t. $\sqsubseteq$. Suppose that the following conditions hold:*

(i) there exist $p, q, r, s, t \geq 0$ such that $p+q+r+s+t < 1$ and $q = r$ or $s = t$, such that

$$d(fx, gy) \preceq pd(x, y) + qd(x, fx) + rd(y, gy) + sd(x, gy) + td(y, fx)$$

for all comparable $x, y \in X$;

(ii) f or g is continuous, or

(ii′) if an nondecreasing sequence $\{x_n\}$ converges to x in X, then $x_n \sqsubseteq x$ for all $n \in \mathbb{N}$.

Then f and g have a common fixed point $x^* \in X$.

The aim of this paper is to make further studies on such problems, and to extend the results in [3, 8]. Inspired by [3, Definition 14], we introduce the following concept of weakly increasing mappings with two variables:

Definition 1.3. *Let $(X, \sqsubseteq)$ be a partially ordered set. Two mappings $F, G : X \times X \to X$ are said to be weakly increasing if*

$$F(x, y) \sqsubseteq G(F(x, y), F(y, x)), \quad G(x, y) \sqsubseteq F(G(x, y), G(y, x))$$

hold for all $(x, y) \in X \times X$.

Example 1.4. *Let $X = [1, \infty)$, endowed with the usual ordering $\leq$. Let $F, G : X \times X \to X$ be defined by $F(x, y) = x + 2y$, $G(x, y) = xy^2$. Then, for all $(x, y) \in X \times X$,*

$$F(x, y) = x + 2y \leq G(F(x, y), F(y, x)) = G(x + 2y, y + 2x) = (x + 2y)(y + 2x)^2$$

and

$$G(x, y) = xy^2 \leq F(G(x, y), G(y, x)) = F(xy^2, x^2y) = xy^2 + 2x^2y.$$

Thus, F and G are two weakly increasing mappings.

2 Main results in ordered metric spaces

Throughout this section, we denote by $(X, \sqsubseteq, d)$ an ordered metric space, i.e., $\sqsubseteq$ is a partial order on the set X, and d is a metric on X. In addition, we call that $(x, y), (u, v) \in X \times X$ are comparable if $x \sqsubseteq u$ and $y \sqsubseteq v$ or $u \sqsubseteq x$ and $v \sqsubseteq y$. We will prove several coupled common fixed point theorems for two weakly increasing mappings.

Theorem 2.1. *Let $(X, \sqsubseteq, d)$ be a complete ordered metric space, and $F, G : X \times X \to X$ be two weakly increasing mappings w.r.t. $\sqsubseteq$. Suppose that the following assumptions hold:*

(i) there exists $\lambda \in [0, \frac{1}{2})$ such that

$$d(F(x, y),\ G(u, v)) \leq \lambda \cdot z$$

for all comparable (x, y), $(u, v) \in X \times X$, where

$$\begin{aligned} z \ = \ & \max\{d(x, u), d(y, v), d(x, F(x, y)), d(y, F(y, x)), d(u, G(u, v)), d(v, G(v, u)), \\ & d(x, G(u, v)), d(y, G(v, u)), d(u, F(x, y)), d(v, F(y, x))\}; \end{aligned}$$

(ii) F or G is continuous, or X has the following property:

(P) if an nondecreasing sequence $\{x_n\}$ converges to x in X, then $x_n \sqsubseteq x$ for all $n \in \mathbb{N}$.

Then F and G has a coupled common fixed point, i.e., there exists $(x^*, y^*) \in X \times X$ such that

$$F(x^*, y^*) = G(x^*, y^*) = x^*$$

and

$$F(y^*, x^*) = G(y^*, x^*) = y^*.$$

Proof. Take $x_0,\ y_0 \in X$. Define two sequences $\{x_n\}$, $\{y_n\}$ in X as follows: $x_{2n+1} = F(x_{2n}, y_{2n})$, $x_{2n+2} = G(x_{2n+1}, y_{2n+1})$, $y_{2n+1} = F(y_{2n}, x_{2n})$ and $y_{2n+2} = G(y_{2n+1}, x_{2n+1})$ for all $n \geq 0$.

Since F and G are weakly increasing, we have

$$\begin{aligned}
x_1 &= F(x_0, y_0) \sqsubseteq G(F(x_0, y_0), F(y_0,\ x_0)) = G(x_1, y_1)\\
&= x_2 \sqsubseteq F(G(x_1, y_1),\ G(y_1, x_1)) = F(x_2, y_2) = x_3 \sqsubseteq \cdots,\\
y_1 &= F(y_0, x_0) \sqsubseteq G(F(y_0, x_0), F(x_0, y_0)) = G(y_1, x_1)\\
&= y_2 \sqsubseteq F(G(y_1, x_1), G(x_1, y_1)) = F(y_2,\ x_2) = y_3 \sqsubseteq \cdots.
\end{aligned}$$

So the sequences $\{x_n\}$, $\{y_n\}$ are nondecreasing.

Since

$$(x_{2n+1}, x_{2n+2}) = (F(x_{2n}, y_{2n}), G(x_{2n+1}, y_{2n+1})), \quad (y_{2n+1}, y_{2n+2}) = (F(y_{2n}, x_{2n}), G(y_{2n+1}, x_{2n+1})),$$

by the condition (i), we have

$$\max\{d(x_{2n+1}, x_{2n+2}), d(y_{2n+1}, y_{2n+2})\} \leq \lambda z, \tag{2.1}$$

where

$$z = \max\{d(x_{2n}, x_{2n+1}), d(y_{2n}, y_{2n+1}), d(x_{2n+1}, x_{2n+2}), d(y_{2n+1}, y_{2n+2}), d(x_{2n}, x_{2n+2}), d(y_{2n}, y_{2n+2})\}.$$

Now, we consider the following three cases:

$1°$ if $z = \max\{d(x_{2n}, x_{2n+1}), d(y_{2n}, y_{2n+1})\}$, then

$$\max\{d(x_{2n+1}, x_{2n+2}), d(y_{2n+1}, y_{2n+2})\} \leq \lambda z \leq \frac{\lambda}{1-\lambda} \max\{d(x_{2n}, x_{2n+1}), d(y_{2n}, y_{2n+1})\};$$

$2°$ if $z = \max\{d(x_{2n+1}, x_{2n+2}), d(y_{2n+1}, y_{2n+2})\}$, then by (2.1), we have

$$\max\{d(x_{2n+1}, x_{2n+2}), d(y_{2n+1}, y_{2n+2})\} = 0 \leq \frac{\lambda}{1-\lambda} \max\{d(x_{2n}, x_{2n+1}), d(y_{2n}, y_{2n+1})\};$$

$3°$ if $z = \max\{d(x_{2n}, x_{2n+2}), d(y_{2n}, y_{2n+2})\}$, it follows from (2.1) that

$$\begin{aligned}
&\max\{d(x_{2n+1}, x_{2n+2}), d(y_{2n+1}, y_{2n+2})\} \leq \lambda z\\
\leq\ &\lambda[\max\{d(x_{2n}, x_{2n+1}), d(y_{2n}, y_{2n+1})\} + \max\{d(x_{2n+1}, x_{2n+2}), d(y_{2n+1}, y_{2n+2})\}],
\end{aligned}$$

which means that

$$\max\{d(x_{2n+1}, x_{2n+2}), d(y_{2n+1}, y_{2n+2})\} \leq \frac{\lambda}{1-\lambda} \max\{d(x_{2n}, x_{2n+1}), d(y_{2n}, y_{2n+1})\}.$$

Thus, in all cases, we have

$$\max\{d(x_{2n+1}, x_{2n+2}), d(y_{2n+1}, y_{2n+2})\} \le \frac{\lambda}{1-\lambda} \max\{d(x_{2n}, x_{2n+1}), d(y_{2n}, y_{2n+1})\}.$$

By a similar proof, one can also show that

$$\max\{d(x_{2n}, x_{2n+1}), d(y_{2n}, y_{2n+1})\} \le \frac{\lambda}{1-\lambda} \max\{d(x_{2n-1}, x_{2n}), d(y_{2n-1}, y_{2n})\}.$$

So we get

$$\max\{d(x_n, x_{n+1}), d(y_n, y_{n+1})\} \le \frac{\lambda}{1-\lambda} \max\{d(x_{n-1}, x_n), d(y_{n-1}, y_n)\}.$$

Since $\lambda \in [0, \frac{1}{2})$, $0 \le \frac{\lambda}{1-\lambda} < 1$. Then, by a standard proof, one can conclude that $\{x_n\}, \{y_n\}$ are Cauchy sequences. Thus, there exist $x^*,\ y^* \in X$ such that $x_n \to x^*$ and $y_n \to y^*$ as $n \to \infty$.

In order to show that x^*, y^* is a coupled common fixed point of F and G, we consider the following three cases:

Case I. F is continuous.

Obviously, $x^* = F(x^*, y^*)$ and $y^* = F(y^*, x^*)$. Noticing that

$$d(x^*, G(x^*, y^*)) = d(F(x^*, y^*), G(x^*, y^*)), \quad d(y^*, G(y^*, x^*)) = d(F(y^*, x^*), G(y^*, x^*)),$$

by (i'), we obtain

$$\max\{d(x^*, G(x^*, y^*)), d(y^*, G(y^*, x^*))\} \le \lambda \max\{d(x^*, G(x^*, y^*)), d(y^*, G(y^*, x^*))\},$$

which yields that

$$x^* = G(x^*, y^*),\ y^* = G(y^*, x^*).$$

Case II. G is continuous.

The proof is similar to that of Case I.

Case III. X has the property (P).

In view of $x_n \sqsubseteq x^*$ and $y_n \sqsubseteq y^*$ for all $n \in \mathbb{N}$, one can use (i') to obtain the following:

$$\max\{d(F(x^*, y^*), x_{2n+2}), d(F(y^*, x^*), y_{2n+2})\} \le \lambda z^*,$$

where

$$\begin{aligned} z^* \quad = \quad & \max\{d(x^*, x_{2n+1}), d(y^*, y_{2n+1}), d(x^*, F(x^*, y^*)), d(y^*, F(y^*, x^*)), d(x_{2n+1}, x_{2n+2}), \\ & d(y_{2n+1}, y_{2n+2}), d(x^*, x_{2n+2}), d(y^*, y_{2n+2}), d(x_{2n+1}, F(x^*, y^*)), d(y_{2n+1}, F(y^*, x^*))\}. \end{aligned}$$

Letting $n \to \infty$, we get

$$\max\{d(x^*, F(x^*, y^*)), d(y^*, F(y^*, x^*))\} \le \lambda \max\{d(x^*, F(x^*, y^*)), d(y^*, F(y^*, x^*))\}.$$

Thus $x^* = F(x^*, y^*)$ and $y^* = F(y^*, x^*)$. Analogously to the above proof, one can also obtain that $x^* = G(x^*, y^*)$ and $y^* = G(y^*, x^*)$. □

Theorem 2.2. *Suppose that all the assumptions of Theorem 2.1 except for (i) are satisfied, and the following assumption holds:*

(i') there exists $\lambda \in [0, 1)$ such that

$$d(F(x, y),\ G(u, v)) \le \lambda \cdot w$$

for all comparable (x, y), $(u, v) \in X \times X$, where

$$\begin{aligned} w \;=\; & \max\{d(x, u), d(y, v), d(x, F(x, y)), d(y, F(y, x)), d(u, G(u, v)), d(v, G(v, u)), \\ & \frac{d(x, G(u, v)) + d(u, F(x, y))}{2}, \frac{d(y, G(v, u)) + d(v, F(y, x))}{2}\}. \end{aligned}$$

Then, the conclusion of Theorem 2.1 also holds.

Proof. Let $\{x_n\}, \{y_n\}$ be as in the proof of Theorem 2.1. By using (i') and the construction of $\{x_n\}, \{y_n\}$, one can conclude

$$\max\{d(x_{2n+1}, x_{2n+2}), d(y_{2n+1}, y_{2n+2})\} \le \lambda w, \tag{2.2}$$

where

$$w = \max\{d(x_{2n}, x_{2n+1}), d(y_{2n}, y_{2n+1}), d(x_{2n+1}, x_{2n+2}), d(y_{2n+1}, y_{2n+2}), \frac{d(x_{2n}, x_{2n+2})}{2}, \frac{d(y_{2n}, y_{2n+2})}{2}\}.$$

Noting that

$$\frac{d(x_{2n}, x_{2n+2})}{2} \le \max\{d(x_{2n}, x_{2n+1}), d(x_{2n+1}, x_{2n+2})\}$$

and

$$\frac{d(y_{2n}, y_{2n+2})}{2} \le \max\{d(y_{2n}, y_{2n+1}), d(y_{2n+1}, y_{2n+2})\},$$

it follows that

$$w = \max\{d(x_{2n}, x_{2n+1}), d(y_{2n}, y_{2n+1}), d(x_{2n+1}, x_{2n+2}), d(y_{2n+1}, y_{2n+2})\}.$$

We also note that if $w = \max\{d(x_{2n+1}, x_{2n+2}), d(y_{2n+1}, y_{2n+2})\}$, then (2.2) yields $w = 0$, and thus $w = \max\{d(x_{2n}, x_{2n+1}), d(y_{2n}, y_{2n+1})\}$. So we conclude

$$w = \max\{d(x_{2n}, x_{2n+1}), d(y_{2n}, y_{2n+1})\}.$$

Then, (2.2) equals to

$$\max\{d(x_{2n+1}, x_{2n+2}), d(y_{2n+1}, y_{2n+2})\} \leq \lambda \max\{d(x_{2n}, x_{2n+1}), d(y_{2n}, y_{2n+1})\}.$$

Similarly, one can also obtain

$$\max\{d(x_{2n}, x_{2n+1}), d(y_{2n}, y_{2n+1})\} \leq \lambda \max\{d(x_{2n-1}, x_{2n}), d(y_{2n-1}, y_{2n})\}.$$

So we get

$$\max\{d(x_n, x_{n+1}), d(y_n, y_{n+1})\} \leq \lambda \max\{d(x_{n-1}, x_n), d(y_{n-1}, y_n)\}.$$

Then, by a standard proof, one can conclude that $\{x_n\}, \{y_n\}$ are Cauchy sequences. Thus, there exist $x^*,\ y^* \in X$ such that $x_n \to x^*$ and $y_n \to y^*$ as $n \to \infty$. The remaining proof is similar to that of Theorem 2.1. So we omit the details.

□

Example 2.3. Let $X = \{1, 2\}$, $\sqsubseteq= \{(1, 1), (2, 2)\}$, $d(x, y) = |x-y|$, and $F = G : X \times X \to X$ defined by

$$F(1, 2) = F(1, 1) = 1,\ F(2, 1) = F(2, 2) = 2.$$

It is easy to verify that all the assumptions of Theorem 2.1-2.2 are satisfied. So F has a coupled fixed point. In fact, $(1, 2)$ is obviously a coupled fixed point of F.

3 Ordered cone metric space cases

In this section, we suppose that E is a Banach space, P is a convex cone in E with $intP \neq \emptyset$, $\preceq$ is the partial ordering induced by P, $e \in intP$, and $\xi_e : E \to \mathbb{R}$ is defined by

$$\xi_e(y) = \inf\{r \in \mathbb{R} : y \in re - P\}, \quad y \in E.$$

First, let us recall some definitions about cone metric space. For more details, we refer the reader to [1–12]. and references therein.

Definition 3.1. *Let X be a nonempty set and P be a cone in a Banach space E. Suppose that a mapping $d : X \times X \to E$ satisfies:*

(i) $\theta \preceq \rho(x, y)$ for all $x, y \in X$ and $\rho(x, y) = \theta$ if and only if $x = y$,where θ is the zero element of P;

(ii) $\rho(x, y) = \rho(y, x)$ *for all* $x, y \in X$*;*

(iii) $\rho(x, y) \preceq \rho(x, z) + \rho(z, y)$ *for all* $x, y, z \in X$*.*

Then ρ *is called a cone metric on* X *and* (X, ρ) *is called a cone metric space.*

Definition 3.2. *Let* (X, ρ) *be a cone metric space, and* $\{x_n\}, \{y_n\}$ *be sequences in* X*.*

(i) Let $x \in X$*. If* $\forall c \gg \theta$*, there exists* $N \in \mathbb{N}$ *such that for all* $n > N$*,* $\rho(x_n, x) \ll c$*, then we call that* $\{x_n\}$ *converges to* x*, and we denote it by* $\lim_{n\to\infty} x_n = x$ *or* $x_n \to x$*,* $n \to \infty$*.*

(ii) If $\forall c \gg \theta$*, there exists* $N \in \mathbb{N}$ *such that for all* $n, m > N$*,* $\rho(x_n, x_m) \ll c$*, then* $\{x_n\}$ *is called a Cauchy sequence in* X*.*

(iii) (X, ρ) *is called complete if every Cauchy sequence in* (X, ρ) *is convergent.*

(iv) A mapping $F : X \times X \to X$ *is called continuous if* $x_n \to x$ *and* $y_n \to y$ *imply that* $F(x_n, y_n) \to F(x, y)$ *as* $n \to \infty$*.*

Next, let us recall some properties about the scalarization function ξ_e.

Theorem 3.3. *The following statements are true:*

(a) $\xi_e(\cdot)$ *is positively homogeneous and continuous on* E*;*

(b) $y, z \in E$ *with* $y \preceq z$ *implies* $\xi_e(y) \le \xi_e(z)$*;*

(c) $\xi_e(y + z) \le \xi_e(y) + \xi_e(z)$ *for all* $y, z \in E$*;*

(d) if (X, ρ) *is a complete cone metric space, then* $(X, \xi_e \circ \rho)$ *is a complete metric space;*

(e) $x_n \to x$ *in* $(X, \rho) \Longleftrightarrow x_n \to x$ *in* $(X, \xi_e \circ \rho)$*, as* $n \to \infty$*.*

Proof. (a)-(b) has been prove in [7]. (e) can be seen from the proof of [7, Theorem 2.2]. □

Now, by using the scalarization function ξ_e, one can deduce many results on cone metric spaces from our theorems in Section 2. For example, we have the following theorem:

Theorem 3.4. *Let* $(X, \sqsubseteq, \rho)$ *be an ordered complete cone metric space, i.e.,* $\sqsubseteq$ *is a partial order on the set* X*, and* ρ *is a complete cone metric on* X *with the underlying cone* P*. Suppose that* $F, G : X \times X \to X$ *are two weakly increasing mappings w.r.t.* $\sqsubseteq$ *satisfying the following assumptions:*

(H1) *there exists* $\lambda \in [0, \frac{1}{2})$ *and*

$$\begin{aligned} z \ \in \ & \{\rho(x,u), \rho(y,v), \rho(x,F(x,y)), \rho(y,F(y,x)), \rho(u,G(u,v)), \rho(v,G(v,u)), \\ & \rho(x,G(u,v)), \rho(y,G(v,u)), \rho(u,F(x,y)), \rho(v,F(y,x))\} \end{aligned}$$

such that

$$d(F(x,y),\ G(u,v)) \preceq \lambda \cdot z$$

for all comparable (x,y), $(u,v) \in X \times X$;

(H2) F *or* G *is continuous, or* X *has the following property:*

(P) *if an nondecreasing sequence* $\{x_n\}$ *converges to* x *in* X, *then* $x_n \sqsubseteq x$ *for all* $n \in \mathbb{N}$.

Then F *and* G *has a coupled common fixed point.*

Proof. Let $d = \xi_e \circ \rho$. By (d) of Theorem 3.3, (X,d) is a complete metric space. Moreover, by (H1) and (a)-(c) of Theorem 3.3, one can show that (i) of Theorem 2.1 holds. In addition, by (H2) and (e) of Theorem 3.3, we know that (ii) of Theorem 2.1 holds. So Theorem 2.1 yields the conclusion. □

Theorem 3.5. *Suppose that all the assumptions of Theorem 3.4 except for (H1) are satisfied, and the following assumption holds:*

(H1') *there exists* $\lambda \in [0,1)$ *and*

$$\begin{aligned} z \ \in \ & \{\rho(x,u), \rho(y,v), \rho(x,F(x,y)), \rho(y,F(y,x)), \rho(u,G(u,v)), \rho(v,G(v,u)), \\ & \frac{\rho(x,G(u,v)) + \rho(u,F(x,y))}{2}, \frac{\rho(y,G(v,u)) + \rho(v,F(y,x))}{2}\} \end{aligned}$$

such that

$$d(F(x,y),\ G(u,v)) \preceq \lambda \cdot z$$

for all comparable (x,y), $(u,v) \in X \times X$.

Then F *and* G *has a coupled common fixed point.*

Proof. By using Theorem 2.2, one can get the conclusion by a similar proof to that of Theorem 3.4. □

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J. COMPUTATIONAL ANALYSIS AND APPLICATIONS, VOL. 15, NO.8, 1391-1402, 2013, COPYRIGHT 2013 EUDOXUS PRESS, LLC

On strictly and semistrictly quasi $\alpha-$preinvex functions*

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Abstract

In this paper, two new classes of generalized convex functions are introduced, which are called strictly quasi $\alpha-$preinvex functions and semistrictly quasi $\alpha-$preinvex functions, respectively. The characterization of quasi $\alpha-$preinvex functions is established under the condition of lower semicontinuity, or upper semicontinuity or semistrict quasi $\alpha-$preinvexity. Furthermore, the characterization of semistrictly quasi $\alpha-$preinvex functions is also obtained under the condition of quasi $\alpha-$preinvexity or lower semiconitinuity. A similar result can also be obtained for strictly quasi $\alpha-$preinvex functions. Finally, an important result stating that 'a local minimum of either a strictly quasi $\alpha-$preinvex functions or a semistrictly quasi $\alpha-$preinvex functions over $\alpha-$invex set is also a global minimum' is established.

Keywords: Convex programming; Quasi $\alpha-$preinvex functions; Semistrictly quasi $\alpha-$preinvex functions; Strictly quasi $\alpha-$preinvex functions; Semicontinuity.

1 Introduction

Convexity and generalized convexity play a central role in mathematical economics, engineering and optimization theory. Therefore, the research on convexity and generalized convexity is one of most important aspects in mathematical programming. In recent years, the concept of convexity has been generalized and extended in several directions using novel and innovative techniques. An important and significant generalization of convexity is the introduction of invexity, preinvexity, semistrictly preinvexity and (semistrictly, strictly) prequasi-invexity, see [1–10] and references therein. Recently, Jeyakumar and Mond in [11, 12] introduced and studied another class of generalized convex functions, which is known as strongly $\alpha-$invex function. Noor and Noor in [13] introduced a new class of generalized convex functions, which is called the strongly $\alpha-$preinvex functions, and established the equivalence among the strongly $\alpha-$preinvex functions, strongly $\alpha-$invex functions and strongly $\alpha\eta-$monotonicity of their differential under some suitable conditions. Fan and Guo in [14] have studied the relationships among (pseudo, quasi) $\alpha-$preinvexity, (strict, strong, pseudo, quasi) $\alpha-$invexity and (strict, strong, pseudo, quasi) $\alpha\eta-$monotonicity in a systematic way.

In this paper, we introduce two new classes of generalized convex functions, which are called strictly quasi $\alpha-$preinvex functions and semistrictly quasi $\alpha-$preinvex functions. We establish the relationships between the quasi $\alpha-$preinvex functions, strictly quasi $\alpha-$preinvex functions and semistrictly quasi $\alpha-$preinvex functions under some suitable and appropriate conditions. Finally, we prove that for general mathematical programming problem, when object function are strictly quasi $\alpha-$preinvex and semistrictly quasi $\alpha-$preinvex , a local minimum of a strictly quasi $\alpha-$preinvex and semistrictly quasi $\alpha-$preinvex functions over an invex set are also a global minimum.

The paper is organized as follows. in Section 2, two new concepts concerning strictly quasi $\alpha-$preinvex functions and semistrictly quasi $\alpha-$preinvex functions are introduced. In Section 3, the characterization of quasi $\alpha-$preinvex functions are introduced under the condition of lower semicontinuity or upper semicontinuity or semistrict quasi $\alpha-$preinvexity. The characterization of strictly quasi $\alpha-$preinvex functions are introduced in Section 4. Applications of two new types of generalized convex functions are given in Section 5.

[0]This work is partially supported by Nanyang Technological University (NTU), Singapore, Temasek Laboratories @ NTU, the National Natural Science Foundation of China (Grant No. 10771228), the Natural Science Foundation Project of CQ CSTC (Grant No. CSTC, 2010BB2090), Education Commission project Research Foundation of Chongqing (Grant No. KJ110617), PR China, and the Program for Innovative Research Team in Higher Educational Institutions of Chongqing, PR China.

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2 Preliminaries

Let H be a real Hilbert space with inner product $\langle .,. \rangle$ and norm $\|.\|$ and K be a nonempty subset of H. Let $f : K \longrightarrow H$ and $\alpha : K \times K \longrightarrow R \setminus \{0\}$ be two real-valued functions and $\eta(.,.) : K \times K \longrightarrow R$ be a vector-valued mapping.

Firstly, we recall the following well-known results and concepts.

Definition 2.1[13]**.** Let $y \in K$. Then the set K is said to be α-invex at y with respect to $\eta(.,.)$ and $\alpha(.,.)$, if, for all $x \in K, t \in [\,0,1]$,

$$y + t\alpha(x,y)\eta(x,y) \in K.$$

K is said to be an α-invex set with respect to η and α if K is α-invex at each $y \in K$. The α-invex set K is also called $\alpha\eta$-connected set. Note that the convex set with $\alpha(x,y) = 1$ and $\eta(x,y) = x - y$ is an invex set, but the converse is not true.

From now on, unless otherwise specified, we assume that K is a nonempty $\alpha-$invex set with respect to η and α.

Definition 2.2[13]**.** The function f on the α-invex set K is said to be α-preinvex with respect to α and η, if

$$f(y + t\alpha(x,y)\eta(x,y)) \le tf(x) + (1-t)f(y), \quad \forall x, y \in K, t \in [0,1].$$

Remark 2.1[13]**.** Every convex function is a preinvex function, but the converse is not true. For example, the function $f(x) = -|x|$ is not a convex function, but it is a preinvex function with respect to η and $\alpha(x,y) = 1$, where

$$\eta(x,y) = \begin{cases} x - y, & if\ x \le 0, y \le 0\ and\ x \ge 0, y \ge 0, \\ y - x, & otherwise. \end{cases}$$

Definition 2.3[13]**.** The function f on the α-invex set K is said to be quasi α-preinvex with respect to α and η, if

$$f(y + t\alpha(x,y)\eta(x,y)) \le \max\{f(x), f(y)\}, \quad \forall x, y \in K, \forall t \in [0,1].$$

Definition 2.4[15]**.** The function f on the α-invex set K is said to be strongly quasi α-preinvex with respect to α and η, if there exists a constant $\beta > 0$ such that

$$f(y + \lambda\alpha(x,y)\eta(x,y)) \le \max\{f(x), f(y)\} - \beta\lambda(1-\lambda)\|\eta(x,y)\|^2, \quad \forall x, y \in K, \quad \forall \lambda \in [0,1].$$

We now introduce two new kinds of generalized convex function termed strictly quasi $\alpha-$preinvex functions and semistrictly quasi $\alpha-$preinvex functions as follows.

Definition 2.5. The function f on the α-invex set K is said to be strictly quasi α-preinvex with respect to α and η, if for any $x, y \in K, x \ne y$, such that

$$f(y + \lambda\alpha(x,y)\eta(x,y)) < \max\{f(x), f(y)\}, \qquad \forall \lambda \in (0,1).$$

Definition 2.6. The function f on the α-invex set K is said to be semistrictly quasi α-preinvex with respect to α and η, if for any $x, y \in K, f(x) \ne f(y)$, such that

$$f(y + \lambda\alpha(x,y)\eta(x,y)) < \max\{f(x), f(y)\}, \qquad \forall \lambda \in (0,1).$$

Remark 2.2. It is obvious that strict quasi $\alpha-$preinvexity implies semistrict quasi $\alpha-$preinvexity as well as quasi $\alpha-$preinvexity. However, quasi $\alpha-$preinvexity does not imply semistrict quasi $\alpha-$preinvexity, and semistrict quasi $\alpha-$preinvexity does not imply quasi $\alpha-$preinvexity.

Example 2.1. This example illustrates that a quasi $\alpha-$preinvex function is not a semistrictly quasi $\alpha-$preinvex function. Let $f(x) = \begin{cases} -x, & if\ \ x > 0, \\ 0, & if\ \ x \le 0, \end{cases}$ and $\eta(x,y) = x - y$, and

$$\alpha(x,y) = \begin{cases} 1, & if\ \ x \ge 0, y \ge 0, \\ 1, & if\ \ x \le 0, y \le 0, \\ -1, & if\ \ x \le 0, y \ge 0, \\ -1, & if\ \ x \ge 0, y \le 0. \end{cases}$$

Then, it is easy to verify that f is a quasi $\alpha-$preinvex function with respect to α and η. However, let $y = -1, x = 1, \lambda = \frac{1}{2}$, we have $f(y) = f(-1) = 0 > -1 = f(1) = f(x)$. That is $f(y) \ne f(x)$. And

$$\begin{aligned} f(y + \lambda\alpha(x,y)\eta(x,y)) &= f((-1) + (1/2)\alpha(1,-1)\eta(1,-1)) = f(-2) = 0 \\ &= \max\{f(1), f(-1)\} = 0. \end{aligned}$$

This shows that f is not a semistrictly quasi $\alpha-$preinvex function for the same α and η.

Example 2.2. This example illustrates that a semistrictly quasi $\alpha-$preinvex function is not a quasi $\alpha-$preinvex function. Let $f(x)=\begin{cases} -|x|, & if \quad |x|\leq 1, \\ -1, & if \quad |x|\geq 1, \end{cases}$ and

$$\eta(x,y)=\begin{cases} x-y, & if \quad x\geq 0, y\geq 0, \\ x-y, & if \quad x\leq 0, y\leq 0, \\ x-y, & if \quad x>1, y<-1, \\ x-y, & if \quad x<-1, y>1, \\ -1, & if \quad -1\leq x\leq 0, y\geq 0, \\ y-x, & if \quad x\geq 0, -1\leq y\leq 0, \\ y-x, & if \quad 0\leq x\leq 1, y\leq 0, \\ y-x, & if \quad x\leq 0, 0\leq y\leq 1. \end{cases} \qquad \alpha(x,y)=\begin{cases} 1, & if \quad x\geq 0, y\geq 0, \\ 1, & if \quad x\leq 0, y\leq 0, \\ 1, & if \quad x>1, y<-1, \\ 1, & if \quad x<-1, y>1, \\ x-y, & if \quad -1\leq x\leq 0, y\geq 0, \\ 1, & if \quad x\geq 0, -1\leq y\leq 0, \\ 1, & if \quad 0\leq x\leq 1, y\leq 0, \\ 1, & if \quad x\leq 0, 0\leq y\leq 1. \end{cases}$$

Then, it is easy to verify that f is a semistrictly quasi $\alpha-$preinvex function with respect to α and η. However, let $x=2, y=-2, \lambda=\frac{1}{2}$. Since

$$\begin{aligned} f(y+\lambda\alpha(x,y)\eta(x,y)) &= f(-2+\tfrac{1}{2}\alpha(2,-2)\eta(2,-2))=f(0)=0 \\ &> -1=f(2)=f(-2)=\max\{f(x),f(y)\}, \end{aligned}$$

f is not a quasi $\alpha-$preinvex function for the same α and η.

Remark 2.3. Example 2.2 also shows that a semistrictly quasi $\alpha-$preinvex function is not necessarily a semistrictly prequasi-invex function.

Definitions 2.3 to 2.6, with $\alpha(x,y)\equiv 1$, reduce to those of perquasi-invex, strongly perquasi-invex, strictly prequasi-invex, semistrictly prequasi-invex functions. See references [6, 7, 9] for details.

Example 2.3. This example illustrates that a quasi $\alpha-$preinvex function is not a strongly quasi $\alpha-$preinvex function. Let $f(x)=\begin{cases} -|x|, & if \quad |x|\leq 1, \\ -1, & if \quad |x|\geq 1, \end{cases}$ and

$$\eta(x,y)=\begin{cases} x-y, & if \quad x\geq 0, y\geq 0, \\ x-y, & if \quad x\leq 0, y\leq 0, \\ y-1, & if \quad x\leq 0, y\geq 0, \\ 1+y, & if \quad x\geq 0, y\leq 0. \end{cases} \qquad \alpha(x,y)=\begin{cases} 1, & if \quad x\geq 0, y\geq 0, \\ 1, & if \quad x\leq 0, y\leq 0, \\ -1, & if \quad x\leq 0, y\geq 0, \\ -1, & if \quad x\geq 0, y\leq 0. \end{cases}$$

Then, it is easy to verify that f is a quasi $\alpha-$preinvex function with respect to α and η. However, for any $\beta>0$, if we let $x=1, y=2, \lambda=\frac{1}{2}$, we get

$$\begin{aligned} f(y+\lambda\alpha(x,y)\eta(x,y)) &= f(2+\tfrac{1}{2}\alpha(1,2)\eta(1,2)=-1 \\ &> \max\{f(1),f(2)\}-\tfrac{1}{2}(1-\tfrac{1}{2})\beta\|(1-2)\|^2=-1-\tfrac{1}{4}\beta. \end{aligned}$$

Thus, f is not a strongly quasi $\alpha-$preinvex function for the same α and η.

Example 2.4. This example illustrates that a strictly quasi $\alpha-$preinvex function is not a strongly quasi $\alpha-$preinvex function. Let $f(x)=-|x|$, and $\eta(x,y)=x-y$, and

$$\alpha(x,y)=\begin{cases} 1, & if \quad x\geq 0, y\geq 0, \\ 1, & if \quad x\leq 0, y\leq 0, \\ -1, & if \quad x\leq 0, y\geq 0, \\ -1, & if \quad x\geq 0, y\leq 0. \end{cases}$$

Then, it is easy to verify that f is a strictly quasi $\alpha-$preinvex function with respect to α and η. However, for any $\beta>0$, if we let $x=\frac{5}{\beta}, y=\frac{1}{\beta}, \lambda=\frac{1}{2}$, we get

$$\begin{aligned} f(y+\lambda\alpha(x,y)\eta(x,y)) &= f(\tfrac{1}{\beta}+\tfrac{1}{2}\cdot 1\cdot(\tfrac{5}{\beta}-\tfrac{1}{\beta}))=-\tfrac{3}{\beta} \\ &> \max\{f(\tfrac{5}{\beta}),f(\tfrac{1}{\beta})\}-\tfrac{1}{2}(1-\tfrac{1}{2})\beta(\tfrac{5}{\beta}-\tfrac{1}{\beta})^2=-\tfrac{5}{\beta}. \end{aligned}$$

Thus, f is not a strongly quasi $\alpha-$preinvex function for the same α and η.

Example 2.5. This example illustrates that a semistrictly quasi $\alpha-$preinvex function is not a strongly quasi

$\alpha-$preinvex function. Let $f(x) = \begin{cases} -|x|, & if \ \ |x| \leq 1, \\ -1, & if \ \ |x| \geq 1, \end{cases}$ and $\eta(x,y) = x - y$, and

$$\alpha(x,y) = \begin{cases} 1, & if \quad x \geq 0, y \geq 0, \\ 1, & if \quad x \leq 0, y \leq 0, \\ 1, & if \quad x > 1, y < -1, \\ 1, & if \quad x < -1, y > 1, \\ -1, & if \quad -1 \leq x \leq 0, y \geq 0, \\ -1, & if \quad x \geq 0, -1 \leq y \leq 0, \\ -1, & if \quad 0 \leq x \leq 1, y \leq 0, \\ -1, & if \quad x \leq 0, 0 \leq y \leq 1. \end{cases}$$

Then, it is easy to verify that f is a semistrictly quasi $\alpha-$preinvex function with respect to α and η. However, for any $\lambda > 0$, if we let $x = 2, y = -2, \lambda = \frac{1}{2}$, we get

$$\begin{aligned} f(y + \lambda\alpha(x,y)\eta(x,y)) &= f((-2) + \tfrac{1}{2}\alpha(2,-2)\eta(2,-2)) = f(0) = 0 \\ &> \max\{f(2), f(-2)\} - \tfrac{1}{4}\beta(2+2)^2 = -1 - 4\beta. \end{aligned}$$

Thus, f is not a strongly quasi $\alpha-$preinvex function for the same α and η.

Remark 2.4. From Example 2.4 and 2.5, we know that strongly quasi $\alpha-$preinvex functions are different from strictly quasi $\alpha-$preinvex functions and semistrictly quasi $\alpha-$preinvex functions and quasi $\alpha-$preinvex functions.

We also need the following assumptions introduced in [13].

Condition A

$$f(y + \alpha(x,y)\eta(x,y)) \leq f(x), \qquad \forall x, y \in K.$$

which plays an important part in studying the properties of the $\alpha-$preinvex ($\alpha-$invex) functions. For $\alpha(x,y) = 1$, Condition A reduces to the following for preinvex functions.

Condition B

$$f(y + \eta(x,y)) \leq f(x), \qquad \forall x, y \in K.$$

For the applications of Condition B see references [9, 16].

Condition C Let $\eta(.,.) : K \times K \longrightarrow R$ and $\alpha(.,.) : K \times K \longrightarrow R\backslash 0$ satisfy the assumptions

$$\begin{aligned} \eta(y, y + \lambda\alpha(x,y)\eta(x,y)) &= -\lambda\eta(x,y), \\ \eta(x, y + \lambda\alpha(x,y)\eta(x,y)) &= (1-\lambda)\eta(x,y), \qquad \forall x, y \in K, \ \lambda \in [\, 0, 1]. \end{aligned}$$

3 Characterizations of quasi $\alpha-$preinvx functions

First of all, we give two important lemmas.

Lemma 3.1[15]**.** Let K be an α-invex set with respect to α and η, for any $x, y \in K, \lambda \in [0,1]$, if α and η satisfy the assumptions

$$\eta(y, y + \lambda\alpha(x,y)\eta(x,y)) = -\lambda\eta(x,y),$$
$$\alpha(x,y) = \alpha(y, y + \lambda\alpha(x,y)\eta(x,y)),$$

then $\forall\lambda_1, \lambda_2 \in [0,1]$ and $\lambda_2 < \lambda_1$, the following equalities hold

(i) $\eta(y + \lambda_1\alpha(x,y)\eta(x,y), y + \lambda_2\alpha(x,y)\eta(x,y)) = (\lambda_1 - \lambda_2\eta(x,y)$,

(ii) $\alpha(x,y) = \alpha(y + \lambda_1\alpha(x,y)\eta(x,y), y + \lambda_2\alpha(x,y)\eta(x,y))$.

Lemma 3.2. Let K be an α-invex set with respect to α and η, and Condition A and C hold. Assume that the following conditions are satisfied:

(i) there exists a $\theta \in (0,1)$ such that, for all $x, y \in K$,

$$f(y + \theta\alpha(x,y)\eta(x,y)) \leq \max\{f(x), f(y)\} \tag{3.1}$$

(ii) for any $x, y \in K, \lambda \in K[0,1]$,

$$\eta(y, y + \lambda\alpha(x,y)\eta(x,y)) = -\lambda\eta(x,y),$$
$$\alpha(x,y) = \alpha(y, y + \lambda\alpha(x,y)\eta(x,y)).$$

Then the set defined by

$$A = \{\lambda \in [0,1] | f(y + \lambda\alpha(x,y)\eta(x,y)) \leq \max\{f(x), f(y)\},\ \forall x, y \in K\}$$

is dense in the interval $[\,0,1]$.

Proof. By contradiction. Suppose that A is not dense in $[\,0,1]$. Then, there exists a $\lambda_0 \in (0,1)$ and a neighborhood $N(\lambda_0)$ of λ_0 such that

$$N(\lambda_0) \cap A = \emptyset. \tag{3.2}$$

From Condition A and (3.1), we have

$$\begin{array}{l} \{\lambda \in A | \lambda \geq \lambda_0\} \neq \emptyset \\ \{\lambda \in A | \lambda \leq \lambda_0\} \neq \emptyset. \end{array}$$

Define

$$\lambda_1 = \inf\{\lambda \in A | \lambda \geq \lambda_0\} \tag{3.3}$$

$$\lambda_2 = \sup\{\lambda \in A | \lambda \leq \lambda_0\} \tag{3.4}$$

Then, by (3.2), we have $0 \leq \lambda_2 < \lambda_1 \leq 1$.

Since $\{\theta, (1-\theta)\} \in (0,1)$, we can choose $u_1, u_2 \in A$ satisfying $u_1 \geq \lambda_1, u_2 \leq \lambda_2$ such that

$$\max\{\theta, (1-\theta)\}(u_1 - u_2) < \lambda_1 - \lambda_2. \tag{3.5}$$

Next, let us consider $\overline{\lambda} = \theta u_1 + (1-\theta)u_2$. From $\lambda_2 < \lambda_1$ and Lemma 3.1, for any $x, y \in K$, we have

$$\begin{aligned} & y + \overline{\lambda}\alpha(x,y)\eta(x,y) \\ = \ & y + (\theta u_1 + (1-\theta)u_2)\alpha(x,y)\eta(x,y) \\ = \ & y + u_2\alpha(x,y)\eta(x,y) + \theta\alpha(x,y) \cdot (u_1 - u_2)\eta(x,y) \\ = \ & y + u_2\alpha(x,y)\eta(x,y) \\ & +\theta\alpha(y + u_1\alpha(x,y)\eta(x,y), y + u_2\alpha(x,y)\eta(x,y))\eta(y + u_1\alpha(x,y)\eta(x,y), y + u_2\alpha(x,y)\eta(x,y)). \end{aligned}$$

Hence, from (3.1) and the fact that $u_1, u_2 \in A$, we get

$$\begin{aligned} & f(y + \overline{\lambda}\alpha(x,y)\eta(x,y)) \\ = \ & f(y + u_2\alpha(x,y)\eta(x,y) \\ & +\theta\alpha(y + u_1\alpha(x,y)\eta(x,y), y + u_2\alpha(x,y)\eta(x,y))\eta(y + u_1\alpha(x,y)\eta(x,y), y + u_2\alpha(x,y)\eta(x,y))) \\ \leq \ & \max\{f(y + u_1\alpha(x,y)\eta(x,y)), f(y + u_2\alpha(x,y)\eta(x,y))\} \\ \leq \ & \max\{\max\{f(x), f(y)\}, \max\{f(x), f(y)\}\} \\ = \ & \max\{f(x), f(y)\}. \end{aligned}$$

That is, $\overline{\lambda} \in A$.

If $\overline{\lambda} \geq \lambda_0$, then it follows from (3.5) that

$$\overline{\lambda} - u_2 = \theta(u_1 - u_2) < \lambda_1 - \lambda_2,$$

and therefore $\overline{\lambda} < \lambda_1$. Because of $\overline{\lambda} \geq \lambda_0$ and $\overline{\lambda} \in A$ this is a contradiction to (3.3). Similarly, $\overline{\lambda} \leq \lambda_0$ provides a contradiction to (3.4). Hence, A is dense in $[\,0,1]$.

Theorem 3.1. Let K be an α-invex set with respect to α and η. If the following assumptions hold:
(i) Condition A and C are satisfied;
(ii) for any $x, y \in K, \theta \in [0,1]$,

$$\alpha(x,y) = \alpha(x, y + \theta\alpha(x,y)\eta(x,y)) = \alpha(y, y + \theta\alpha(x,y)\eta(x,y));$$

(iii) f is an upper semicontinuous function;
then f is quasi $\alpha-$preinvex function on K if and only if exists a $\theta \in (0,1)$, such that, for all $x, y \in K$

$$f(y + \theta\alpha(x,y)\eta(x,y)) \leq \max\{f(x), f(y)\}.$$

Proof. The necessity is obvious from Definition of quasi $\alpha-$preinvex functions. We only prove the sufficiency. Suppose that f is not quasi $\alpha-$preinvex functions on K. Then, there exist $x, y \in K$ and $\overline{\lambda} \in (0,1)$ such that

$$f(y + \overline{\lambda}\alpha(x,y)\eta(x,y)) > \max\{f(x), f(y)\}. \tag{3.6}$$

Let

$$\begin{aligned} z &= y + \overline{\lambda}\alpha(x,y)\eta(x,y), \\ A &= \{\lambda \in [0,1] | f(y + \lambda\alpha(x,y)\eta(x,y)) \le \max\{f(x), f(y)\}, \forall x, y \in K\}. \end{aligned}$$

From Lemma 3.2, there exists a $\{\lambda_n\} \subset A$, $\lambda_n < \overline{\lambda}$ such that $\lambda_n \to \overline{\lambda},\ n \to \infty$.
Define $y_n = z - \frac{\lambda_n}{1-\lambda_n}\alpha(x,z)\eta(x,z)$. From Condition C and (ii), we have $y_n = y + \frac{\overline{\lambda}-\lambda_n}{1-\lambda_n}\alpha(x,y)\eta(x,y)$. Then,

$$y_n \to y, \qquad n \to \infty.$$

Since K is an $\alpha-$invex set, it follows that, for sufficiently large n, $y_n \in K$.
Again from Condition C and (ii), we get

$$\begin{aligned} & y_n + \lambda_n\alpha(x,y_n)\eta(x,y_n) \\ =& y + \frac{\overline{\lambda}-\lambda_n}{1-\lambda_n}\alpha(x,y)\eta(x,y) \\ & + \lambda_n\alpha(x, y + \frac{\overline{\lambda}-\lambda_n}{1-\lambda_n}\alpha(x,y)\eta(x,y))\eta(x, y + \frac{\overline{\lambda}-\lambda_n}{1-\lambda_n}\alpha(x,y)\eta(x,y)) \\ =& y + \frac{\overline{\lambda}-\lambda_n}{1-\lambda_n}\alpha(x,y)\eta(x,y) + \lambda_n \cdot \frac{1-\overline{\lambda}}{1-\lambda_n}\alpha(x,y)\eta(x,y) \\ =& y + \overline{\lambda}\alpha(x,y)\eta(x,y) \\ =& z. \end{aligned} \tag{3.7}$$

By the upper semicontinuity of f on K, for any $\varepsilon > 0$, there exists an $N > 0$ such that

$$f(y_n) \le f(y) + \varepsilon, \qquad for \quad n > N.$$

Therefore, from (3.7) and $\lambda_n \in A$, we have

$$\begin{aligned} f(z) &= f(y_n + \lambda_n\alpha(x,y_n)\eta(x,y_n)) \\ &\le \max\{f(x), f(y_n)\} \\ &\le \max\{f(x), f(y) + \varepsilon\}, \qquad for \quad n > N. \end{aligned}$$

Since $\varepsilon > 0$ is arbitrarily small, we have

$$f(z) \le \max\{f(x), f(y)\},$$

which contradicts the inequality (3.6). Thus, f is a quasi $\alpha-$preinvex function for same α and η on K.
Remark 3.1. By [15, example 3.1], there exist α and η that satisfy both Condition C and the equality $\alpha(x,y) = \alpha(x, y + \theta\alpha(x,y)\eta(x,y)) = \alpha(y, y + \theta\alpha(x,y)\eta(x,y))$. For example, when $\alpha(x,y) \equiv 1$, the Condition C above is exectly the same as Condition C in [5].
Theorem 3.2. Let K be an α-invex set with respect to η and α. If the following assumptions hold:
(i) Condition A and C are satisfied;
(ii) for any $x, y \in K, \theta \in [0,1]$,

$$\alpha(x,y) = \alpha(x, y + \theta\alpha(x,y)\eta(x,y)) = \alpha(y, y + \theta\alpha(x,y)\eta(x,y));$$

(iii) f is lower semicontinuous functions;
then f is quasi $\alpha-$preinvex functions on K if and only if for any $x, y \in K$, there exists a $\theta \in (0,1)$ such that

$$f(y + \theta\alpha(x,y)\eta(x,y)) \le \max\{f(x), f(y)\}.$$

Proof. The necessity is obvious from Definition of quasi $\alpha-$preinvex functions. We only prove the sufficiency. By contradiction, we assume that there exist distinct $x, y \in K$ and $\overline{\theta} \in (0,1)$ such that

$$f(y + \overline{\theta}\alpha(x,y)\eta(x,y)) > \max\{f(x), f(y)\}.$$

Let

$$\begin{aligned} z &= y + \overline{\theta}\alpha(x,y)\eta(x,y), \\ x_t &= z + t\alpha(x,z)\eta(x,z). \end{aligned}$$

From Condition C and (ii), we have

$$\begin{aligned} x_t &= y + \overline{\theta}\alpha(x,y)\eta(x,y) + t\alpha(x, y + \overline{\theta}\alpha(x,y)\eta(x,y))\eta(x, y + \overline{\theta}\alpha(x,y)\eta(x,y)) \\ &= y + \overline{\theta}\alpha(x,y)\eta(x,y) + t\alpha(x,y) \cdot (1-\overline{\theta})\eta(x,y) \\ &= y + [\overline{\theta} + t(1-\overline{\theta})]\alpha(x,y)\eta(x,y). \end{aligned}$$

Let

$$\begin{aligned} B &= \{x_t \in K | t \in (0,1], f(x_t) \leq \max\{f(x), f(y)\}\}, \\ u &= inf\{t \in (0,1] | x_t \in B\}. \end{aligned}$$

It is obvious that $x_1 \in B$ from Condition A, but $x_0 \notin B$. Thus, $x_t \notin B,\ 0 \leq t < u$, and there exist $t_n \geq u, x_{t_n} \in B$ (from Lemma 3.2), such that

$$t_n \to u, \quad u \to \infty.$$

Since f is a lower semicontinuous function, we have

$$f(x_u) \leq \liminf_{n\to\infty} f(x_{t_n}) \leq \max\{f(x), f(y)\}.$$

Hence, $x_u \in B$.
Similarly, let $y_t = z + (1-t)\alpha(y,z)\eta(y,z)$. From Condition C and (ii), we have

$$y_t = y + t\bar{\theta}\alpha(x,y)\eta(x,y).$$

Let

$$\begin{aligned} D &= \{y_t \in K | t \in [0,1), f(y_t) = f(y + t\bar{\theta}\alpha(x,y)\eta(x,y)) \leq \max\{f(x), f(y)\}\}, \\ v &= \sup\{t \in [0,1) | y_t \in D\}. \end{aligned}$$

It is obvious that

$$\begin{aligned} y_0 &= y \in D, \\ y_1 &= y + \bar{\theta}\alpha(y,z)\eta(y,z) = z \notin D, \\ y_t &\notin D, \qquad v < t \leq 1, \end{aligned}$$

and there exist $t_n \leq v, y_{t_n} \in D$ (from Lemma 3.2), such that

$$t_n \to v, \qquad n \to \infty.$$

Since f is a lower semicontinuous function, we have

$$f(y_n) \leq \liminf_{n\to\infty} f(y_{t_n}) \leq \max\{f(x), f(y)\}.$$

Hence, $y_v \in D$.
Let

$$\begin{aligned} \theta_1 &= v\bar{\theta}, \\ \theta_2 &= \bar{\theta} + u - u\bar{\theta}. \end{aligned}$$

Then, $0 \leq \theta_1 < \bar{\theta} < \theta_2 \leq 1$.
Now, from Condition C and (ii), we have

$$\begin{aligned} & x_u + \lambda\alpha(y_v, x_u)\eta(y_v, x_u) \\ =& y + \theta_2\alpha(x,y)\eta(x,y) \\ & + \lambda\alpha(y + \theta_1\alpha(x,y)\eta(x,y), y + \theta_2\alpha(x,y)\eta(x,y)) \\ & \cdot \eta(y + \theta_1\alpha(x,y)\eta(x,y), y + \theta_2\alpha(x,y)\eta(x,y)) \\ =& y + \theta_2\alpha(x,y)\eta(x,y) + \lambda\alpha(x,y)\cdot(\theta_1 - \theta_2)\eta(x,y) \\ =& y + [\lambda\theta_1 + (1-\lambda)\theta_2]\alpha(x,y)\eta(x,y), \qquad \forall\lambda \in [0,1]. \end{aligned}$$

Hence, from the definitions of θ_1 and θ_2, we have

$$\begin{aligned} f(x_u + \lambda\alpha(y_v, x_u)\eta(y_v, x_u)) &= f\{y + [\lambda\theta_1 + (1-\lambda)\theta_2]\alpha(x,y)\eta(x,y)\} \\ &> \max\{f(x), f(y)\} \\ &\geq \max\{f(y_v), f(x_u)\}, \quad \forall\lambda \in (0,1), \end{aligned}$$

contradicting the assumptions of the theorem.
Theorem 3.3. Let K be an $\alpha-$invex set with respect to α and η. If the following assumptions hold:
(i) Condition C is satisfied;
(ii) for any $x, y \in K, \theta \in [0,1]$,

$$\alpha(x,y) = \alpha(x, y + \theta\alpha(x,y)\eta(x,y)) = \alpha(y, y + \theta\alpha(x,y)\eta(x,y));$$

(iii) f is a semistrictly $\alpha-$preinvex functions;
Then, f is a quasi $\alpha-$preinvex function on K if and only if the following condition is satified:
there exists a $\theta \in (0,1)$ such that, for all $x, y \in K$,

$$f(y+\theta\alpha(x,y)\eta(x,y)) \leq \max\{f(x), f(y)\}. \tag{3.8}$$

Proof. The necessity is obvious from Definition of quasi $\alpha-$preinvex functions. We prove the sufficiency. Suppose that there exist $x, y \in K$ and $\lambda \in (0,1)$ such that

$$f(y+\lambda\alpha(x,y)\eta(x,y)) > \max\{f(x), f(y)\}.$$

Without loss of generality, assume that $f(x) \geq f(y)$ and let $z = y+\lambda\alpha(x,y)\eta(x,y)$. Then,

$$f(z) > f(x). \tag{3.9}$$

If $f(x) > f(y)$, it follows from the semistrict quasi $\alpha-$ preinvexity of f that

$$f(z) < f(x),$$

contradicting (3.9).
If $f(x) = f(y)$, then (3.9) implies that

$$f(z) > f(x) = f(y). \tag{3.10}$$

There are two cases to be considered.
Case 1 $0 < \lambda < \theta < 1$. Let $z_1 = y + \frac{\lambda}{\theta}\alpha(x,y)\eta(x,y)$. Thus, from Condition C and (ii), we have

$$\begin{aligned}
& y+\theta\alpha(z_1,y)\eta(z_1,y) \\
= \;& y+\theta\alpha(y+\tfrac{\lambda}{\theta}\alpha(x,y)\eta(x,y),y)\eta(y+\tfrac{\lambda}{\theta}\alpha(x,y)\eta(x,y),y) \\
= \;& y+\theta\alpha(x,y)\eta(y+\tfrac{\lambda}{\theta}\alpha(x,y)\eta(x,y),y+\tfrac{\lambda}{\theta}\alpha(x,y)\eta(x,y)-\tfrac{\lambda}{\theta}\alpha(x,y)\eta(x,y)) \\
= \;& y+\theta\alpha(x,y)\eta(y+\tfrac{\lambda}{\theta}\alpha(x,y)\eta(x,y),y+\tfrac{\lambda}{\theta}\alpha(x,y)\eta(x,y) \\
& +\alpha(y,y+\tfrac{\lambda}{\theta}\alpha(x,y)\eta(x,y))\eta(y,y+\tfrac{\lambda}{\theta}\alpha(x,y)\eta(x,y))) \\
= \;& y-\theta\eta(y,y+\tfrac{\lambda}{\theta}\alpha(x,y)\eta(x,y)) \\
= \;& y+\theta\alpha(x,y)\eta(x,y) \\
= \;& z.
\end{aligned}$$

According to (3.8), we have $f(z) \leq \max\{f(z_1), f(y)\}$. From (3.10) and the above inequality, it follows that

$$f(z) \leq f(z_1). \tag{3.11}$$

Let $b = \frac{\lambda(1-\theta)}{\theta(1-\lambda)}$. Since $0 < \lambda < \theta < 1$, it is easy to show that $0 < b < 1$. Thus, from Condition C and (ii), we have

$$\begin{aligned}
& z+b\alpha(x,z)\eta(x,z) \\
= \;& y+\lambda\alpha(x,y)\eta(x,y)+b\alpha(x,y+\lambda\alpha(x,y)\eta(x,y))\eta(x,y+\lambda\alpha(x,y)\eta(x,y)) \\
= \;& y+[\lambda+b(1-\lambda)]\alpha(x,y)\eta(x,y) \\
= \;& y+[\lambda+\lambda\cdot\tfrac{(1-\theta)}{\theta}]\alpha(x,y)\eta(x,y) \\
= \;& y+\tfrac{\lambda}{\theta}\alpha(x,y)\eta(x,y) \\
= \;& z_1.
\end{aligned}$$

Since f is a semistrictly quasi $\alpha-$preinvex function, it follows from inequality (3.10) and the above equality that

$$f(z_1) < \max\{f(x), f(z)\} = f(z),$$

contradicting (3.11).
Case 2 $0 < \theta < \lambda < 1$. In this case, we still get a contradiction by just exchanging the roles of θ and $1-\theta$ and the roles of λ and $\lambda-\theta$ in Case 1.
Theorem 3.4. Let K be an α-invex set with respect to α and η. If the following assumptions hold:
(i) Condition A and C are statisfied;
(ii) for any $x, y \in K, \theta \in [0,1]$,

$$\alpha(x,y) = \alpha(x,y+\theta\alpha(x,y)\eta(x,y)) = \alpha(y,y+\theta\alpha(x,y)\eta(x,y));$$

(iii) f is lower semicontinuous functions and if there exists a $\theta \in (0,1)$ such that, for every $x, y \in K$, $f(x) \neq f(y)$ implies

$$f(y+(1-\theta)\alpha(x,y)\eta(x,y)) < \max\{f(x), f(y)\},) \tag{3.12}$$

then f is a quasi $\alpha-$preinvex function for same η and α on K.
Proof. By Theorem 3.2, we need only to show that, for each $x, y \in K$, there exists a $\lambda \in (0,1)$ such that

$$f(y+\lambda\alpha(x,y)\eta(x,y)) \leq \max\{f(x), f(y)\}, \tag{3.13}$$

By contradiction, we assume that there exist $x, y \in K$ such that

$$f(y+\lambda\alpha(x,y)\eta(x,y)) > \max\{f(x), f(y)\}, \quad \forall\lambda \in (0,1). \tag{3.14}$$

If $f(x) \neq f(y)$, it follows from (3.12) that

$$f(y+(1-\theta)\alpha(x,y)\eta(x,y)) < \max\{f(x), f(y)\},$$

which contradicts (3.14).
If $f(x) = f(y)$, then (3.14) implies

$$f(y+\lambda\alpha(x,y)\eta(x,y)) > f(x) = f(y), \quad \forall\lambda \in (0,1). \tag{3.15}$$

By (3.15), we obtain

$$\begin{aligned} & f(y+\lambda\alpha(x,y)\eta(x,y)+(1-\theta)\alpha(x,y+\lambda\alpha(x,y)\eta(x,y))\eta(x,y+\lambda\alpha(x,y)\eta(x,y))) \\ = \; & f(y+\lambda\alpha(x,y)\eta(x,y)+(1-\theta)\alpha(x,y)\cdot(1-\lambda)\eta(x,y)) \\ = \; & f(y+[\lambda+(1-\theta)(1-\lambda)]\alpha(x,y)\eta(x,y) \\ > \; & f(y), \qquad \forall\lambda \in (0,1). \end{aligned} \tag{3.16}$$

And, from (3.12) and (3.15), we have

$$\begin{aligned} & f[y+\lambda\alpha(x,y)\eta(x,y)+(1-\theta)\alpha(x,y+\lambda\alpha(x,y)\eta(x,y))\eta(x,y+\lambda\alpha(x,y)\eta(x,y))] \\ < \; & \max\{f(x), f(y+\lambda\alpha(x,y)\eta(x,y))\} \\ = \; & f(y+\lambda\alpha(x,y)\eta(x,y)), \quad \forall\lambda \in (0,1). \end{aligned} \tag{3.17}$$

Again by (3.12),(3.16), (3.17), we have

$$\begin{aligned} & f(y+\theta\gamma\alpha(x,y)\eta(x,y)) \\ = \; & f(y+\gamma\alpha(x,y)\eta(x,y)-(1-\theta)\gamma\alpha(x,y)\eta(x,y)) \\ = \; & f(y+\gamma\alpha(x,y)\eta(x,y)+(1-\theta)\alpha(y,y+\gamma\alpha(x,y)\eta(x,y))\eta(y,y+\gamma\alpha(x,y)\eta(x,y))) \\ < \; & \max\{f(y), f(y+\gamma\alpha(x,y)\eta(x,y))\} \\ = \; & f(y+\gamma\alpha(x,y)\eta(x,y)) \\ < \; & f(y+\lambda\alpha(x,y)\eta(x,y)), \quad \forall\lambda \in (0,1), \end{aligned}$$

where $\gamma = \lambda + (1-\theta)(1-\lambda)$.
Let $\lambda = \frac{\theta}{1+\theta} \in (0,1)$. Then, the above inequality implies

$$f(y+\frac{\theta}{1+\theta}\alpha(x,y)\eta(x,y)) < f(y+\frac{\theta}{1+\theta}\alpha(x,y)\eta(x,y)),$$

which is a contradiction.

4 Characterizations of Strictly quasi $\alpha-$preinvex Functions

Theorem 4.1. Let K be an α-invex set with respect to α and η. If the following assumptions hold:
(i) Condition C is satisfied;
(ii) for any $x, y \in K, \theta \in [0,1]$,

$$\alpha(x,y) = \alpha(x, y+\theta\alpha(x,y)\eta(x,y)) = \alpha(y, y+\theta\alpha(x,y)\eta(x,y));$$

Then f is a strictly quasi $\alpha-$preinvex function on K if and only if the following two conditions hold:
(a) f is a quasi $\alpha-$preinvex function on K;
(b) there exists an $\theta \in (0,1)$ such that, for every pair of distinct points $x, y \in K$,

$$f(y+\theta\alpha(x,y)\eta(x,y)) < \max\{f(x), f(y)\}. \tag{4.1}$$

Proof. The necessity is obvious from Definition 2.3 and 2.5. We prove the sufficiency. Suppose that f is not a strictly quasi $\alpha-$preinvex function for the same α and η on K. Then, there exist $x, y \in K, x \neq y, \lambda \in (0,1)$ such that

$$f(y+\lambda\alpha(x,y)\eta(x,y)) \geq \max\{f(x), f(y)\}.$$

Since f is quasi $\alpha-$preinvex function, we have

$$f(y+\lambda\alpha(x,y)\eta(x,y)) \leq \max\{f(x), f(y)\}.$$

Hence,

$$f(y+\lambda\alpha(x,y)\eta(x,y)) = \max\{f(x), f(y)\}. \tag{4.2}$$

Let us choose β_1, β_2 so that

$$0 < \beta_1 < \lambda < \beta_2 < 1,$$

where $\lambda = \theta\beta_1 + (1-\theta)\beta_2$.
Let

$$\begin{aligned} \overline{x} &= y+\beta_1\alpha(x,y)\eta(x,y), \\ \overline{y} &= y+\beta_2\alpha(x,y)\eta(x,y). \end{aligned}$$

Then, from Condition C and (ii), we get

$$\begin{aligned} & \overline{y}+\theta\alpha(\overline{x},\overline{y})\eta(\overline{x},\overline{y}) \\ = \ & y+\beta_2\alpha(x,y)\eta(x,y) \\ & +\theta\alpha(y+\beta_1\alpha(x,y)\eta(x,y), y+\beta_2\alpha(x,y)\eta(x,y))\eta(y+\beta_1\alpha(x,y)\eta(x,y), y+\beta_2\alpha(x,y)\eta(x,y)) \\ = \ & y+\beta_2\alpha(x,y)\eta(x,y)+\theta\alpha(x,y)\cdot(\beta_1-\beta_2)\eta(x,y) \\ = \ & y+\lambda\alpha(x,y)\eta(x,y). \end{aligned}$$

That is,

$$\overline{y}+\theta\alpha(\overline{x},\overline{y})\eta(\overline{x},\overline{y}) = y+\lambda\alpha(x,y)\eta(x,y). \tag{4.3}$$

Again, since f is quasi $\alpha-$preinvex function, we have

$$f(\overline{x}) \leq \max\{f(x), f(y)\}, \tag{4.4}$$

$$f(\overline{y}) \leq \max\{f(x), f(y)\}. \tag{4.5}$$

By (4.1) and (4.3)-(4.5), we have

$$f(y+\lambda\alpha(x,y)\eta(x,y)) < \max\{f(x), f(y)\},$$

which contradicts the inequality (4.2).
Theorem 4.2. Let f be a lower semicontinuous function and satisfy Condition A, and $\alpha(x,y) = \alpha(x, y+\theta\alpha(x,y)\eta(x,y)) = \alpha(y, y+\theta\alpha(x,y)\eta(x,y)), \forall x,y \in K, \theta \in [0,1]$. Then, f is a strictly quasi $\alpha-$preinvex function on K if and only if the following condition hold:
there exists an $\theta \in (0,1)$, for every pair of distinct points $x, y \in K$, we have

$$f(y+\theta\alpha(x,y)\eta(x,y)) < \max\{f(x), f(y)\}.$$

Theorem 4.3. Let K be an α-invex set with respect to α and η, and η satisfy Condition C, and $\alpha(x,y) = \alpha(x, y+\theta\alpha(x,y)\eta(x,y)) = \alpha(y, y+\theta\alpha(x,y)\eta(x,y)), \forall x,y \in K, \theta \in [0,1]$. Then, f is a strictly quasi $\alpha-$preinvex function on K if and only if f is a semistrictly quasi $\alpha-$preinvex function and the following condition hold:
there exists an $\theta \in (0,1)$, for every pair of distinct points $x, y \in K$, we have

$$f(y+\theta\alpha(x,y)\eta(x,y)) < \max\{f(x), f(y)\}. \tag{4.6}$$

Proof. The necessity is obvious from Definition 2.5 and 2.6. We prove the sufficiency. Since f is a semistrictly quasi $\alpha-$preinvex function, it suffices to show that $f(x) = f(y), x \neq y$, implies

$$f(y+\lambda\alpha(x,y)\eta(x,y)) < \max\{f(x), f(y)\}, \quad \forall\lambda \in (0,1).$$

From (4.6) and for each $x, y \in K, x \neq y$, we have

$$f(y+\theta\alpha(x,y)\eta(x,y)) < f(x) = f(y). \tag{4.7}$$

Let $\overline{x} = y+\theta\alpha(x,y)\eta(x,y)$. Let $\lambda \in (0,1)$. If $\lambda < \theta$, then, $\mu = (\theta-\lambda)/\theta \in (0,1)$.
From Condition C and (ii), we have

$$\overline{x}+\mu\alpha(y,\overline{x})\eta(y,\overline{x}) = y+\lambda\alpha(x,y)\eta(x,y).$$

Since f is semistrictly quasi $\alpha-$preinvex functions for same η and α on K and (4.7) holds, we have

$$\begin{aligned} f(y+\lambda\alpha(x,y)\eta(x,y)) &= f(\overline{x}+\mu\alpha(y,\overline{x})\eta(y,\overline{x})) \\ &< \max\{f(y), f(\overline{x})\} \\ &= f(y). \end{aligned}$$

If $\lambda > \theta$, then

$$\nu = (\lambda-\theta)/(1-\theta) \in (0,1).$$

From Condition C and (ii), we have

$$\overline{x}+\nu\alpha(x,\overline{x})\eta(x,\overline{x}) = y+\lambda\alpha(x,y)\eta(x,y).$$

Since f is semistrictly quasi $\alpha-$preinvex function on K and (4.7) holds, we have

$$\begin{aligned} f(y+\lambda\alpha(x,y)\eta(x,y)) &= f(\overline{x}+\nu\alpha(x,\overline{x})\eta(x,\overline{x})) \\ &< \max\{f(x), f(\overline{x})\} \\ &= f(x). \end{aligned}$$

This completes the proof.

5 Applications of Strictly and Semistrictly quasi $\alpha-$preinvex Functions

Let the problem of minimizing $f(x)$ subject to $x \in K$ be denoted by (P). The following two theorems show that a local minimum of a strictly quasi $\alpha-$preinvex functions and semistrictly quasi $\alpha-$preinvex functions over an $\alpha-$invex set are also a global minimum.

Theorem 5.1. Let K be a nonempty $\alpha-$invex set with respect to α and η, and f be a strictly quasi $\alpha-$preinvex for the same α and η on K. If $\overline{x} \in K$ is a local minimum to the problem (P), then $\overline{x}$ is a global minimum.

Proof. Assume that $\overline{x} \in K$ is a local minimum to the problem (P). Then there exists an ε-neighborhood $N_\varepsilon(\overline{x}) \subset K$ around $\overline{x}$ such that

$$f(\overline{x}) \le f(x), \quad \forall x \in K \cap N_\varepsilon(\overline{x}). \tag{5.1}$$

Suppose that $\overline{x}$ is not a global minimum of (P), then there exists a $x^* \in K$ such that

$$f(x^*) < f(\overline{x}).$$

Since K is a nonempty $\alpha-$invex set with respect to α and η, and f is strictly quasi $\alpha-$preinvex function, for any $\lambda \in (0,1)$, $\overline{x}+\lambda\alpha(x^*,\overline{x})\eta(x^*,\overline{x}) \in K$, we have

$$\begin{aligned} f(\overline{x}+\lambda\alpha(x^*,\overline{x})\eta(x^*,\overline{x})) &< \max\{f(x^*), f(\overline{x})\} \\ &< f(\overline{x}) \end{aligned}$$

i.e., for any $\lambda \in (0,1)$, we have

$$f(\overline{x}+\lambda\alpha(x^*,\overline{x})\eta(x^*,\overline{x})) < f(\overline{x}).$$

Thus, for a sufficiently small $\lambda > 0$, we have

$$\overline{x}+\lambda\alpha(x^*,\overline{x})\eta(x^*,\overline{x}) \in K \cap N_\varepsilon(\overline{x}),$$

which is a contradiction to (5.1). This completes the proof.

Theorem 5.2. Let K be a nonempty $\alpha-$invex set with respect to α and η, and f be a semistrictly quasi $\alpha-$preinvex for the same α and η on K. If $\overline{x} \in K$ is a local minimum to the problem (P), then $\overline{x}$ is a global minimum.

Proof. Assume that $\overline{x} \in K$ is a local minimum to the problem (P). Then there exists an ε-neighborhood $N_\varepsilon(\overline{x}) \subset K$ around $\overline{x}$ such that

$$f(\overline{x}) \le f(x), \quad \forall x \in K \cap N_\varepsilon(\overline{x}). \tag{5.2}$$

Suppose that $\overline{x}$ is not a global minimum of (P), then there exists an $x^* \in K$ such that

$$f(x^*) < f(\overline{x}).$$

Since K is a nonempty $\alpha-$invex set with respect to η and α, and f is semistrictly quasi $\alpha-$preinvex function, for any $\lambda \in (0,1)$, $\overline{x}+\lambda\alpha(x^*,\overline{x})\eta(x^*,\overline{x}) \in K$, we have

$$\begin{aligned} f(\overline{x}+\lambda\alpha(x^*,\overline{x})\eta(x^*,\overline{x})) &< \max\{f(x^*), f(\overline{x})\} \\ &< f(\overline{x}) \end{aligned}$$

i.e., for any $\lambda \in (0,1)$, we have

$$f(\overline{x}+\lambda\alpha(x^*,\overline{x})\eta(x^*,\overline{x})) < f(\overline{x}).$$

Thus, for a sufficiently small $\lambda > 0$, we have

$$\overline{x}+\lambda\alpha(x^*,\overline{x})\eta(x^*,\overline{x}) \in K \cap N_\varepsilon(\overline{x}),$$

which is a contradiction to (5.2). This completes the proof.

Remark 5.1. Theorem 5.1 and 5.2 illustrat that strictly quasi $\alpha-$preinvex functions and semistrictly quasi $\alpha-$preinvex functions are very important in mathematical programming.

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J. COMPUTATIONAL ANALYSIS AND APPLICATIONS, VOL. 15, NO.8, 1403-1412, 2013, COPYRIGHT 2013 EUDOXUS PRESS, LLC

ON STABILITY OF FUNCTIONAL INEQUALITIES AT RANDOM LATTICE φ-NORMED SPACES

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ABSTRACT. We establish some stability results concerning the following functional inequalities

$$\| f(x) + f(y) + f(z) \| \leq \|f(x + y + z)\|$$

and

$$\| f(x) + f(y) + 2f(z) \| \leq \left\|2f\left(\frac{x+y}{2} + z\right)\right\|$$

in the setting of latticetic random φ-normed spaces.

1. Introduction and preliminaries

Let $\mathcal{L} = (L, \geq_L)$ be a complete lattice, i.e., a partially ordered set in which every nonempty subset admits supremum and infimum, and $0_{\mathcal{L}} = \inf L$, $1_{\mathcal{L}} = \sup L$. The space of latticetic random distribution functions, denoted by Δ^+_L, is defined as the set of all left continuous non-decreasing mappings $F : \mathbb{R} \cup \{-\infty, +\infty\} \to L$ with $F(0) = 0_{\mathcal{L}}$, $F(+\infty) = 1_{\mathcal{L}}$.

$D^+_L \subseteq \Delta^+_L$ is defined as $D^+_L = \{F \in \Delta^+_L : l^- F(+\infty) = 1_{\mathcal{L}}\}$, where $l^- f(x)$ denotes the left limit of the function f at the point x. The space Δ^+_L is partially ordered by the usual point-wise ordering of functions, i.e., $F \geq G$ if and only if $F(t) \geq_L G(t)$ for all t in $\mathbb{R}$. The maximal element for Δ^+_L in this order is the distribution function given by

$$\varepsilon_0(t) = \begin{cases} 0_{\mathcal{L}}, & \text{if } t \leq 0, \\ 1_{\mathcal{L}}, & \text{if } t > 0. \end{cases}$$

The concept of Menger probabilistic φ-normed space was introduced by Goleţ in [1].

Let φ be a function defined on the real field $\mathbb{R}$ into itself, with the following properties:

(a) $\varphi(-t) = \varphi(t)$ for every $t \in \mathbb{R}$;

(b) $\varphi(1) = 1$;

(c) φ is strictly increasing and continuous on $[0, \infty)$, $\varphi(0) = 0$ and $\lim_{\alpha\to\infty} \varphi(\alpha) = \infty$;

(d) $\varphi(st) = \varphi(s)\varphi(t)$ for every $t, s > 0$.

An example of such functions is: $\varphi(t) = |t|^p$, $p \in (0, \infty)$ (see [2, Theorem 1.49]).

2010 *Mathematics Subject Classification.* Primary 39B82.

Key words and phrases. Stability, random Banach space, random φ-normed space, lattice, additive functional inequality, additive functional equation, generalized Hyers–Ulam–Rassias stability.

♣This work was supported by Daejin University Rearch Grants in 2012.

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Definition 1.1. A *latticetic random φ-normed space* is a triple $(X, \mu, \wedge)$, where X is a vector space and μ is a mapping from X into D_L^+ (for $x \in X$, the function $\mu(x)$ is denoted by μ_x, and $\mu_x(t)$ is the value μ_x at $t \in \mathbb{R}$) such that the following conditions hold:

(LRN1) $\mu_x(t) = \varepsilon_0(t)$ for all $t > 0$ if and only if $x = 0$;
(LRN2) $\mu_{\alpha x}(t) = \mu_x\left(\frac{t}{\varphi(\alpha)}\right)$ for all x in X, $\alpha \neq 0$ and $t \geq 0$;
(LRN3) $\mu_{x+y}(t+s) \geq_L \wedge(\mu_x(t), \mu_y(s))$ for all $x, y \in X$ and $t, s \geq 0$.

We note that from (LPN2) it follows $\mu_{-x}(t) = \mu_x(t) \quad (x \in X, t \geq 0)$.

It is also worth noting that latticetic random φ-normed spaces include, in a natural way, p-normed spaces ([1, 3]).

Example 1.2. Let $L = [0,1] \times [0,1]$ and operation $\geq_L$ be defined by:

$$L = \{(a_1, a_2) : (a_1, a_2) \in [0,1] \times [0,1] \text{ and } a_1 + a_2 \leq 1\},$$

$$(b_1, b_2) \geq_L (a_1, a_2) \Longleftrightarrow a_1 \leq b_1,\ a_2 \geq b_2, \quad \forall a = (a_1, a_2), b = (b_1, b_2) \in L.$$

Then $(L, \geq_L)$ is a complete lattice (see [4]). In this complete lattice, we denote its units by $0_L = (0,1)$ and $1_L = (1,0)$. Let $(X, \|\cdot\|)$ be a normed space. Let μ be a mapping defined by

$$\mu_x(t) = \Big(\frac{t}{t + \|x\|^p}, \frac{\|x\|^p}{t + \|x\|^p}\Big), \quad \forall t \in \mathbb{R}^+,\ 0 < p \leq 1.$$

Then $(X, \mu, \wedge)$ is a latticetic random φ-normed spaces. Note that, here, $\varphi(\alpha) = \alpha^p$.

Definition 1.3. Let $(X, \mu, \wedge)$ be a latticetic random φ-normed spaces.

(1) A sequence (x_n) in X is said to be *convergent* to x in X if, for every $0 < t \in \mathbb{R}$ the sequence $(\mu_{x_n - x}(t))$ is order convergent to $1_{\mathcal{L}}$.

(2) A sequence (x_n) in X is called *Cauchy sequence* if, for every $0 < t \in \mathbb{R}$ the sequence $(\mu_{x_n - x_m}(t))$ is order convergent to $1_{\mathcal{L}}$ whenever n, m tend to ∞.

(3) A latticetic random φ-normed spaces $(X, \mu, \wedge)$ is said to be *complete* if and only if every Cauchy sequence in X is order convergent to a point in X.

Theorem 1.4. *If $(X, \mu, \wedge)$ is a latticetic random φ-normed space and $\{x_n\}$ is a sequence such that $x_n \to x$, then $\lim_{n\to\infty} \mu_{x_n}(t) = \mu_x(t)$.*

Proof. The proof is the same as classical random normed spaces, see [5]. □

Lemma 1.5. *Let $(X, \mu, \wedge)$ be a latticetic random φ-normed space and $x \in X$. If*

$$\mu_x(t) = C, \quad \textit{for all } t > 0,$$

then $C = 1_{\mathcal{L}}$ and $x = 0$.

Proof. Let $\mu_x(t) = C$ for all $t > 0$. Since $Ran(\mu) \subseteq D_L^+$, we have $C = 1_{\mathcal{L}}$ and by (LRN1) we conclude that $x = 0$. □

The generalized Hyers-Ulam–Rassias stability of the functional inequality (1.1) has been proved by Fechner [6] and Gilányi [7]. Gilányi [8] showed that if f satisfies the functional inequality

$$\| 2f(x) + 2f(y) - f(x-y) \| \leq \| f(x+y) \| \tag{1.1}$$

then f also satisfies the Jordan-von Neumann functional equation

$$2f(x) + 2f(y) = f(x-y) + f(x+y),$$

see also [9]. Park, Cho and Han [10] investigated the Cauchy additive functional inequality

$$\| f(x) + f(y) + f(z) \| \leq \| f(x+y+z) \| \tag{1.2}$$

and the Cauchy-Jensen additive functional inequality

$$\| f(x) + f(y) + 2f(z) \| \leq \| 2f\left(\frac{x+y}{2} + z\right) \| \tag{1.3}$$

and proved the generalized Hyers-Ulam–Rassias stability of the functional inequalities (1.2) and (1.3) in Banach spaces. We also mention here the paper [11]. The stability of the Cauchy additive functional equation in the settings of fuzzy, probabilistic and random normed spaces and random φ-normed spaces has been recently investigated by Mirmostafaee, Mirzavaziri and Moslehian [12, 13], Alsina [14], Miheţ [15], Miheţ and Radu [16] and Miheţ, Saadati and Vaezpour [3, 17, 18].

The aims of this paper are a synthesis of these two theories, probabilistic normed space [5] and vector-lattice-normed space [19, 20] respectively, named by latticetic random φ-normed spaces and to prove the generalized Hyers-Ulam–Rassias stability of the functional inequalities (1.2) and (1.3) in these spaces.

For more details on this preliminary part, the reader is referred to [21], [22], [23], [24], [25], [26], [27].

2. Main results

We start our work with the main result in a latticetic random φ-normed space.

Lemma 2.1. *Let X be a linear space, $(Z, \mu, \wedge)$ be a latticetic random φ-normed space and $f : X \longrightarrow Z$ be a function such that*

$$\mu_{f(x)+f(y)+f(z)}(t) \geq_L \mu_{f(x+y+z)}\left(\frac{t}{\varphi(2)}\right) \quad (x, y, z \in X, t > 0). \tag{2.1}$$

Then f is Cauchy additive, i.e., $f(x+y) = f(x) + f(y)$ for all $x, y \in X$.

Proof. Putting $x = y = z = 0$ in (2.1), we obtain

$$\mu_{3f(0)}(t) \geq_L \mu_{f(0)}\left(\frac{t}{\varphi(3)}\right) \geq_L \mu_{f(0)}\left(\frac{t}{\varphi(2)}\right) \quad (t > 0).$$

By Lemma 1.5, it follows that $f(0) = 0$. Putting $y = -x$ and $z = 0$ in (2.1), one obtains

$$\mu_{f(x)+f(-x)}(t) \geq_L \mu_{f(0)}\left(\frac{t}{\varphi(2)}\right) = \mu_0\left(\frac{t}{\varphi(2)}\right) = 1_{\mathcal{L}} \quad (t > 0),$$

hence

$$f(x) = -f(-x) \quad (x \in X).$$

Putting $z=-x-y$ in (2.1) we deduce that

$$\begin{aligned}\mu_{f(x)+f(y)-f(x+y)}(t) &= \mu_{f(x)+f(y)+f(-x-y)}(t)\\ &\geq_L \mu_{f(0)}\left(\frac{t}{\varphi(2)}\right)=\mu_0\left(\frac{t}{\varphi(2)}\right)=1_{\mathcal{L}}.\end{aligned}$$

and thus, from (LRN1),

$$f(x)+f(y)=f(x+y),\ \forall x,y\in X.$$

□

Similarly one can prove the following

Lemma 2.2. *Let X be a linear space, $(Z,\mu,\wedge)$ be a latticetic random φ-normed space and $f:X\longrightarrow Z$ be a function such that*

$$\mu_{f(x)+f(y)+2f(z)}(t)\geq_L \mu_{2f(\frac{x+y}{2}+z)}\left(\frac{\varphi(2)t}{\varphi(3)}\right)\quad (x,y,z\in X,t>0). \tag{2.2}$$

Then f is Cauchy additive.

Theorem 2.3. *Let X be a linear space, Φ be a mapping from X^3 to D_L^+ ($\Phi(x,y,z)(t)$ is denoted by $\Phi_{x,y,z}(t)$), such that for some $0<\alpha<\varphi(2)$,*

$$\Phi_{2x,2y,2z}(\alpha t)\geq_L \Phi_{x,y,z}(t)\quad (x,y,z\in X,t>0) \tag{2.3}$$

and $(Y,\mu,\wedge)$ be a complete a latticetic random φ-normed space.

If $f:X\to Y$ is an odd mapping satisfying the inequality

$$\wedge(\mu_{f(x)+f(y)+f(z)}(t),\mu_{f(x+y+z)}(t))\geq_L \Phi_{x,y,z}(t)\quad (x,y,z\in X,t>0), \tag{2.4}$$

then there exists a unique Cauchy additive mapping $A:X\to Y$ such that

$$\mu_{f(x)-A(x)}(t)\geq_L \Phi_{x,x,-2x}((\varphi(2)-\alpha)t)\quad (x\in X,t>0). \tag{2.5}$$

Proof. Putting $x=y$ and $z=-2x$ in (2.4) we get

$$\begin{aligned}\mu_{2f(x)-f(2x)}(t) &= \wedge(\mu_{2f(x)-f(2x)}(t),1_{\mathcal{L}})\\ &\geq_L \wedge(\mu_{2f(x)-f(2x)}(t),\mu_{f(0)}(t))\\ &\geq_L \Phi_{x,x,-2x}(t)\quad (x\in X,t>0).\end{aligned} \tag{2.6}$$

From (2.6) we have

$$\mu_{\frac{f(2x)}{2}-f(x)}\left(\frac{t}{\varphi(2)}\right)=\mu_{2f(x)-f(2x)}(t)\geq_L \Phi_{x,x,-2x}(t)\quad (x\in X,t>0). \tag{2.7}$$

Replacing x by 2^nx in (2.7), and using (2.3) we obtain

$$\mu_{\frac{f(2^{n+1}x)}{2^{n+1}}-\frac{f(2^nx)}{2^n}}\left(\frac{t}{\varphi(2^{n+1})}\right)\geq_L \Phi_{2^nx,2^nx,-2^{n+1}x}(t)\quad (x\in X,t>0,n\in\mathbb{N}), \tag{2.8}$$

that is,

$$\begin{aligned}\mu_{\frac{f(2^{n+1}x)}{2^{n+1}}-\frac{f(2^nx)}{2^n}}(t) &\geq_L \Phi_{2^nx,2^nx,-2^{n+1}x}(\varphi(2^{n+1})t) \\ &\geq_L \Phi_{x,x,-2x}\left(\frac{\varphi(2^{n+1})t}{\alpha^n}\right) \quad (x \in X, t > 0, n \in \mathbb{N})\end{aligned} \tag{2.9}$$

Since $\frac{f(2^nx)}{2^n} - f(x) = \sum_{k=0}^{n-1}(\frac{f(2^{k+1}x)}{2^{k+1}} - \frac{f(2^kx)}{2^k})$, by (2.9) we have

$$\mu_{\frac{f(2^nx)}{2^n}-f(x)}\left(t\sum_{k=0}^{n}\frac{\alpha^k}{\varphi(2^{k+1})}\right) \geq_L (\wedge)_{k=0}^{n-1}\Phi_{x,x,-2x}(t) = \Phi_{x,x,-2x}(t)$$

that is,

$$\mu_{\frac{f(2^nx)}{2^n}-f(x)}(t) \geq_L \Phi_{x,x,-2x}\left(\frac{t}{\sum_{k=0}^{n}\frac{\alpha^k}{\varphi(2^{k+1})}}\right) \tag{2.10}$$

By replacing x with 2^mx in (2.10) we obtain:

$$\begin{aligned}\mu_{\frac{f(2^{n+m}x)}{2^{n+m}}-\frac{f(2^mx)}{2^m}}(t) &\geq_L \Phi_{2^mx,2^mx,-2^{m+1}x}\left(\frac{t}{\sum_{k=0}^{n}\frac{\alpha^k}{\varphi(2)^{m+k+1}}}\right) \\ &\geq_L \Phi_{x,x,-2x}\left(\frac{t}{\sum_{k=0}^{n}\frac{\alpha^{m+k}}{\varphi(2)^{m+k+1}}}\right) \\ &\geq_L \Phi_{x,x,-2x}\left(\frac{t}{\sum_{k=m}^{n+m}\frac{\alpha^k}{\varphi(2)^{k+1}}}\right).\end{aligned} \tag{2.11}$$

As $\Phi_{x,x,-2x}\left(\frac{t}{\sum_{k=m}^{n+m}\frac{\alpha^k}{\varphi(2)^{k+1}}}\right)$ tends to $1_{\mathcal{L}}$ as m,n tend to ∞, we conclude that $(\frac{f(2^nx)}{2^n})$ is a Cauchy sequence in $(Y,\mu,\wedge)$. Since $(Y,\mu,\wedge)$ is a complete latticetic random φ-normed space, this sequence converges to some point $A(x) \in Y$. Fix $x \in X$ and put $m = 0$ in (2.11) to obtain

$$\mu_{\frac{f(2^nx)}{2^n}-f(x)}(t) \geq_L \Phi_{x,x,-2x}\left(\frac{t}{\sum_{k=0}^{n}\frac{\alpha^k}{\varphi(2)^{k+1}}}\right), \tag{2.12}$$

from which we obtain for every $t, \delta > 0$

$$\begin{aligned}\mu_{A(x)-f(x)}(t+\delta) &\geq_L \wedge\left(\mu_{A(x)-\frac{f(2^nx)}{2^n}}(\delta), \mu_{\frac{f(2^nx)}{2^n}-f(x)}(t)\right) \\ &\geq_L \wedge\left(\mu_{A(x)-\frac{f(2^nx)}{2^n}}(\delta), \Phi_{x,x,-2x}\left(\frac{t}{\sum_{k=0}^{n}\frac{\alpha^k}{\varphi(2)^{k+1}}}\right)\right).\end{aligned} \tag{2.13}$$

Taking the limit as $n \longrightarrow \infty$ and using (2.13) we get

$$\mu_{A(x)-f(x)}(t+\delta) \geq_L \Phi_{x,x,-2x}(t(\varphi(2)-\alpha)). \tag{2.14}$$

Since δ was arbitrary, by taking $\delta \longrightarrow 0$ one obtains

$$\mu_{A(x)-f(x)}(t) \geq_L \Phi_{x,x,-2x}(t(\varphi(2)-\alpha)).$$

Now, we show that the mapping A is Cauchy additive:

$$\begin{aligned}
\mu_{A(x)+A(y)+A(z)}(t) \quad \geq_L \quad & \wedge\Bigg(\mu_{A(x)-\frac{f(2^n x)}{2^n}}\left(\frac{t\left(1-\frac{1}{\varphi(2)}\right)}{8}\right), \mu_{A(y)-\frac{f(2^n y)}{2^n}}\left(\frac{t\left(1-\frac{1}{\varphi(2)}\right)}{8}\right)\\
&, \quad \mu_{A(z)-\frac{f(2^n z)}{2^n}}\left(\frac{t\left(1-\frac{1}{\varphi(2)}\right)}{8}\right), \mu_{A(x+y+z)-\frac{f(2^n(x+y+z))}{2^n}}\left(\frac{t\left(1-\frac{1}{\varphi(2)}\right)}{8}\right)\\
&, \quad \mu_{\frac{f(2^n(x+y+z))}{2^n}-\frac{f(2^n x)}{2^n}-\frac{f(2^n y)}{2^n}-\frac{f(2^n z)}{2^n}}\left(\frac{t\left(1-\frac{1}{\varphi(2)}\right)}{2}\right)\\
&, \quad \mu_{A(x+y+z)}\left(\frac{t}{\varphi(2)}\right)\Bigg)
\end{aligned} \tag{2.15}$$

for all $x, y, z \in X$ and for all $t > 0$. The first four terms on the right-hand side of the above inequality tend to $1_{\mathcal{L}}$ as $n \longrightarrow \infty$. Also, from (LRN3),

$$\begin{aligned}
&\mu_{\frac{f(2^n(x+y+z))}{2^n}-\frac{f(2^n x)}{2^n}-\frac{f(2^n y)}{2^n}-\frac{f(2^n z)}{2^n}}\left(\frac{t\left(1-\frac{1}{\varphi(2)}\right)}{2}\right)\\
\geq_L \quad & \wedge\Big(\mu_{f(2^n x)+f(2^n y)+f(2^n z)}\Big(\frac{\varphi(2)^n}{4}\Big(1-\frac{1}{\varphi(2)}\Big)t\Big), \mu_{f(2^n(x+y+z))}\Big(\frac{\varphi(2)^n}{4}\Big(1-\frac{1}{\varphi(2)}\Big)t\Big)\\
\geq_L \quad & \Phi_{2^n x, 2^n y, 2^n z}\Big(\frac{\varphi(2)^n}{4}\Big(1-\frac{1}{\varphi(2)}\Big)t\Big)\\
\geq_L \quad & \Phi_{x,y,z}\Big(\frac{\varphi(2)^n}{4\alpha^n}\Big(1-\frac{1}{\varphi(2)}\Big)t\Big),
\end{aligned}$$

that is, the fifth term also tends to $1_{\mathcal{L}}$ when n tends to ∞. Therefore, we have

$$\mu_{A(x)+A(y)+A(z)}(t) \geq_L \mu_{A(x+y+z)}\left(\frac{t}{\varphi(2)}\right),$$

hence by Lemma 2.1 we conclude that the mapping A is Cauchy additive.

To prove the uniqueness of the Cauchy additive function A, assume that there exists a Cauchy additive function $B : X \longrightarrow Y$ which satisfies (2.5). Fix $x \in X$. Clearly $A(2^n x) = 2^n A(x)$ and $B(2^n x) = 2^n B(x)$ for all $n \in \mathbb{N}$. It follows from (2.5) that

$$\begin{aligned}
\mu_{A(x)-B(x)}(t) \quad &= \quad \mu_{\frac{A(2^n x)}{2^n}-\frac{B(2^n x)}{2^n}}(t)\\
&\geq_L \quad \wedge\left(\mu_{\frac{A(2^n x)}{2^n}-\frac{f(2^n x)}{2^n}}\left(\frac{t}{2}\right), \mu_{\frac{B(2^n x)}{2^n}-\frac{f(2^n x)}{2^n}}\left(\frac{t}{2}\right)\right)\\
&\geq_L \quad \Phi_{2^n x, 2^n x, -2^{n+1} x}\Big(\frac{\varphi(2^n)(\varphi(2)-\alpha)t}{2}\Big)\\
&\geq_L \quad \Phi_{x,x,-2x}\left(\left(\frac{\varphi(2)}{\alpha}\right)^n \frac{(\varphi(2)-\alpha)t}{2}\right).
\end{aligned}$$

Since $\alpha < \varphi(2)$, we get

$$\lim_{n\to\infty} \Phi_{x,x,-2x}\left(\left(\frac{\varphi(2)}{\alpha}\right)^n \frac{(\varphi(2)-\alpha)t}{2}\right) = 1_{\mathcal{L}}.$$

Therefore $\mu_{A(x)-B(x)}(t) = 1_{\mathcal{L}}$ for all $t > 0$, whence $A(x) = B(x)$. $\square$

Corollary 2.4. *Consider Example 1.2. If $f : X \to Y$ is a mapping such that, for some $p < 1$,*

$$\begin{aligned}&\wedge(\mu_{f(x)+f(y)+f(z)}(t), \mu_{f(x+y+z)}(t))\\ \geq_L \quad &\Big(\frac{t}{t+(\|x\|^p+\|y\|^p+\|z\|^p)}, \frac{\|x\|^p+\|y\|^p+\|z\|^p}{t+(\|x\|^p+\|y\|^p+\|z\|^p)}\Big) \quad (x, y, z \in X, t > 0),\end{aligned}$$

then there exists a unique Cauchy additive mapping $A : X \to Y$ such that

$$\mu_{f(x)-A(x)}(t) \geq_L \Big(\frac{(2-2^p)t}{(2-2^p)t+(2+2^p)\|x\|^p}, \frac{(2+2^p)\|x\|^p}{(2-2^p)t+(2+2^p)\|x\|^p}\Big).$$

for all $x \in X$ and $t > 0$.

Proof. Let $\Phi : X^3 \longrightarrow D_L^+$ be defined by

$$\Phi_{x,y,z}(t) = \Big(\frac{t}{t+(\|x\|^p+\|y\|^p+\|z\|^p)}, \frac{\|x\|^p+\|y\|^p+\|z\|^p}{t+(\|x\|^p+\|y\|^p+\|z\|^p)}\Big).$$

Then the corollary is followed from Theorem 2.3 with $\alpha = 2^p$. □

Corollary 2.5. *Consider Example 1.2. If $f : X \to Y$ is a mapping such that*

$$\wedge(\mu_{f(x)+f(y)+f(z)}(t), \mu_{f(x+y+z)}(t)) \geq_L \Big(\frac{t}{t+\varepsilon}, \frac{\varepsilon}{t+\varepsilon}\Big) \quad (x, y, z \in X, t > 0)$$

and $f(0) = 0$, then there exists a unique Cauchy additive mapping $A : X \to Y$ such that

$$\mu_{f(x)-A(x)}(t) \geq_L \Big(\frac{t}{t+\varepsilon}, \frac{\varepsilon}{t+\varepsilon}\Big).$$

for all $x \in X$ and $t > 0$.

Proof. Let $\Phi : X^3 \longrightarrow D_L^+$ be defined by

$$\Phi_{x,y,z}(t) = \Big(\frac{t}{t+\varepsilon}, \frac{\varepsilon}{t+\varepsilon}\Big).$$

Then the corollary is followed from Theorem 2.3 with $\alpha = 1$. □

Theorem 2.6. *Let X be a linear space, Φ be a mapping from $X^3 \times [0, \infty)$ to D_L^+ such that for some $0 < \alpha < \varphi(3)$,*

$$\Phi_{3x,3y,3z}(\alpha t) \geq_L \Phi_{x,y,z}(t) \quad (x, y, z \in X, t > 0). \tag{2.16}$$

Let $(Y, \mu, \wedge)$ be a complete latticetic random φ-normed space. If $f : X \to Y$ is an odd mapping such that

$$\wedge(\mu_{f(x)+f(y)+2f(z)}(t), \mu_{f(\frac{x+y}{2}+z)}(t)) \geq_L \Phi_{x,y,z}(t) \quad (x, y, z \in X, t > 0), \tag{2.17}$$

then there exists a unique Cauchy additive mapping $A : X \to Y$ such that

$$\mu_{f(x)-A(x)}(t) \geq_L \Phi_{x,-3x,x}((\varphi(3) - \alpha)t) \quad (x \in X, t > 0). \tag{2.18}$$

Proof. As the proof is similar to that of the preceding theorem, we only sketch it.

Putting $y = -3x$ and $z = x$ in (2.17) we get

$$\mu_{3f(x)-f(3x)}(t) \geq_L \Phi_{x,-3x,x}(t) \quad (x \in X, t > 0). \tag{2.19}$$

From this relation it follows

$$\mu_{\frac{f(3^n x)}{3^n}-f(x)}(t) \geq_L \Phi_{x,-3x,x}\left(\frac{t}{\sum_{k=0}^{n}\frac{\alpha^k}{\varphi(3)^{k+1}}}\right) \tag{2.20}$$

and then, as in the proof of Theorem 2.3,

$$\mu_{\frac{f(3^{n+m}x)}{3^{n+m}}-\frac{f(3^m x)}{3^m}}(t) \geq_L \Phi_{x,-3x,x}\left(\frac{t}{\sum_{k=m}^{n+m}\frac{\alpha^k}{\varphi(3)^{k+1}}}\right),$$

proving that, for every x, $(\frac{f(3^n x)}{3^n})$ is a Cauchy sequence in $(Y,\mu,\wedge)$. Denote $A(x)\in Y$ its limit. From

$$\mu_{\frac{f(3^n x)}{3^n}-f(x)}(t) \geq_L \Phi_{x,-3x,x}\left(\frac{t}{\sum_{k=0}^{n}\frac{\alpha^k}{\varphi(3)^{k+1}}}\right) \tag{2.21}$$

and

$$\begin{aligned}\mu_{A(x)-f(x)}(t+\delta) \quad &\geq_L \quad \wedge\left(\mu_{A(x)-\frac{f(3^n x)}{3^n}}(\delta), \mu_{\frac{f(3^n x)}{3^n}-f(x)}(t)\right) \\ &\geq_L \quad \wedge\left(\mu_{A(x)-\frac{f(3^n x)}{3^n}}(\delta), \Phi_{x,-3x,x}\left(\frac{t}{\sum_{k=0}^{n}\frac{\alpha^k}{\varphi(3)^{k+1}}}\right)\right)\end{aligned} \tag{2.22}$$

we obtain

$$\mu_{A(x)-f(x)}(t) \geq_L \Phi_{x,-3x,x}(t(\varphi(3)-\alpha)).$$

The additivity of A follows from

$$\begin{aligned}\mu_{A(x)+A(y)+2A(z)}(t) \quad &\geq_L \quad \wedge\Bigg(\mu_{A(x)-\frac{f(3^n x)}{3^n}}\left(\frac{\left(1-\frac{\varphi(2)}{\varphi(3)}\right)t}{12}\right), \mu_{A(y)-\frac{f(3^n y)}{3^n}}\left(\frac{\left(1-\frac{\varphi(2)}{\varphi(3)}\right)t}{12}\right) \\ &, \quad \mu_{2A(z)-2\frac{f(3^n z)}{3^n}}\left(\frac{\left(1-\frac{\varphi(2)}{\varphi(3)}\right)t}{12}\right), \mu_{2A(\frac{x+y}{2}+z)-\frac{2f(3^n(\frac{x+y}{2}+z))}{3^n}}\left(\frac{\left(1-\frac{\varphi(2)}{\varphi(3)}\right)t}{12}\right) \\ &, \quad \mu_{\frac{2f(3^n(\frac{x+y}{2}+z))}{3^n}-\frac{f(3^n x)}{3^n}-\frac{f(3^n y)}{3^n}-\frac{2f(3^n z)}{3^n}}\left(\frac{\left(1-\frac{\varphi(2)}{\varphi(3)}\right)2t}{3}\right) \\ &, \quad \mu_{2A(\frac{x+y}{2}+z)}\left(\frac{\varphi(2)t}{\varphi(3)}\right)\Bigg) \quad (x,y,z\in X, t>0)\end{aligned} \tag{2.23}$$

and

$$\begin{aligned}
&\mu_{\frac{2f(3^n(\frac{x+y}{2}+z))}{3^n}-\frac{f(3^n x)}{3^n}-\frac{f(3^n y)}{3^n}-\frac{2f(3^n z)}{3^n}}\left(\frac{\left(1-\frac{\varphi(2)}{\varphi(3)}\right)2t}{3}\right)\\
\geq_L\quad & \wedge\left(\mu_{f(3^n x)+f(3^n y)+2f(3^n z)}\left(\frac{\left(1-\frac{\varphi(2)}{\varphi(3)}\right)\varphi(3)^n t}{3}\right), \mu_{2f(3^n(\frac{x+y}{2}+z))}\left(\frac{\left(1-\frac{\varphi(2)}{\varphi(3)}\right)\varphi(3)^n t}{3}\right)\right)\\
\geq_L\quad & \Phi_{3^n x,3^n y,3^n z}\left(\frac{\left(1-\frac{\varphi(2)}{\varphi(3)}\right)\varphi(3)^n t}{3}\right)\\
\geq_L\quad & \Phi_{x,y,z}\left(\frac{\left(1-\frac{\varphi(2)}{\varphi(3)}\right)\varphi(3)^n t}{3\alpha^n}\right),
\end{aligned}$$

by using Lemma 2.2.

Finally, the uniqueness of the Cauchy additive mapping A subject (2.18) follows from

$$\begin{aligned}
\mu_{A(x)-B(x)}(t) &= \mu_{\frac{A(3^n x)}{3^n}-\frac{B(3^n x)}{3^n}}(t)\\
&\geq_L \wedge\left(\mu_{\frac{A(3^n x)}{3^n}-\frac{f(3^n x)}{3^n}}\left(\frac{t}{2}\right), \mu_{\frac{B(3^n x)}{3^n}-\frac{f(3^n x)}{3^n}}\left(\frac{t}{2}\right)\right)\\
&\geq_L \Phi_{3^n x,-3^{n+1}x,2^n x}\left(\frac{\varphi(3)^n(\varphi(3)-\alpha)}{2}t\right)\\
&\geq_L \Phi_{x,-3x,x}\left(\left(\frac{\varphi(3)}{\alpha}\right)^n\frac{(\varphi(3)-\alpha)t}{2}\right).
\end{aligned}$$

□

Acknowledgements

The authors would like to thank the two referees and the area editor for giving useful comments and suggestions for the improvement of this paper. This work was supported by Daejin University Rearch Grants in 2012.

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J. COMPUTATIONAL ANALYSIS AND APPLICATIONS, VOL. 15, NO.8, 1413-1423, 2013, COPYRIGHT 2013 EUDOXUS PRESS, LLC

On bi-Cubic functional equations

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Abstract. In this paper, we investigate the solution and Hyers-Ulam stability of the following bi-cubic functional equation

$$f(2x+y,2z+w)+f(2x-y,2z-w)=2f(x+y,z+w)+2f(x-y,z-w)+12f(x,z)$$

in Banach spaces.

1. Introduction

We say a functional equation (ξ) is *stable* if any function g satisfying the equation (ξ) approximately is near to a true solution of (ξ). It seems that the stability problem of functional equations had been first raised by Ulam (cf. [13, 14]). In 1941 this problem was solved by Hyers [6] in the case of Banach spaces. This type of stability is called the Hyers–Ulam stability. In 1978, Th. M. Rassias [10] extended the Hyers–Ulam stability (see [5]). This type of stability is called Hyers–Ulam–Rassias stability.

The functional equation

$$h(x+y)+h(x-y)=2h(x)+2h(y) \tag{1.1}$$

is the quadratic functional equation and every solution of (1.1) is said to be a quadratic mapping. The general solution and Hyers–Ulam–Rassias stability of (1.1) are established in [1, 12, 3].

The function $f(x)=ax+bx^2$ satisfies the following functional equation

$$h(x+y+z)+h(x)+h(y)+h(z)=h(x+y)+h(y+z)+h(z+x). \tag{1.2}$$

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[0] 2010 Mathematics Subject Classification: 39B82, 39B52.

[0] Keywords: cubic and bi-cubic functional equation; Hyers-Ulam stability.

Hence the functional equation (1.2) is said to be additive–quadratic. Pl. Kannappan [8] proved that, the function h is quadratic if and only if there exists a unique symmetric bi–additive mapping B such that $h(x) = B(x,x)$ and h is additive–quadratic if and only if there exist a unique symmetric bi–additive mapping B and a unique additive mapping A such that $h(x) = A(x) + B(x,x)$.

The functional equation

$$g(2x+y) + g(2x-y) = 2g(x+y) + 2g(x-y) + 12g(x) \tag{1.3}$$

is called the cubic functional equation and every solution of the cubic functional equation is said to be a cubic function. The function $g(x) = x^3$ satisfies (1.3). Jun and Kim [7] established the general solution and Hyers–Ulam–Rassias stability of the functional equation (1.3). They proved that a function g between real vector spaces X and Y is a cubic function if there exists a unique function $C : X \times X \times X \longrightarrow Y$ such that $g(x) = C(x,x,x)$ for all $x \in X$, where C is symmetric for each fixed one variable and additive for each fixed two variables. The mapping C is given by

$$C(x,y,z) = \frac{1}{24}(g(x+y+z) - g(-x+y+z) - g(x+y-z) - g(x-y+z))$$

for all $x,y,z \in X$.
The stability problem of various cubic functional equations have been extensively investigated by number of authors [4, 7, 9, 11].

The functional equation

$$f(2x+y, 2z+w) + f(2x-y, 2z-w) = 2f(x+y, z+w) + 2f(x-y, z-w) + 12f(x,z) \tag{1.4}$$

is called the bi-cubic functional equation and every solution of (1.4) is called a bi–cubic function. For instance, let X be a real algebra. If the mapping $f : X \times X \longrightarrow X$ is given by

$$f(x,y) = ax^3 + bx^2y + cxy^2 + dy^3$$

for all $x,y \in X$ and $a,b,c,d \in \mathbb{R}$, then it is easy to show that f is a bi–cubic function.

In this paper, we investigate the general solution and the Hyers–Ulam stability of the functional equation (1.4) and generalize the results in [2].

2. Main Results

In this section, we investigate solution and stability of the bi-cubic functional equation (1.4). Moreover, we establish the Hyers–Ulam stability of (1.4).

We start our work by the following theorem.

Theorem 2.1. *Let X and Y be real vector spaces and $f : X \times X \to Y$ be a mapping satisfying* (1.4)*. We define $g : X \to Y$ by $g(x) = f(x,x)$ for all $x \in X$. Then g satisfies the functional equation* (1.3).

Proof. We have

$$\begin{aligned} g(2x+y)+g(2x-y) &= f(2x+y,2x+y)+f(2x-y,2x-y) \\ &= 2f(x+y,x+y)+2f(x-y,x-y)+12f(x,x) \\ &= 2g(x+y)+2g(x-y)+12g(x). \end{aligned}$$

□

Example 2.2. *Assume that X is a real algebra and $D : X \longrightarrow X$ is a derivation on X. We define a mapping $f : X \times X \longrightarrow X$ by*

$$f(x,y) = D(x^2y) = x^2D(y) + D(x^2)y = x^2D(y) + xD(x)y + D(x)xy$$

for all $x, y \in X$. It is easy to see that f satisfies (1.4). Now we define $g : X \to X$ by

$$g(x) = x^2D(x) + (Dx^2)x = x^2Dx + x(Dx)x + D(x)x^2.$$

It follows from theorem 2.1 that g satisfies (1.3).

Assume that X and Y are real vector spaces. The mapping $f : X \times X \to Y$ is called a two variables odd function, if $f(-x,-y) = -f(x,y)$ for all $x, y \in X$.

Remark 2.3. The bi-cubic function f that satisfies (1.4) is a two variables odd function. Putting $x = y = z = w = 0$ in (1.4), we get $f(0,0) = 0$. Letting $x = z = 0$ in (1.4), gives $f(y,w) + f(-y,-w) = 2f(y,w) + 2f(-y,-w)$. Hence $f(-y,-w) = -f(y,w)$.

The cubic functional equation (1.3) induces the bi-cubic functional equation (1.4) with an additional condition.

Theorem 2.4. *Assume that $a, b, c, d \in \mathbb{R}$ and X, Y are real vector spaces. Suppose $g : X \to Y$ is a function satisfying (1.3). If $f : X \times X \to Y$ is the mapping given by*

$$f(x,y) = ag(x) + \frac{b}{6}(g(x+y) - g(x-y) - 2g(y)) + \frac{c}{6}(g(x+y) + g(x-y) - 2g(x)) + dg(y) \quad (2.1)$$

then f satisfies the equality (1.4). Furthermore, $f(x,x) = g(x)$ if and only if $a+b+c+d = 1$.

Proof. Because g satisfies (1.3), we get from (2.1) that

$$\begin{aligned}
f(2x+y,2z+w) &+ f(2x-y,2z-w)\\
&= ag(2x+y)+\frac{b}{6}(g(2x+y+2z+w)-g(2x+y-2z-w)-2g(2z+w))\\
&+ \frac{c}{6}(g(2x+y+2z+w)+g(2x+y-2z-w)-2g(2x+y))+dg(2z+w)\\
&+ ag(2x-y)+\frac{b}{6}(g(2x-y+2z-w)-g(2x-y-2z+w)-2g(2z-w))\\
&+ \frac{c}{6}(g(2x-y+2z-w)+g(2x-y-2z+w)-2g(2x-y))+dg(2z-w)\\
&= a(2g(x+y)+2g(x-y)+12g(x))\\
&+ \frac{b}{6}(2g(x+z+y+w)+2g(x+z-y-w)+12g(x+z)-2g(x-z+y-w)\\
&- 2g(x-z-y+w)-12g(x-z)-4g(z+w)-4g(z-w)-24g(z))\\
&+ \frac{c}{6}(2g(x+z+y+w)+2g(x+z-y-w)+12g(x+z)+2g(x-z+y-w)\\
&+ 2g(x-z-y+w)+12g(x-z)-4g(x+y)-4g(x-y)-24g(x))\\
&+ d(2g(z+w)+2g(z-w)+12g(z))\\
&= 2ag(x+y)+\frac{b}{6}(2g(x+y+z+w)-2g(x+y-z-w)-4g(z+w))\\
&+ \frac{c}{6}(2g(x+y+z+w)+2g(x+y-z-w)-4g(x+y))+2dg(z+w)\\
&+ 2ag(x-y)+\frac{b}{6}(2g(x-y+z-w)-2g(x-y-z+w)-4g(z-w))\\
&+ \frac{c}{6}(2g(x-y+z-w)+2g(x-y-z+w)-4g(x-y))+2dg(z-w)\\
&+ 12ag(x)+\frac{b}{6}(12g(x+z)-12g(x-z)-24g(z))\\
&+ \frac{c}{6}(12g(x+z)+12g(x-z)-24g(x))+12dg(z)\\
&= 2f(x+y,z+w)+2f(x-y,z-w)+12f(x,z).
\end{aligned}$$

for all $x,y\in X$. Letting $x=y=0$ in (1.3), we have $g(0)=0$. Letting $y=0$ in (1.3), we obtain that $g(2x)=8g(x)$ for all $x\in X$. Therefore, we see that

$$\begin{aligned}
f(x,x) &= ag(x)+\frac{b}{6}(g(2x)-2g(x))+\frac{c}{6}(g(2x)-2g(x))+dg(x)\\
&= ag(x)+bg(x)+cg(x)+dg(x)=(a+b+c+d)g(x)
\end{aligned}$$

for all $x\in X$. □

In the following theorem, we find two necessary conditions for (1.4).

Theorem 2.5. *Let X, Y be real vector spaces and $f : X \times X \to Y$ be a mapping which satisfies (1.4). Then the following equalities hold.*

$$f(x+t-y, z+p-w) + f(x+t+y, z+p+w) = f(t+y, p+w) + f(t-y, p-w) + f(x+y, z+w) + f(x-y, z-w) + 2f(x+t, z+p) - 2f(x,z) - 2f(t,p) \tag{2.2}$$

$$f(x+t+y, z+p+w) - f(x-t-y, z-p-w) = f(x+t, z+p) - f(x-t, z-p) + f(x+y, z+w) - f(x-y, z-w) + 2f(y+t, p+w) - 2f(y,w) - 2f(t,p) \tag{2.3}$$

for all $x, y, z, p, t, w \in X$.

Proof. Setting $y = w = 0$ in (1.4), we obtain that $f(2x, 2z) = 8f(x,z)$. Replacing y by $2y$ and w by $2w$ in (1.4) and by using Remark 2.3, we get

$$f(2y+x, 2w+z) - f(2y-x, 2w-z) = 4f(x+y, z+w) + 4f(x-y, z-w) - 6f(x,z) \tag{2.4}$$

for all $x, y, z, w \in X$. Interchange x with y and z with w in (2.4), we get

$$f(2x+y, 2z+w) - f(2x-y, 2z-w) = 4f(x+y, z+w) - 4f(x-y, z-w) - 6f(y,w) \tag{2.5}$$

for all $x, y, z, w \in X$. By adding (1.4), (2.5), we have

$$f(2x+y, 2z+w) = 3f(x+y, z+w) - f(x-y, z-w) + 6f(x,z) - 3f(y,w) \tag{2.6}$$

for all $x, y, z, w \in X$. If we subtract (2.5) from (1.4), we get

$$f(2x-y, 2z-w) = -f(x+y, z+w) + 3f(x-y, z-w) + 6f(x,z) + 3f(y,w) \tag{2.7}$$

for all $x, y, z, w \in X$. We substitute $x+t$ in place of x and $z+p$ in place of z in (2.7) to get

$$f(2x+2t-y, 2z+2p-w) = -f(x+t+y, z+p+w) + 3f(x+t-y, z+p-w) + 6f(x+t, z+p) + 3f(y,w) \tag{2.8}$$

for all $x, y, z, t, p, w \in X$. We substitute $x+y$ in place of y and $z+w$ in place of w in (2.8) to obtain

$$f(x+2t-y, z+2p-w) = -f(2x+t+y, 2z+p+w) + 3f(t-y, p-w) + 6f(x+t, z+p) + 3f(x+y, z+w) \tag{2.9}$$

for all $x, y, z, t, p, w \in X$. We substitute t in place of x, $x-y$ in place of y, p in place of z and $z-w$ in place of w in (2.6), to get

$$f(2t+x-y, 2p+z-w) = 3f(t+x-y, p+z-w) - f(t-x+y, p-z+w) + 6f(t,p) - 3f(x-y, z-w) \tag{2.10}$$

for all $x, y, z, t, p, w \in X$. We substitute $t+y$ in place of y and $p+w$ in place of w in (2.6) to obtain

$$f(2x+t+y, 2z+p+w) = 3f(x+t+y, z+p+w) - f(x-t-y, z-p-w) + 6f(x,z) - 3f(t+y, p+w) \tag{2.11}$$

for all $x, y, z, t, p, w \in X$. Now by using (2.10) and (2.11) in (2.9), and dividing by 3 and oddness of f we obtain (2.2). If in the equality (2.2) interchange x with y and z with w and using oddness of f, we obtain the equality (2.3). This completes the proof of Theorem. □

Lemma 2.6. *Let X, Y be real vector spaces. Then $f : X \to Y$ is a solution of (1.2) if and only if it satisfies the following equation*

$$f(2x + y) + 2f(x) + f(y) = f(2x) + 2f(x + y) \tag{2.12}$$

for all $x, y \in X$.

Proof. Putting $x = z$ in (1.2) we obtain (2.12). Conversely, suppose that f satisfies the functional equation (2.12). We show that f satisfies (1.2). Interchanging x with y in (2.12), we get

$$f(2y + x) = f(2y) + 2f(x + y) - 2f(y) - f(x) \tag{2.13}$$

for all $x, y \in X$. Replacing y by $2y$ in (2.12), we have

$$f(2x + 2y) = f(2x) + 2f(2y + x) - 2f(x) - f(2y) \tag{2.14}$$

for all $x, y \in X$. It follows from (2.13) and (2.14) that

$$f(2x + 2y) = f(2x) + f(2y) + 4f(x + y) - 4f(y) - 4f(x) \tag{2.15}$$

for all $x, y \in X$. Replacing x by $x + z$ in (2.12), we obtain

$$f(2x + 2z + y) = f(2x + 2z) + 2f(x + y + z) - 2f(x + z) - f(y) \tag{2.16}$$

for all $x, y, z \in X$. Interchanging z with y in (2.16) to get

$$f(2x + 2y + z) = f(2x + 2y) + 2f(x + y + z) - 2f(x + y) - f(z) \tag{2.17}$$

for all $x, y, z \in X$. By employing (2.17) and (2.15), we have

$$f(2x + 2y + z) = 2f(x + y + z) + 2f(x + y) + f(2x) + f(2y) - 4f(x) - 4f(y) - f(z) \tag{2.18}$$

for all $x, y, z \in X$. Replacing x by $x + z$ in (2.13) to obtain

$$f(2y + x + z) = f(2y) + 2f(x + y + z) - 2f(y) - f(x + z) \tag{2.19}$$

for all $x, y, z \in X$. Replacing y by $y + z$ in (2.12), we get

$$f(2x + y + z) = f(2x) + 2f(x + y + z) - 2f(x) - f(y + z) \tag{2.20}$$

for all $x, y, z \in X$. Replacing y by $2y$ in (2.20) to get

$$f(2x + 2y + z) = f(2x) + 2f(2y + x + z) - 2f(x) - f(2y + z) \tag{2.21}$$

for all $x, y, z \in X$. Replacing x by z in (2.13), we have

$$f(2y + z) = f(2y) + 2f(z + y) - 2f(y) - f(z) \tag{2.22}$$

for all $y, z \in X$. By applying (2.18), (2.19) and (2.22) in (2.21) and divide by 2, we obtain (1.2). □

Theorem 2.7. *Let X, Y be real vector spaces. If a mapping $f : X \times X \to Y$ satisfies (1.4), then there exist mappings $S_1, S_2 : X \times X \times X \longrightarrow Y$ and $g : X \times X \longrightarrow Y$ such that*

$$f(x, y) = S_1(x, x, x) + g(x, y) + S_2(y, y, y) \tag{2.23}$$

for all $x, y \in X$. Where S_1, S_2 are symmetric for each fixed one variable and additive for fixed two variables, and g is an additive-quadratic for each fixed one variable.

Proof. Suppose that f is a solution of (1.4). Define $f_1, f_2 : X \to Y$ by $f_1(x) = f(x, 0)$, $f_2(y) = f(0, y)$. One can easily verify that f_1, f_2 are cubic. By [7] there exist two mappings $S_1, S_2 : X \times X \times X \to Y$ such that $f_1(x) = S_1(x, x, x)$ and $f_2(y) = S_2(y, y, y)$ for all $x, y \in X$ and S_1, S_2 have the properties mentioned in the theorem. Define $g : X \times X \to Y$ by $g(x, y) = f(x, y) - f(x, 0) - f(0, y)$. We show that g is an additive-quadratic for each fixed y. Putting $y = z = p = 0$ In (2.3) to get

$$\begin{aligned} f(x+t, w) - f(x-t, -w) = f(x+t, 0) - f(x-t, 0) + \\ f(x, w) - f(x, -w) + 2f(t, w) - 2f(0, w) - 2f(t, 0) \end{aligned} \tag{2.24}$$

for all $x, t, w \in X$. Interchanging w by y in (2.24) to obtain

$$\begin{aligned} f(x+t, y) - f(x-t, -y) - 2f(t, y) - f(x+t, 0) + \\ f(x-t, 0) + 2f(t, 0) = f(x, y) - 2f(0, y) - f(x, -y) \end{aligned} \tag{2.25}$$

for all $x, t, y \in X$. It follows from (2.25) that

$$\begin{aligned} (3f(x+t, y) - f(x-t, -y) + 6f(x, 0) - 3f(t, y)) - (3f(x+t, 0) - f(x-t, 0) + \\ 6f(x, 0) - 3f(t, 0)) + 2f(x, y) - 2f(x, 0) + f(t, y) - f(t, 0) - f(0, y) = \\ 2f(x+t, y) - 2f(x+t, 0) + (3f(x, y) - f(x, -y) + 6f(x, 0) - 3f(0, y)) - 8f(x, 0) \end{aligned} \tag{2.26}$$

for all $x, t, y \in X$. By applying (2.6) and (2.26), we obtain

$$\begin{aligned} f(2x+t, y) - f(2x+t, 0) - f(0, y) + 2f(x, y) - 2f(x, 0) - 2f(0, y) + f(t, y) - f(t, 0) - \\ f(0, y) = 2f(x+t, y) - 2f(x+t, 0) - 2f(0, y) + f(2x, y) - f(2x, 0) - f(0, y) \end{aligned} \tag{2.27}$$

for all $x, t, y \in X$. It follows from definition of g and (2.27) that

$$g(2x+t, y) + 2g(x, y) + g(t, y) = 2g(x+t, y) + g(2x, y)$$

for all $x, t, y \in X$. Hence for fixed y, by Lemma 2.6, $g(., y)$ is an additive-quadratic mapping. By the same method, we prove that for fixed x, $g(x, .)$ is an additive-quadratic mapping. $\square$

Theorem 2.8. *Let X, Y be real vector spaces. If we define $f : X \times X \to Y$ by*

$$f(x, y) = S_1(x, x, x) + B_1(x, x)A_1(y) + B_2(y, y)A_2(x) + S_2(y, y, y), \tag{2.28}$$

then f satisfies (1.4), where $S_1, S_2 : X \times X \times X \to Y$ are symmetric functions for each fixed one variable and additive for fixed two variables and $B_1, B_2 : X \times X \to Y$ are symmetric functions and additive for each fixed one variable and $A_1, A_2 : X \to Y$ are additive mappings.

Proof. It follows from (2.28) that

$$\begin{aligned}
f(2x+y,2z+w) &+ f(2x-y,2z-w) = 16S_1(x,x,x)+12S_1(x,y,y)\\
&+ 16B_1(x,x)A_1(z)+4B_1(y,y)A_1(z)++8B_1(x,y)A_1(w)\\
&+ 16B_2(z,z)A_2(x)+4B_2(w,w)A_2(x)+8B_2(z,w)A_2(y)\\
&+ 16S_2(z,z,z)+12S_2(z,w,w)\\
&= 2S_1(x,x,x)+6S_1(x,x,y)+6S_1(x,y,y)+2S_1(y,y,y)\\
&+ 2B_1(x,x)A_1(z)+4B_1(x,y)A_1(z)+2B_1(y,y)A_1(z)\\
&+ 2B_1(x,x)A_1(w)+4B_1(x,y)A_1(w)+2B_1(y,y)A_1(w)\\
&+ 2B_2(z,z)A_2(x)+4B_2(z,w)A_2(x)+2B_2(w,w)A_2(x)\\
&+ 2B_2(z,z)A_2(y)+4B_2(z,w)A_2(y)+2B_2(w,w)A_2(y)\\
&+ 2S_2(z,z,z)+6S_2(z,z,w)+6S_2(z,w,w)+2S_2(w,w,w)\\
&+ 2S_1(x,x,x)-6S_1(x,x,y)+6S_1(x,y,y)-2S_1(y,y,y)\\
&+ 2B_1(x,x)A_1(z)-4B_1(x,y)A_1(z)+2B_1(y,y)A_1(z)\\
&- 2B_1(x,x)A_1(w)+4B_1(x,y)A_1(w)-2B_1(y,y)A_1(w)\\
&+ 2B_2(z,z)A_2(x)-4B_2(z,w)A_2(x)+2B_2(w,w)A_2(x)\\
&- 2B_2(z,z)A_2(y)+4B_2(z,w)A_2(y)-2B_2(w,w)A_2(y)\\
&+ 2S_2(z,z,z)-6S_2(z,z,w)+6S_2(z,w,w)-2S_2(w,w,w)\\
&+ 12S_1(x,x,x)+12B_1(x,x)A_1(z)+12B_2(z,z)A_2(x)+12S_2(z,z,z)\\
&= 2f(x+y,z+w)+2f(x-y,z-w)+12f(x,z).
\end{aligned}$$

for all $x,y,z,w \in X$. This completes the proof. □

Theorem 2.9. *Let X, Y be unital real algebras. Let $f : X \times X \to Y$ be a mapping satisfying (1.4). Define $g : X \times X \to Y$ by $g(x,y) = f(x,y) - f(x,0) - f(0,y)$ and $A_1, A_2, h_1, h_2 : X \to Y$ by*

$$A_1(y) = g(1,y) + g(-1,y) \qquad A_2(x) = g(x,1) + g(x,-1)$$

$$h_1(x) = \frac{1}{2}(g(x,1) - g(x,-1)) \qquad h_2(y) = \frac{1}{2}(g(1,y) - g(-1,y))$$

for all $x,y \in X$. Then h_1, h_2 are quadratic mappings and A_1, A_2 are additive.

Proof. First, we show that A_2 is additive. In (2.2), we put $y = z = p = 0, w = 1$, and we obtain

$$\begin{aligned}
f(x+t,-1)+f(x+t,1) &= f(t,1)+f(t,-1)+f(x,1)+\\
&f(x,-1)+2f(x+t,0)-2f(x,0)-2f(t,0) \qquad (2.29)
\end{aligned}$$

for all $t, x \in X$. By Remark 2.3 and (2.29), we conclude that

$$\begin{aligned} f(x+t,-1) - f(x+t,0) - f(0,-1) + f(x+t,1) - f(x+t,0) - f(0,1) = \\ f(x,1) - f(x,0) - f(0,1) + f(x,-1) - f(x,0) - f(0,-1) + \\ f(t,1) - f(t,0) - f(0,1) + f(t,-1) - f(t,0) - f(0,-1) \end{aligned} \tag{2.30}$$

for all $t, x \in X$. By definition of g and (2.30), we get

$$g(x+t,-1) + g(x+t,1) = g(x,1) + g(x,-1) + g(t,1) + g(t,-1) \tag{2.31}$$

for all $x, t \in X$. We conclude that A_2 is additive. Similarly, it is proved that A_1 is additive. Now we show that h_1 is a quadratic mapping. In (2.3) putting $y = z = p = 0, w = -1$, we have

$$\begin{aligned} f(x+t,-1) - f(x-t,1) = f(x+t,0) - f(x-t,0) + \\ f(x,-1) - f(x,1) + 2f(t,-1) - 2f(0,-1) - 2f(t,0) \end{aligned} \tag{2.32}$$

for all $x, t \in X$. Putting $y = z = p = 0, w = 1$ in (2.3) to get

$$\begin{aligned} f(x+t,1) - f(x-t,-1) = f(x+t,0) - f(x-t,0) + \\ f(x,1) - f(x,-1) + 2f(t,1) - 2f(0,1) - 2f(t,0) \end{aligned} \tag{2.33}$$

for all $x, t \in X$. If we subtract (2.32) from (2.33), it follows that

$$\begin{aligned} f(x+t,1) - f(x+t,-1) + f(x-t,1) - f(x-t,-1) = \\ 2(f(x,1) + f(t,1) - f(x,-1) - f(t,-1) - f(0,1) + f(0,-1)) \end{aligned} \tag{2.34}$$

for all $x, t \in X$. By using of (2.34), we conclude that

$$\begin{aligned} f(x+t,1) - f(x+t,0) - f(0,1) - f(x+t,-1) + f(x+t,0) + f(0,-1) + \\ f(x-t,1) - f(x-t,0) - f(0,1) - f(x-t,-1) + f(x-t,0) + f(0,-1) = \\ 2(f(x,1) - f(x,0) - f(0,1) - f(x,-1) + f(x,0) + f(0,-1)) + \\ 2(f(t,1) - f(t,0) - f(0,1) - f(t,-1) + f(t,0) + f(0,-1)) \end{aligned} \tag{2.35}$$

for all $x, t \in X$. By definition of g and (2.35), we have

$$\begin{aligned} g(x+t,1) - g(x+t,-1) + g(x-t,1) - g(x-t,-1) = \\ 2(g(x,1) - g(x,-1) + g(t,1) - g(t,-1)) \end{aligned} \tag{2.36}$$

for all $x, t \in X$. By dividing both sides of (2.36) by 2, we conclude that h_1 is a quadratic. similarly, we can show that h_2 is quadratic. □

Now by using the idea of Gavruta [5], we exhibit the stability of functional equation (1.4). In the sequel we assume that X is a real vector space and Y is a real Banach space. We define the difference operator $D_f : X^4 \longrightarrow Y$ by

$$\begin{aligned} D_f(x,y,z,w) = f(2x+y, 2z+w) + f(2x-y, 2z-w) - \\ 2f(x+y, z+w) - 2f(x-y, z-w) - 12f(x,z) \end{aligned}$$

for all $x, y, z, w \in X$

Theorem 2.10. *Let $f : X \times X \longrightarrow Y$ be a two variables odd function and let*

$$\|D_f(x, y, z, w)\| \leq \varepsilon \tag{2.37}$$

for all $x, y, z, w \in X$. Then the limit

$$T(x, z) := \lim_{n\to\infty} \frac{1}{8^n} f(2^n x, 2^n z)$$

exists for all $x, z \in X$ and $T : X \times X \to Y$ is a unique bi-cubic function satisfying

$$\|f(x, z) - T(x, z)\| \leq \frac{\varepsilon}{14} \tag{2.38}$$

for all $x, z \in X$.

Proof. Putting $y = w = 0$ in (2.37), we get

$$\|f(2x, 2z) - 8f(x, z)\| \leq \frac{\varepsilon}{2} \tag{2.39}$$

for all $x, z \in X$. Dividing both sides of (2.39) by 8, we obtain

$$\|\frac{1}{8} f(2x, 2z) - f(x, z)\| \leq \frac{\varepsilon}{2} \times \frac{1}{8} \tag{2.40}$$

for all $x, z \in X$. If we replace x and z by $2^j x$ and $2^j z$ respectively, in (2.40), and then divide both sides of inequality by 8^j, we get

$$\|\frac{1}{8^{j+1}} f(2^{j+1}x, 2^{j+1}z) - \frac{1}{8^j} f(2^j x, 2^j z)\| \leq \frac{\varepsilon}{2} \times \frac{1}{8^{j+1}} \tag{2.41}$$

for all $x, z \in X$. It follows from (2.41) that

$$\|\frac{1}{8^m} f(2^m x, 2^m z) - \frac{1}{8^k} f(2^k x, 2^k z)\| \leq \frac{\varepsilon}{2} \sum_{j=k}^{m-1} \frac{1}{8^{j+1}} \tag{2.42}$$

for all non-negative integers m, k with $k < m$ and all $x, z \in X$. By (2.42), the sequence $\{\frac{1}{8^j} f(2^j x, 2^j z)\}$ is a Cauchy sequence for all $x, z \in X$. By completeness of Y, the sequence $\{\frac{1}{8^j} f(2^j x, 2^j z)\}$ converges for all $x, z \in X$. Define $T : X \times X \to Y$ by $T(x, z) = \lim_{n\to\infty} \frac{1}{8^n} f(2^n x, 2^n z)$ for all $x, z \in X$. Then we have $T(2x, 2z) = 8T(x, z)$. It follows from (2.37) that

$$\|D_T(x, y, z, w)\| = \lim_{n\to\infty} \frac{1}{8^n} \|f(2^{n+1}x + 2^n y, 2^{n+1}z + 2^n w) + f(2^{n+1}x - 2^n y, 2^{n+1}z - 2^n w) - 2f(2^n x + 2^n y, 2^n z + 2^n w) - 2f(2^n x - 2^n y, 2^n z - 2^n w) - 12f(2^n x, 2^n z)\| \leq \lim_{n\to\infty} \frac{\varepsilon}{8^n} = 0 \tag{2.43}$$

for all $x, y, z, w \in X$. Hence by (2.43) the function T satisfies the equation (1.4). Thus T is a bi-cubic function. Letting $k = 0$ and passing the limit $m \to \infty$ in (2.42) we receive (2.38).

To prove the uniqueness of T, suppose that $T' : X \times X \to Y$ is another bi-cubic function satisfying (2.38). We have

$$\|T(x,z) - T'(x,z)\| = \lim_{n\to\infty} \frac{1}{8^n}\|T(2^n x, 2^n z) - T'(2^n x, 2^n z)\|$$

$$\leq \lim_{n\to\infty} \frac{1}{8^n}\|f(2^n x, 2^n z) - T(2^n x, 2^n z)\| + \lim_{n\to\infty} \frac{1}{8^n}\|f(2^n x, 2^n z) - T'(2^n x, 2^n z)\|$$

$$\leq \lim_{n\to\infty} \frac{\varepsilon}{7} \times \frac{1}{8^n} = 0$$

for all $x, z \in X$. This means that $T = T'$. □

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J. COMPUTATIONAL ANALYSIS AND APPLICATIONS, VOL. 15, NO.8, 1424-1429, 2013, COPYRIGHT 2013 EUDOXUS PRESS, LLC

A note on the second kind generalized q-Euler polynomials

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Abstract. In this paper we introduce the second kind generalized q-Euler numbers $E_{n,\chi,q}$ and polynomials $E_{n,\chi,q}(x)$. We obtain the Witt-type formulae of the second kind generalized q-Euler numbers $E_{n,\chi,q}$ and polynomials $E_{n,\chi,q}(x)$ attached to χ.

Key words: The second kind Euler numbers and polynomials, the second kind q-Euler numbers and q-Euler polynomials, the second kind generalized q-Euler numbers and polynomials

1. INTRODUCTION

Throughout this paper we use the following notations. By $\mathbb{Z}_p$ we denote the ring of p-adic rational integers, $\mathbb{Q}$ denotes the field of rational numbers, $\mathbb{Q}_p$ denotes the field of p-adic rational numbers, $\mathbb{C}$ denotes the complex number field, and $\mathbb{C}_p$ denotes the completion of algebraic closure of $\mathbb{Q}_p$. Let ν_p be the normalized exponential valuation of $\mathbb{C}_p$ with $|p|_p = p^{-\nu_p(p)} = p^{-1}$. When one talks of q-extension, q is considered in many ways such as an indeterminate, a complex number $q \in \mathbb{C}$, or p-adic number $q \in \mathbb{C}_p$. If $q \in \mathbb{C}$ one normally assume that $|q| < 1$. If $q \in \mathbb{C}_p$, we normally assume that $|q-1|_p < p^{-\frac{1}{p-1}}$ so that $q^x = \exp(x \log q)$ for $|x|_p \leq 1$.

Let $UD(\mathbb{Z}_p)$ be the space of uniformly differentiable function on $\mathbb{Z}_p$. For $g \in UD(\mathbb{Z}_p)$ the fermionic p-adic invariant q-integral on $\mathbb{Z}_p$ is defined by Kim as follows:

$$I_{-q}(g) = \int_{\mathbb{Z}_p} g(x) d\mu_{-q}(x) = \lim_{N\to\infty} \frac{1}{[p^N]_{-q}} \sum_{x=0}^{p^N-1} g(x)(-q)^x, \text{ see [3, 4] }. \tag{1.1}$$

If we take $g_n(x) = g(x+n)$ in (1.1), then we see that

$$q^n I_q(g_n) + (-1)^{n-1} I_q(g) = [2]_q \sum_{l=0}^{n-1} (-1)^{n-1-l} q^l g(l). \tag{1.2}$$

Let a fixed positive integer d with $(p,d) = 1$, set

$$\begin{aligned} X &= X_d = \varprojlim_N (\mathbb{Z}/dp^N\mathbb{Z}), X_1 = \mathbb{Z}_p, \\ X^* &= \bigcup_{\substack{0<a<dp \\ (a,p)=1}} a + dp\mathbb{Z}_p, \\ a + dp^N\mathbb{Z}_p &= \{x \in X \mid x \equiv a \pmod{dp^N}\}, \end{aligned}$$

where $a \in \mathbb{Z}$ satisfies the condition $0 \leq a < dp^N$.

It is easy to see that

$$I_{-q}(g) = \int_X g(x) d\mu_{-q}(x) = \int_{\mathbb{Z}_p} g(x) d\mu_{-q}(x). \tag{1.3}$$

For $g \in UD(\mathbb{Z}_p)$, the fermionic p-adic invariant integral on $\mathbb{Z}_p$ is defined by

$$I_{-1}(g) = \int_X g(x) d\mu_{-1}(x) = \int_{\mathbb{Z}_p} g(x) d\mu_{-1}(x) = \lim_{N\to\infty} \sum_{x=0}^{p^N-1} g(x)(-1)^x. \tag{1.4}$$

If we take $g_n(x) = g(x+n)$ in (1.4), then we see that

$$I_{-1}(g_n) = (-1)^n I_{-1}(g) + 2\sum_{l=0}^{n-1}(-1)^{n-1-l}g(l). \tag{1.5}$$

First, we introduce the second kind Euler numbers and Euler polynomials(see [5]). Ryoo [5] investigated the zeros of the second kind Euler polynomials $E_n(x)$. The second kind Euler numbers E_n are defined by the generating function:

$$F(t) = \frac{2e^t}{e^{2t}+1} = \sum_{n=0}^{\infty} E_n \frac{t^n}{n!} \quad (|t| < \frac{\pi}{2}),$$

where we use the technique method notation by replacing E^n by $E_n (n \geq 0)$ symbolically. We consider the second kind Euler polynomials $E_n(x)$ as follows:

$$F(x,t) = \frac{2e^t}{e^{2t}+1}e^{xt} = \sum_{n=0}^{\infty} E_n(x)\frac{t^n}{n!}. \tag{1.6}$$

Note that $E_n(x) = \sum_{k=0}^{n}\binom{n}{k}E_k x^{n-k}$. In the special case $x = 0$, we define $E_n(0) = E_n$.

In [8], we observed the zeros of the second kind q-Euler polynomials $E_{n,q}(x)$. The second kind q-Euler numbers $E_{n,q}$ are defined by the generating function:

$$F_q(t) = \frac{2e^t}{qe^{2t}+1} = \sum_{n=0}^{\infty} E_{n,q}\frac{t^n}{n!}, \tag{1.7}$$

We consider the second kind q-Euler polynomials $E_{n,q}(x)$ as follows:

$$F_q(x,t) = \frac{2e^t}{qe^{2t}+1}e^{xt} = \sum_{n=0}^{\infty} E_{n,q}(x)\frac{t^n}{n!}. \tag{1.8}$$

Many mathematicians have studied Euler numbers and Euler polynomials(see [1-10]). The purpose of this paper is to construct the second kind generalized q-Euler polynomials $E_{n,\chi,q}(x)$ attached to χ and derive a new l-series which interpolates the second kind generalized q-Euler polynomials $E_{n,\chi,q}(x)$.

2. The second kind generalized q-Euler numbers and polynomials

In this section, our goal is to give generating functions of the second kind generalized q-Euler numbers and polynomials. These numbers will be used to prove the analytic continuation of the l-series. Let q be a complex number with $|q| < 1$. Let χ be Dirichlet's character with conductor $d \in \mathbb{N}$ with $d \equiv 1 (\text{mod } 2)$. Then the second kind generalized q-Euler numbers associated with associated with χ, $E_{n,\chi,q}$, are defined by the following generating function

$$F_{\chi,q}(t) = \frac{2\sum_{a=0}^{d-1}\chi(a)(-1)^a q^a e^{(2a+1)t}}{q^d e^{2dt}+1} = \sum_{n=0}^{\infty} E_{n,\chi,q}\frac{t^n}{n!}. \tag{2.1}$$

We now consider the second kind generalized q-Euler polynomials associated with χ, $E_{n,\chi,q}(x)$, are also defined by

$$F_{\chi,q}(x,t) = \frac{2\sum_{a=0}^{d-1}\chi(a)(-1)^a q^a e^{(2a+1)t}}{q^d e^{2dt}+1}e^{xt} = \sum_{n=0}^{\infty} E_{n,\chi,q}(x)\frac{t^n}{n!}. \tag{2.2}$$

When $\chi = \chi^0$, above (2.1) and (2.2) will become the corresponding definitions of the second kind Euler numbers E_n and polynomials $E_n(x)$.

Since

$$\begin{aligned}&\frac{2\sum_{a=0}^{d-1}\chi(a)(-1)^a q^a e^{(2a+1)t}}{q^d e^{2dt}+1}e^{xt}\\&=\sum_{a=0}^{d-1}\chi(a)(-1)^a q^a\left(\frac{2e^{dt}e^{(\frac{2a+1+x-d}{d})dt}}{q^d e^{2dt}+1}\right)\\&=\sum_{m=0}^{\infty}\left(d^m\sum_{a=0}^{d-1}\chi(a)(-1)^a q^a E_{m,q^d}\left(\frac{2a+1+x-d}{d}\right)\right)\frac{t^m}{m!},\end{aligned}$$

we have the following theorem.

Theorem 1. Let χ be Dirichlet's character with conductor $d \in \mathbb{N}$ with $d \equiv 1 (\text{mod } 2)$. Then we have

$$(1)\ E_{n,\chi,q}(x) = d^m\sum_{a=0}^{d-1}\chi(a)(-1)^a q^a E_{m,q^d}\left(\frac{2a+1+x-d}{d}\right),$$

$$(2)\ E_{n,\chi,q} = d^m\sum_{a=0}^{d-1}\chi(a)(-1)^a q^a E_{m,q^d}\left(\frac{2a+1-d}{d}\right),$$

$$(3)\ E_{n,\chi,q}(x) = \sum_{l=0}^{n}\binom{n}{l}E_{l,\chi,q}x^{n-l}.$$

For $n \in \mathbb{N}$ with $n \equiv 0(\text{mod } 2)$, we have

$$\begin{aligned}&\frac{-2\sum_{a=0}^{d-1}\chi(a)(-1)^a q^a e^{(2a+1)t}}{q^d e^{2dt}+1}q^{nd}e^{2ndt}+\frac{2\sum_{a=0}^{d-1}\chi(a)(-1)^a q^a e^{(2a+1)t}}{q^d e^{2dt}+1}\\&=\sum_{m=0}^{\infty}\left(2\sum_{a=0}^{nd-1}\chi(a)(-1)^a q^a(2a+1)^m\right)\frac{t^m}{m!}\end{aligned}$$

By comparing coefficients of $\frac{t^m}{m!}$ in the above equation, we have the following theorem:

Theorem 2. Let χ be Dirichlet's character with conductor $d \in \mathbb{N}$ with $d \equiv 1$ (mod 2), n a positive even integer, and $m \in \mathbb{N}$. Then we have

$$2\sum_{a=0}^{nd-1}\chi(a)(-1)^a q^a(2a+1)^m = E_{m,\chi,q} - q^{nd}E_{m,\chi,q}(2nd).$$

Next, we introduce the second kind l-series and two variable l-series.

Definition 3. For $s \in \mathbb{C}$, define two variable l-series as

$$l_q(s, x|\chi) = 2\sum_{m=0}^{\infty}\frac{(-1)^m\chi(m)q^m}{(2m+1+x)^s}.$$

By using (2.2), we easily see that

$$\begin{aligned} F_{\chi,q}(x,t) &= \frac{2\sum_{a=0}^{d-1}\chi(a)(-1)^a q^a e^{(2a+1)t}}{q^d e^{2dt}+1}e^{xt} \\ &= 2\sum_{a=0}^{d-1}\chi(a)(-1)^a q^a e^{(2a+1+x)t}\sum_{l=0}^{\infty}(-1)^l q^{ld}e^{2dlt} \\ &= 2\sum_{a=0}^{d-1}\sum_{l=0}^{\infty}\chi(a)(-1)^{a+dl}q^{a+dl}e^{(2a+1+x+dl)t} \\ &= 2\sum_{m=0}^{\infty}\chi(m)(-1)^m q^m e^{(2m+1+x)t}. \end{aligned}$$

Then we have

$$\left(\frac{d}{dt}\right)^k F_{\chi,q}(x,t)\bigg|_{t=0} = 2\sum_{n=0}^{\infty}\chi(n)(-1)^n q^n(2n+1+x)^k, \tag{2.3}$$

and

$$\left(\frac{d}{dt}\right)^k \left(\sum_{n=0}^{\infty}E_{n,\chi,q}(x)\frac{t^n}{n!}\right)\Big|_{t=0} = E_{k,\chi,q}(x),\ \text{ for } k\in\mathbb{N}. \tag{2.4}$$

By (2.3), (2.4), we have the following theorem.

Theorem 4. For any positive integer k, we have

$$E_{k,\chi,q}(x) = l_q(-k,x|\chi).$$

Definition 5. For $s\in\mathbb{C}$, define l-series as

$$l_q(s\mid\chi) = 2\sum_{m=0}^{\infty}\frac{(-1)^m\chi(m)q^m}{(2m+1)^s}.$$

By simple calculation, we have the following theorem.

Theorem 6. For any positive integer k, we have

$$l_q(-k\mid\chi) = E_{k,\chi,q}.$$

3. Witt-type formulae on $\mathbb{Z}_p$ in p-adic number field

Our primary aim in this section is to obtain the Witt-type formulae of the second kind generalized q-Euler numbers $E_{n,\chi,q}$ and polynomials $E_{n,\chi,q}(x)$ attached to χ. We assume that $q\in\mathbb{C}_p$ with $|q-1|_p<1$. Let χ be the primitive Dirichlet character with conductor $d\in\mathbb{N}$ with $d\equiv 1(\mathrm{mod}\ 2)$. Let $g(y)=\chi(y)q^y e^{(2y+1+x)t}$. By (1.4), we derive

$$\begin{aligned} I_1\left(\chi(y)q^y e^{(2y+1+x)t}\right) &= \int_X \chi(y)q^y e^{(2y+1+x)t}d\mu_{-1}(y) \\ &= \frac{2\sum_{a=0}^{d-1}\chi(a)(-1)^a q^a e^{(2a+1)t}}{q^d e^{2dt}+1}e^{xt} \\ &= \sum_{n=0}^{\infty}E_{n,\chi,q}(x)\frac{t^n}{n!}. \end{aligned} \tag{3.1}$$

By using Taylor series of $e^{(2y+1+x)t}$ in the above equation (3.1), we obtain

$$\sum_{n=0}^{\infty}\left(\int_X \chi(y)q^y(2y+1+x)^n d\mu_{-1}(y)\right)\frac{t^n}{n!}=\sum_{n=0}^{\infty}E_{n,\chi,q}(x)\frac{t^n}{n!}.$$

By comparing coefficients of $\frac{t^n}{n!}$ in the above equation, we have the Witt formula for the second kind generalized q- Euler polynomials attached to χ as follows:

Theorem 7. For positive integers n, we have

$$E_{n,\chi,q}(x)=\int_X \chi(y)q^y(2y+1+x)^n d\mu_{-1}(y). \tag{3.2}$$

Observe that for $x=0$, the equation (3.2) reduces to (3.3).

Corollary 8. For positive integers n, we have

$$E_{n,\chi,q}=\int_X \chi(y)q^y(2y+1)^n d\mu_{-1}(y). \tag{3.3}$$

By (3.1) and (1.5), we have the following theorem:

Theorem 9. For positive integers n, we have

$$q^{nd}E_{m,\chi,q}(2nd)-(-1)^n E_{m,\chi,q}=2\sum_{l=0}^{nd-1}(-1)^{n-1-l}\chi(l)q^l(2l+1)^m.$$

ACKNOWLEDGEMENT

This paper has been supported by the 2013 Hannam University Research Fund.

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J. COMPUTATIONAL ANALYSIS AND APPLICATIONS, VOL. 15, NO.8, 1430-1443, 2013, COPYRIGHT 2013 EUDOXUS PRESS, LLC

Analytic Approximation of Time-Fractional Diffusion-Wave Equation Based on Connection of Fractional and Ordinary Calculus

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Abstract

In this paper, we present a connection between fractional and ordinary derivative, which can be used in various fields of science and engineering deal with dynamical systems for solving fractional ordinary and partial differential equations. Some examples are given to show ability of the method for solving the fractional nonlinear equations.

Keywords: Diffusion Equation; Fractional Calculus; Fractional Partial Differential Equation

AMS subject classifications: 35Kxx, 34K37, 35R11

1 Introduction

The theory of the fractional derivatives (FD) has a long history, but the application of FD goes back to the 19th century. For example, Caputo and Mainardi found good agreement with experimental results when using FD for the description of viscoelastic materials [3]. Recently, many works from various fields of science have been described by fractional differential equation, for example, the time-fractional diffusion-wave equation (TFDWE) and the space-fractional diffusion equation (SFDE) have been widely researched [2]. A fractional diffusion equation can be interpreted a fractional Fick law replacing the classical Fick law, which describes transport processes with a long memory [6].

Authors have considered FD of Reimann-Liouville, Caputo and Grounwald-Letnikov and their applications having different points of views of definitions [15]. Some approximations for these fractional derivatives and Laplace transform of fractional derivative are also considered [7, 8]. Because of the wide application of fractional derivatives—fractional ordinary differential equation (FODE) and fractional partial differential equation (FPDE)—in the various fields of science and engineering, the connection between fractional and ordinary derivative (OD), for solving related problems is important. A little works have been done in this field[1]. In this paper, we are going to overcome this problem by providing a robust connection between FD and OD.

In this paper, we provide a strategy for obtaining an analytic approximation of the SFDE and TFDWE. Specifically, we employ analytic approximation method—homogony perturbation method—to compute the fundamental solutions of the SFDE and TFDWE [2]. These three methods offer efficient approaches for solving nonlinear problems.

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[1]In the time of writing this paper, we could not find any work.

[2]We only use the homogony perturbation method, the other analytic methods as Adomian decomposition method, and variational iteration method can be used.

We have organized our presentation as follows. In Sections 2, we will present a review of the homotopy perturbation method (HPM). In Section 3, we provide a connection between FD and OD. Finally, some experiments to clarify the methods are provided in Sections 4.

2 Homotopy Perturbation Method

The principals of the HPM and its applicability for various kinds of differential equations are given in [9, 10]. For convenience of the reader, we will present a review of the HPM. To achieve our goal, we consider the nonlinear differential equation

$$L(u) + N(u)) = f(r), \qquad r \in \Omega, \tag{1}$$

with boundary conditions

$$B(u, \frac{\partial u}{\partial n}) = 0, \qquad r \in \Gamma,$$

where L is a linear operator, while N is nonlinear operator, B is boundary operator, Γ is the boundary of the domain Ω and $f(r)$ is known analytic function. By the homotopy technique proposed by He in [9, 10], we construct a homotopy of equation (1), $v(r,p) : \Omega \times [0,1] \to \mathbf{R}$ which satisfies

$$\mathrm{H}(v,p) = (1-p)[L(v) - L(u_0)] + p[L(v) + N(v) - f(r)] = 0$$

or

$$\mathrm{H}(v,p) = L(v) - L(u_0) + pL(u_0) + p[N(v) - f(r)] = 0,$$

where $r \in \Omega$ and $p \in [0,1]$ is an impeding parameter, u_0 is an initial approximation which satisfies the boundary conditions. The changing process of p from zero to unity is just that of $v(r,p)$ from u_0 to u(r). In topology, this called deformation, $L(v) - L(u_0)$ and $L(v) + N(v) - f(r)$ are homotopic.

We assume that the solution of equation (1) can be expressed as

$$v = p^0 v_0 + p^1 v_1 + p^2 v_2 + p^3 v_3 + \cdots, \tag{2}$$

so, the approximate solution of equation (1) can be obtained as follows

$$u = \lim_{p \to 1} v = v_0 + v_1 + v_2 + v_3 + \cdots. \tag{3}$$

It is well known that the series (3) is convergent for most of the cases and also the rate of convergence is dependent on $L(u)$, see [10, 13].

3 The Connection of Fractional and Ordinary Calculus

In this section we will reach a formula that it provide a robust connection between fractional and ordinary derivatives. Suppose $0 < \alpha < 1$, based on binomial series we will have

$$(1-L)^\alpha = 1 - \alpha L - \frac{\alpha(1-\alpha)}{2!} - \cdots = \sum_{j=0}^{\infty} \alpha_j L^j, \tag{4}$$

where the sequence $\{\alpha_j\}_{j=0}^{\infty}$ is obtained from the following recurrence relation:

$$\alpha_0 = 0, \qquad \alpha_j = \frac{j-1-\alpha}{j}\alpha_{j-1}, \qquad j = 1, 2, \cdots.$$

Now, let D^{α} and f be the fractional derivative of order α and arbitrary function, respectively. According to the equation (4), the following approximation can be obtained:

$$\begin{aligned} D^{\alpha}f &= (1-1+D)^{\alpha}f, \\ &= \left(\sum_{j=0}^{\infty}\alpha_j L^j\right)f, \\ &= \left(\sum_{j=0}^{\infty}\alpha_j \sum_{i=0}^{j}(-1)^i C_j^i D^i\right)f, \end{aligned} \tag{5}$$

where $L = (1-D)$.

So, we will have

$$D^{\alpha}f \simeq \left(\sum_{j=0}^{k}\alpha_j \sum_{i=0}^{j}(-1)^i C_j^i D^i\right)f. \tag{6}$$

In equation (6), the fractional derivative $D^{\alpha}f$ is approximated based on a sequence of ordinary derivative. In fact, equation (6) provide a connection between fractional and ordinary derivatives. Now, we will compute some approximations of fractional derivative for different amount of k

- $k = 0, \quad \to \quad D^{\alpha}f \simeq \alpha_0 D^0 f.$
- $k = 1, \quad \to \quad D^{\alpha}f \simeq (\alpha_0+\alpha_1)D^0 f - \alpha_1 D^1 f.$
- $k = 2, \quad \to \quad D^{\alpha}f \simeq (\alpha_0+\alpha_1+\alpha_2)D^0 f - (\alpha_1+2\alpha_2)D^1 f + \alpha_2 D^2 f.$

$\vdots$

$$k = n, \quad \to \quad D^{\alpha}f \simeq (\sum_{i=0}^{n}\alpha_i)D^0 f - (\sum_{i=1}^{n} i\alpha_i)D^1 f + \cdots + (-1)^n \alpha_n D^n f. \tag{7}$$

In equation (7), the coefficients and the sign of coefficients are obtained from Fig1 to Fig3. The following algorithm can be arranged for the getting the approximation formula of fractional derivatives:

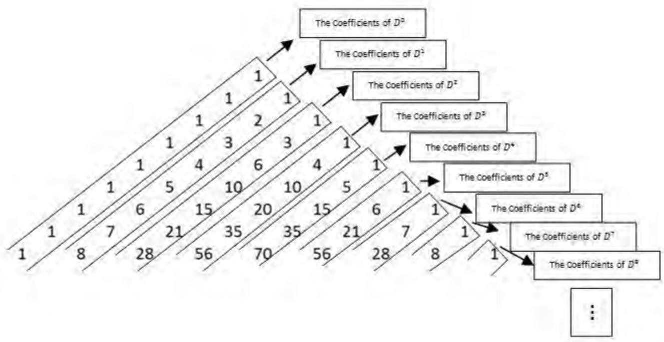

Figure 1: The coefficients of ordinary derivatives in the approximation of fractional derivatives.

Algorithm 1:

- **Step 1.** Select the numbers of series terms.

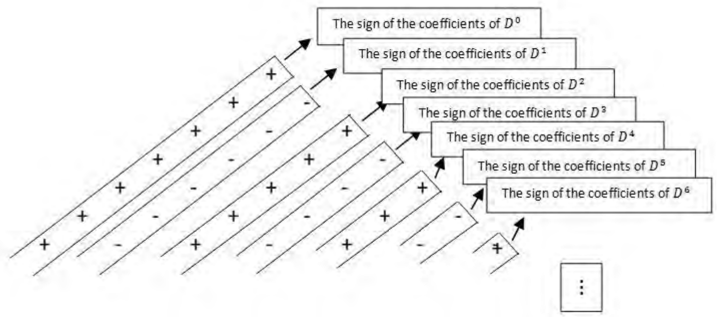

Figure 2: The sign of the coefficients of ordinary derivatives in the approximation of fractional derivatives.

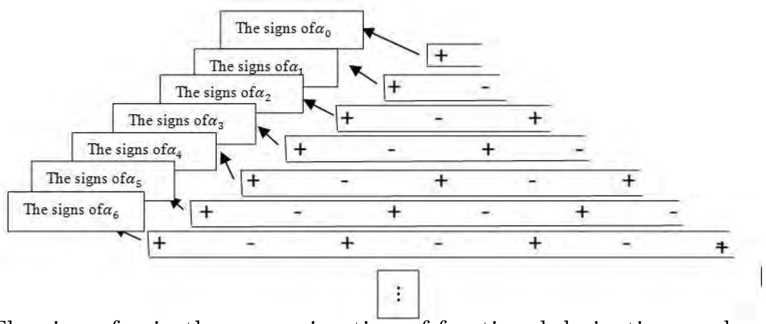

Figure 3: The sign of $\alpha_i$in the approximation of fractional derivatives and number of series terms.

- **Step 2.** Find the coefficients of terms from Fig1.
- **Step 3.** Find the sign of the coefficients from Fig2.

To show the efficiency of the described approximation, we apply some experiments used extensively in many natural processes in physics [12], finance [1] and hydrology [2] are tested. We summarize the described procedure for solving a problem in Algorithm 2:

Algorithm 2:

- **Step 1.** Select the number of series's terms.
- **Step 2.** Find the approximate formula from Algorithm 1.
- **Step 3.** Find the equivalent problem for solving.
- **Step 4.** Select an analytic methods as HPM, VIM, and ADM.
- **Step 5.** Find the analytic approximation solution of the problem.

4 Application

In this section, we derive the analytic solution of SFDE and TFDWE by using the homotopy perturbation method.

Example 1 *We first consider*

$$\frac{\partial^\alpha u(x,t)}{\partial t^\alpha} = \frac{\partial^2 u(x,t)}{\partial x^2}, \qquad 0 < x < \pi, \qquad t \geq 0, \qquad 1 < \alpha \leq 2, \tag{8}$$

where the initial and boundary condition are $u(x,0) = \sin(x)$ and $u(0,t) = u(\pi,t) = 0$, respectively.

Now, we will use Algorithm 2 for getting the solution of equation (8). If $k = 0$ the equivalent differential equation will be

$$\frac{\partial u(x,t)}{\partial t} = \frac{\partial^2 u(x,t)}{\partial x^2}. \tag{9}$$

To solve equation (9) with initial conditions $u(x,0) = \sin(x)$ and $u_t(x,0) = 0$, according to the HPM, we construct the following homotopy:

$$(1-p)\left(\frac{\partial v}{\partial t} - \frac{\partial u_0}{\partial t}\right) + p\left(\frac{\partial v}{\partial t} - \frac{\partial^2 v}{\partial x^2}\right) = 0. \tag{10}$$

Substituting equation (2) into equation (10), and comparing coefficients of terms with identical powers of p, leads to:

$$p^0 : \frac{\partial v_0}{\partial t} - \frac{\partial u_0}{\partial t} = 0$$

$$p^1 : \frac{\partial v_1}{\partial t} = \frac{\partial^2 v_0}{\partial x^2} - \frac{\partial u_0}{\partial t}, \qquad v_1(x,0) = 0$$

$$p^2 : \frac{\partial v_2}{\partial t} = \frac{\partial^2 v_1}{\partial x^2}, \qquad v_2(x,0) = 0,$$

$$\vdots$$

$$p^n : \frac{\partial v_n}{\partial t} = D\frac{\partial^2 v_{n-1}}{\partial x^2}, \qquad v_n(x,0) = 0.$$

For simplicity, we take $v_0(x,t) = u_0(x,t) = \sin(x)$. According to the above equations, we derive the following recurrence equation

$$\begin{aligned}
v_1(x,t) &= -\sin(x) \times t, \\
v_2(x,t) &= \sin(x) \times \frac{1}{2}t^2, \\
v_3(x,t) &= -\sin(x) \times \frac{1}{6}t^3, \\
&\vdots \\
v_n(x,t) &= (-1)^n \sin(x) \times \frac{1}{\Gamma(n+1)}t^n.
\end{aligned}$$

Therefore

$$u(x,t) = \sum_{i=0}^{\infty} (-1)^i \sin(x) \times \frac{1}{\Gamma(i+1)} t^i.$$

If $k=1$, from Algorithm 2 we derive

$$\frac{\partial^{\alpha} u}{\partial t^{\alpha}} \simeq (\alpha_0+\alpha_1)u-\alpha_1\frac{\partial u}{\partial t}.$$

So, the equivalent differential equation of equation (8) will be

$$(\alpha_0+\alpha_1)u-\alpha_1\frac{\partial u}{\partial t}=\frac{\partial^2 u}{\partial x^2}. \tag{11}$$

To solve equation (11) with initial conditions $u(x,0)=\sin(x)$ and $u_t(x,0)=0$, according to the HPM, we construct the following homotopy:

$$\alpha_1\frac{\partial v}{\partial t}=p\left((\alpha_0+\alpha_1)v-\frac{\partial^2 u}{\partial x^2}\right). \tag{12}$$

Substituting equation (2) into equation (12), and comparing coefficients of terms with identical powers of p, leads to:

$$\begin{aligned}
&p^0:\alpha_1\frac{\partial v_0}{\partial t}=0,\\
&p^1:\alpha_1\frac{\partial v_1}{\partial t}=(\alpha_0+\alpha_1)v_0-\frac{\partial^2 v_0}{\partial x^2}, \qquad v_1(x,0)=0,\\
&p^2:\alpha_1\frac{\partial v_2}{\partial t}=(\alpha_0+\alpha_1)v_1-\frac{\partial^2 v_1}{\partial x^2}, \qquad v_2(x,0)=0,\\
&\vdots\\
&p^n:\alpha_1\frac{\partial v_n}{\partial t}=(\alpha_0+\alpha_1)v_{n-1}-\frac{\partial^2 v_{n-1}}{\partial x^2}, \qquad v_n(x,0).
\end{aligned}$$

For simplicity, we take $v_0(x,t)=u_0(x,t)=\sin(x)$. According to the above equations, we derive the following recurrence equation

$$\begin{aligned}
v_1(x,t)&=\left(\left(\frac{\alpha_0+\alpha_1+1}{\alpha_1}\right)\sin(x)\right)\times t,\\
v_2(x,t)&=\left(\left(\frac{\alpha_0+\alpha_1+1}{\alpha_1}\right)^2\sin(x)\right)\times \frac{1}{2}t^2,\\
v_3(x,t)&=\left(\left(\frac{\alpha_0+\alpha_1+1}{\alpha_1}\right)^3\sin(x)\right)\times \frac{1}{6}t^3,\\
&\vdots\\
v_n(x,t)&=\left(\left(\frac{\alpha_0+\alpha_1+1}{\alpha_1}\right)^n\sin(x)\right)\times \frac{1}{\Gamma(n+1)}t^n.
\end{aligned}$$

Therefore

$$u(x,t)=\sum_{i=0}^{\infty}\left(\left(\frac{\alpha_0+\alpha_1+1}{\alpha_1}\right)^i\sin(x)\right)\times\frac{1}{\Gamma(i+1)}t^i.$$

If $k=2$, the equivalent differential equation of equation (8) will be obtained. Using HPM we get the analytic solution for $k=2$. So, Algorithm 2 provide a procedure for getting the analytic solution of equation (8). Also, the solution can be verified through substitution in equation (8).

Example 2 *We first consider:*

$$\frac{\partial^\alpha u(x,t)}{\partial t^\alpha} = \frac{\partial^2 u(x,t)}{\partial x^2}, \qquad 0 < x < 2, \qquad t \geq 0, \qquad 1 < \alpha \leq 2, \tag{13}$$

with the initial condition $u(x,0) = f(x)$, *and* $u_t(x,0) = 0$ *where*

$$f(x) = \begin{cases} x, & 0 \leq x \leq 1, \\ 2 - x, & 1 \leq x \leq 2, \end{cases} \tag{14}$$

and boundary condition $u(0,t) = u(2,t) = 0$.

Since $f(x)$ *is a periodic function with period 2. The Fourier series of* $f(x)$ *in* $[0,2]$ *can be obtained as*

$$f(x) = \sum_{n=1}^{\infty} \left(\frac{8(-1)^{n-1}}{(2n-1)^2\pi^2} \right) \sin\left(\frac{(2n-1)\pi x}{2} \right),$$

so,we will have

$$u_0(x) = u(x,0) + t u_t(x,0) = \sum_{n=1}^{\infty} \left(\frac{8(-1)^{n-1}}{(2n-1)^2\pi^2} \right) \sin\left(\frac{(2n-1)\pi x}{2} \right). \tag{15}$$

Now, we will use Algorithm 2 for getting the solution of equation (13). If $k = 0$ *the equivalent differential equation will be*

$$\frac{\partial u(x,t)}{\partial t} = \frac{\partial^2 u(x,t)}{\partial x^2}. \tag{16}$$

To solve equation (16) with initial conditions (15), according to the HPM, we construct the following homotopy:

$$(1-p)\left(\frac{\partial v}{\partial t} - \frac{\partial u_0}{\partial t} \right) + p\left(\frac{\partial v}{\partial t} - \frac{\partial^2 v}{\partial x^2} \right) = 0. \tag{17}$$

Substituting equation (2) into equation (17), and comparing coefficients of terms with identical powers of p, *leads to:*

$$p^0 : \frac{\partial v_0}{\partial t} - \frac{\partial u_0}{\partial t} = 0,$$

$$p^1 : \frac{\partial v_1}{\partial t} = \frac{\partial^2 v_0}{\partial x^2} - \frac{\partial u_0}{\partial t}, \qquad v_1(x,0) = 0,$$

$$p^2 : \frac{\partial v_2}{\partial t} = \frac{\partial^2 v_1}{\partial x^2}, \qquad v_2(x,0) = 0,$$

$$\vdots$$

$$p^n : \frac{\partial v_n}{\partial t} = D\frac{\partial^2 v_{n-1}}{\partial x^2}, \qquad v_n(x,0) = 0.$$

For simplicity, we take $v_0(x,t) = u_0(x)$. According to the above equations, we derive the following recurrence equation

$$\begin{aligned}
v_1(x,t) &= -\sum_{n=1}^{\infty}\left(\frac{(2n-1)\pi x}{2}\right)^2 \times \left(\frac{8(-1)^{n-1}}{(2n-1)^2\pi^2}\right)\sin\left(\frac{(2n-1)\pi x}{2}\right) \times t,\\
v_2(x,t) &= \sum_{n=1}^{\infty}\left(\frac{(2n-1)\pi x}{2}\right)^4 \times \left(\frac{8(-1)^{n-1}}{(2n-1)^2\pi^2}\right)\sin\left(\frac{(2n-1)\pi x}{2}\right) \times \frac{1}{2}t^2,\\
&\vdots\\
v_k(x,t) &= (-1)^k\sum_{n=1}^{\infty}\left(\frac{(2n-1)\pi x}{2}\right)^{2k} \times \left(\frac{8(-1)^{n-1}}{(2n-1)^2\pi^2}\right)\\
&\times \sin\left(\frac{(2n-1)\pi x}{2}\right) \times \frac{1}{\Gamma(k+1)}t^k.
\end{aligned}$$

Therefore

$$\begin{aligned}
u(x,t) &= \sum_{k=1}^{\infty}\left((-1)^k\sum_{n=1}^{\infty}\left[\left(\frac{(2n-1)\pi x}{2}\right)^{2k} \times \left(\frac{8(-1)^{n-1}}{(2n-1)^2\pi^2}\right)\right.\right.\\
&\times \left.\left.\sin\left(\frac{(2n-1)\pi x}{2}\right)\right] \times \frac{1}{\Gamma(k+1)}t^k\right).
\end{aligned} \tag{18}$$

If $k = 1$, from Algorithm 1 we derive

$$\frac{\partial^\alpha u}{\partial t^\alpha} \simeq (\alpha_0+\alpha_1)u - \alpha_1\frac{\partial u}{\partial t}.$$

So, the equivalent differential equation of equation (8) will be

$$(\alpha_0+\alpha_1)u - \alpha_1\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2}. \tag{19}$$

To solve equation (19) with initial conditions $u_0(x)$, according to the HPM, we construct the following homotopy:

$$\alpha_1\frac{\partial v}{\partial t} = p\left((\alpha_0+\alpha_1)v - \frac{\partial^2 u}{\partial x^2}\right). \tag{20}$$

Substituting equation (2) into equation (20), and comparing coefficients of terms with identical powers of p, leads to:

$$\begin{aligned}
&p^0 : \alpha_1\frac{\partial v_0}{\partial t} = 0,\\
&p^1 : \alpha_1\frac{\partial v_1}{\partial t} = (\alpha_0+\alpha_1)v_0 - \frac{\partial^2 v_0}{\partial x^2}, \qquad v_1(x,0)=0,\\
&p^2 : \alpha_1\frac{\partial v_2}{\partial t} = (\alpha_0+\alpha_1)v_1 - \frac{\partial^2 v_1}{\partial x^2}, \qquad v_2(x,0)=0,\\
&\vdots
\end{aligned}$$

$$p^n : \alpha_1 \frac{\partial v_n}{\partial t} = (\alpha_0 + \alpha_1) v_{n-1} - \frac{\partial^2 v_{n-1}}{\partial x^2}, \qquad v_n(x, 0).$$

For simplicity, we take $v_0(x,t) = u_0(x)$. *According to the above equations, we derive the following recurrence equation*

$$\begin{aligned} v_1(x,t) \quad &= \sum_{n=1}^{\infty} \left(\frac{2^2(\alpha_0+\alpha_1) + ((2n-1)\pi x)^2}{2^2 \alpha_1} \right) \\ &\times \left(\frac{8(-1)^{n-1}}{(2n-1)^2\pi^2} \right) \sin\left(\frac{(2n-1)\pi x}{2} \right) \times t, \\ v_2(x,t) \quad &= \sum_{n=1}^{\infty} \left(\frac{2^2(\alpha_0+\alpha_1) + ((2n-1)\pi x)^2}{2^2 \alpha_1} \right)^2 \\ &\times \left(\frac{8(-1)^{n-1}}{(2n-1)^2\pi^2} \right) \sin\left(\frac{(2n-1)\pi x}{2} \right) \times \frac{1}{2} t^2, \\ &\vdots \\ v_k(x,t) \quad &= \sum_{n=1}^{\infty} \left(\frac{2^2(\alpha_0+\alpha_1) + ((2n-1)\pi x)^2}{2^2 \alpha_1} \right)^k \\ &\times \left(\frac{8(-1)^{n-1}}{(2n-1)^2\pi^2} \right) \sin\left(\frac{(2n-1)\pi x}{2} \right) \times \frac{1}{\Gamma(k+1)} t^k. \end{aligned}$$

Therefore

$$\begin{aligned} u(x,t) \quad = \quad & \sum_{k=1}^{\infty} \sum_{n=1}^{\infty} \left[\left(\frac{2^2(\alpha_0+\alpha_1) + ((2n-1)\pi x)^2}{2^2 \alpha_1} \right)^k \right. \\ & \left. \times \left(\frac{8(-1)^{n-1}}{(2n-1)^2\pi^2} \right) \sin\left(\frac{(2n-1)\pi x}{2} \right) \right] \times \frac{1}{\Gamma(k+1)} t^k. \end{aligned}$$

If $k = 2$, *the equivalent differential equation of equation (13) will be obtained. Using HPM we get the analytic solution for* $k = 2$. *So, Algorithm 1 provide a procedure for getting the analytic solution of equation (13). Also, the solution can be verified through substitution in equation (13).*

Example 3 *Let us consider (1+1)-dimensional nonlinear fractional equation:*

$$\begin{gathered} \frac{\partial^\alpha u(x,t)}{\partial t^\alpha} - \gamma^2 \frac{\partial^2 u(x,t)}{\partial x^2} + c^2 u(x,t) - \sigma u^3(x,t) = 0, \\ t \geq 0, \qquad 1 < \alpha \leq 2, \end{gathered} \tag{21}$$

with initial conditions $u(x,0) = \varepsilon \cos(kx)$, *and* $u_t(x,0) = 0$.

Now, we will use Algorithm 1 for getting the solution of equation (21). If $k = 0$ *the equivalent differential equation will be*

$$\frac{\partial u(x,t)}{\partial t} = \gamma^2 \frac{\partial^2 u(x,t)}{\partial x^2} - c^2 u(x,t) + \sigma u^3(x,t). \tag{22}$$

To solve equation (22) with initial conditions $u(x,0) = \varepsilon \cos(kx)$, *and* $u_t(x,0) = 0$, *according to the HPM, we construct the following homotopy:*

$$\frac{\partial u(x,t)}{\partial t} = p \left(\gamma^2 \frac{\partial^2 u(x,t)}{\partial x^2} - c^2 u(x,t) + \sigma u^3(x,t) \right). \tag{23}$$

Substituting equation (2) into equation (23), and comparing coefficients of terms with identical powers of p, leads to:

$$p^0 : \frac{\partial v_0}{\partial t} = 0,$$

$$p^1 : \frac{\partial v_1}{\partial t} = \gamma^2 \frac{\partial^2 v_0(x,t)}{\partial x^2} - c^2 v_0(x,t) + \sigma v_0^3(x,t), \qquad v_1(x,0) = 0,$$

$$p^2 : \frac{\partial v_2}{\partial t} = \gamma^2 \frac{\partial^2 v_1(x,t)}{\partial x^2} - c^2 v_1(x,t) + \sigma v_1^3(x,t), \qquad v_2(x,0) = 0,$$

$$\vdots$$

$$p^n : \frac{\partial v_n}{\partial t} = \gamma^2 \frac{\partial^2 v_{n-1}(x,t)}{\partial x^2} - c^2 v_{n-1}(x,t) + \sigma v_{n-1}^3(x,t), \qquad v_n(x,0) = 0.$$

For simplicity, we take $v_0(x,t) = u_0(x)$. According to the above equations, we derive the following recurrence equation

$$\begin{aligned}
v_0(x,t) \quad & = (\varepsilon \cos(kx)), \\
v_1(x,t) \quad & = \left((-\gamma^2 k^2 + c^2)\varepsilon \cos(kx) + \varepsilon^3 \cos^3(kx)\right) \times t, \\
v_2(x,t) \quad & = \left[\left(-\varepsilon^2 k^2(-\gamma^2 k^2 + c^2)\cos(kx)\right) + \left(3\varepsilon^3 k^2 \cos^3(kx)\right.\right. \\
& \left. -6k^2 \sin^2(kx)\cos(kx)\right), \\
& -c^2 \left((-\gamma^2 k^2 + c^2)\varepsilon \cos(kx) + \varepsilon^3 \cos^3(kx)\right) \times \frac{1}{\Gamma(3)} t^2, \\
& \left. +\sigma \left((-\gamma^2 k^2 + c^2)\varepsilon \cos(kx) + \varepsilon^3 \cos^3(kx)\right)^3\right] \times \frac{1}{\Gamma(3)} t^2, \\
& \vdots
\end{aligned} \tag{24}$$

and so on.

If $k = 1$, from Algorithm 2 we derive

$$\frac{\partial^\alpha u}{\partial t^\alpha} \simeq (\alpha_0 + \alpha_1)u - \alpha_1 \frac{\partial u}{\partial t}.$$

So, the equivalent differential equation of equation (21) will be

$$(\alpha_0 + \alpha_1)u - \alpha_1 \frac{\partial u}{\partial t} = \gamma^2 \frac{\partial^2 u(x,t)}{\partial x^2} - c^2 u(x,t) + \sigma u^3(x,t). \tag{25}$$

Using HPM we get the analytic solution for $k = 1$. So, Algorithm 2 provide a procedure for getting the analytic solution of equation (21). Also, the solution can be verified through substitution in equation (21).

Example 4 *Consider space-fractional diffusion equation [5]*

$$\frac{\partial u}{\partial t} = C \frac{\partial^\alpha}{\partial x^\alpha} u(x,t), \quad x \in \mathbf{R}, t > 0, 0 < \alpha < 2, \tag{26}$$

subject to initial condition $u(x,0) = f(x)$, and $C$is positive coefficient. We can see from Theorem 1 of [?], that the fundamental solution K(x,t) of equation (26) is the density of the stable distribution $S_\alpha((-Ct\cos(\frac{\alpha\pi}{2}))^{\frac{1}{\alpha}}, 1, 0)$, where the initial condition is $u(x,0) = \delta(x)$.

Now, we will use Algorithm 2 for getting the solution of equation (26). If $k = 0$, the equivalent differential equation of (6) will be

$$\frac{\partial}{\partial t}u(x,t) \simeq Cu(x,t), \quad x \in \mathbf{R}, t > 0, 0 < \alpha < 2. \tag{27}$$

The problem (27) is a linear ordinary differential equation of first order. So, from the initial condition,the solution of it will be

$$u(x,t) \simeq \exp(t)\delta(x).$$

If $k = 1$, the equivalent differential equation of (26) will be

$$\frac{\partial}{\partial t}u(x,t) \simeq (\alpha_0 + \alpha_1)u(x,t) - \alpha_1\frac{\partial}{\partial x}u(x,t), \quad x \in \mathbf{R}, t > 0, 0 < \alpha < 2. \tag{28}$$

Now, we will use the analytic methods for getting the analytic solution of problem (28). To solve equation (28) with initial condition $u(x,0) = \delta(x)$, according to the homotopy perturbation technique, we construct the following homotopy:

$$(1-p)\left(\frac{\partial v}{\partial t} - \frac{\partial u_0}{\partial t}\right) + p\left(\frac{\partial v}{\partial t} - C\left((\alpha_0 + \alpha_1)v(x,t) - \alpha_1\frac{\partial}{\partial x}v(x,t)\right)\right) = 0, \tag{29}$$

Substituting equation (2) into equation (29), and comparing coefficients of terms with identical powers of p, leads to:

$$p^0 : \frac{\partial v_0}{\partial t} - \frac{\partial u_0}{\partial t} = 0,$$

$$p^1 : \frac{\partial v_1}{\partial t} = -\frac{\partial v_0}{\partial t} + C\left((\alpha_0 + \alpha_1)v_0(x,t) - \alpha_1\frac{\partial}{\partial x}v_0(x,t)\right), \qquad v_1(x,0) = 0,$$

$$p^2 : \frac{\partial v_2}{\partial t} = C\left((\alpha_0 + \alpha_1)v_1(x,t) - \alpha_1\frac{\partial}{\partial x}v_1(x,t)\right), \qquad v_2(x,0) = 0,$$

$$\vdots$$

$$p^n : \frac{\partial v_n}{\partial t} = C\left((\alpha_0 + \alpha_1)v_{n-1}(x,t) - \alpha_1\frac{\partial}{\partial x}v_{n-1}(x,t)\right), \qquad v_n(x,0) = 0,$$

For simplicity we take $v_0(x,t) = u_0(x,t) = \delta(x)$. So we derive the following recurrent relation

$$\begin{aligned} v_1(x,t) \quad &= \int_0^t \left(C\left((\alpha_0 + \alpha_1)v_0(x,t) - \alpha_1\frac{\partial}{\partial x}v_0(x,t)\right)\right)dt \\ &= C\left((\alpha_0 + \alpha_1)\delta(x) - \alpha_1\frac{\partial}{\partial x}\delta(x)\right) \times t, \\ &\vdots \end{aligned}$$

and so on.

Therefore

$$u(x,t)=\lim_{n\to\infty} u_n(x,t)=\delta(x)+\sum_{k=1}^{\infty}\int_0^t\left(C\left((\alpha_0+\alpha_1)v_{k-1}(x,t)-\alpha_1\frac{\partial}{\partial x}v_{k-1}(x,t)\right)\right)dt.$$

If $k=2$, the equivalent differential equation of (6) will be

$$\begin{aligned}\frac{\partial}{\partial t}u(x,t) \quad &\simeq C\left((\alpha_0+\alpha_1+\alpha_2)u(x,t)-(\alpha_1+2\alpha_2)\frac{\partial}{\partial x}u(x,t)+\alpha_2\frac{\partial^2}{\partial x^2}u(x,t)\right),\\ &x\in\mathbf{R}, t>0, 0<\alpha<2,\end{aligned}$$

using HPM we get the analytic solution for $k=2$. So, Algorithm 2 provide a procedure for getting the analytic solution of equation (26).

5 Conclusion

In this paper, we have shown that the HPM can be used successfully for finding the solutions of space-fractional partial differential equation based on connection of FC and OD. It may be concluded that this technique is very powerful and efficient in finding the analytical solutions SFDE and TFDWE. Some experiments supported the theoritical results.

A Fractional Calculus

Fractional calculus goes back to the beginning of the theory of differential calculus and deals with the generalization of standard integrals and derivatives to a non-integer, or even complex order [14, 16, 15].

In this section we give the basic definitions and some properties of the fractional calculus. More detailed information may be found in the book by Samko et al. [16] and [11].

Let $\Omega=[a,b](\infty<a<b<\infty)$ be a finite interval on the real axis $\mathbf{R}$. The Riemann-Liouville fractional integrals $I^{\alpha}_{a^+}$ and $I^{\alpha}_{b^-}$ of order $\alpha\in\mathbf{C}$ $(\Re(\alpha)>0)$ are defined by

$$(I^{\alpha}_{a^+}f)(x)=\frac{1}{\Gamma(\alpha)}\int_a^x\frac{f(t)dt}{(x-t)^{1-\alpha}} \qquad (x>a,\Re(\alpha)>0),$$

and

$$(I^{\alpha}_{b^-}f)(x)=\frac{1}{\Gamma(\alpha)}\int_x^b\frac{f(t)dt}{(t-x)^{1-\alpha}} \qquad (x<b,\Re(\alpha)>0),$$

respectively. Here $\Gamma(\alpha)$ is the Gamma function.These integrals are called the left-sided and the right-sided fractional integrals.

The Riemann-Liouville fractional derivatives $D^{\alpha}_{a^+}y$ and $D^{\alpha}_{b^-}y$ of order $\alpha\in\mathbf{C}$ $(\Re(\alpha)\geq 0)$ are defined by

$$(D^{\alpha}_{a^+}y)(x)=(\frac{d}{dx})^n(I^{n-\alpha}_{b^-}y)(x)$$

$$=\frac{1}{\Gamma(n-\alpha)}(\frac{d}{dx})^n\int_a^x\frac{y(t)dt}{(x-t)^{\alpha-n+1}}, \qquad (x>a, n=[\Re(\alpha)]+1),$$

and

$$(D^{\alpha}_{b-}y)(x) = (-\frac{d}{dx})^n (I^{n-\alpha}_{b^-}y)(x)$$

$$= \frac{1}{\Gamma(n-\alpha)}(-\frac{d}{dx})^n \int_x^b \frac{y(t)dt}{(t-x)^{\alpha-n+1}}, \qquad (x < b, n = [\Re(\alpha)] + 1),$$

respectively, where $[\Re(\alpha)]$ means the integral part of $\Re(\alpha)$. If $0 < \Re(\alpha) < 1$, then

$$(D^{\alpha}_{a+}y)(x) = \frac{1}{\Gamma(1-\alpha)} \frac{d}{dx} \int_a^x \frac{y(t)dt}{(x-t)^{\alpha-[\Re(\alpha)]}} \qquad (x > a, 0 < \Re(\alpha) < 1),$$

$$(D^{\alpha}_{b-}y)(x) = -\frac{1}{\Gamma(1-\alpha)} \frac{d}{dx} \int_x^b \frac{y(t)dt}{(t-x)^{\alpha-[\Re(\alpha)]}} \qquad (x < b, 0 < \Re(\alpha) < 1).$$

For $f \in c_{\mu}$, $\mu \geq -1$, $\alpha, \beta \geq 0$ and $\gamma > -1$ the following properties will be easily obtained:

- $I^{\alpha} I^{\beta} f(x) = I^{\alpha+\beta} f(x)$,
- $I^{\alpha} I^{\beta} f(x) = I^{\beta} I^{\alpha} f(x)$.

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J. COMPUTATIONAL ANALYSIS AND APPLICATIONS, VOL. 15, NO.8, 1444-1455, 2013, COPYRIGHT 2013 EUDOXUS PRESS, LLC

Higher order duality in nondifferentiable fractional programming involving generalized convexity

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Abstract The purpose of this paper is to consider a class of nondifferentiable multiobjective fractional programming problems in which every component of the objective and constraints functions contains a term involving the support function of a compact convex set. Usual duality theorems are established for two types of higher order dual models under the assumptions of higher order $(F, \alpha, \rho, d) - V-$type I functions.

Keywords: Fractional programming; Nondifferentiable programming; Support function; Generalized convexity; Higher order duality

1. Introduction

In recent years, optimality conditions and duality theory for nondifferentiable multiobjective fractional programming problems involving different kinds of generalized convexity assumptions have received much attention by many authors [6, 7, 8, 9] and the references therein. Under the assumption of (C, α, ρ, d) convexity, Long [9] established sufficient optimality conditions and duality results for a nondifferentiable multiobjective fractional programming problem in which every component of the objective function contains a term involving the support function of a compact convex set.

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The research of the first author partially supported by the Deanship of Scientific Research , King Fahd University of Petroleum and Minerals under internal project IN111015.

Second and higher-order duality in nonlinear programming has been studied in the last few years by many researchers. One practical advantage of second and higher-order duality is that it provides tighter bounds for the value of objective function of the primal problem when approximations are used because there are more parameters involved. Mangasarian [10] first formulated a class of higher-order dual problems for a nonlinear programming problem. Mond and Zhang [11] obtained duality results for various higher-order dual problems under higher-order invexity assumptions. Motivated by the various kinds of generalized convexity Liang *et al.* [7], introduced a unified form of generalized convexity called (F, α, ρ, d)-convex function. Gulati and Agarwal [2] introduced second order (F, α, ρ, d)-V-type I functions for a multiobjective programming problem and proved duality results involving aforesaid functions.

Recently, Suneja *et al.* [12] formulated higher order Mond-Weir and Schaible type dual programs for a nondifferentiable multiobjective fractional programming problem where the objective functions and the constraints contain support function of compact convex sets in R^n and established weak and strong duality results involving higher order (F, ρ, σ)-type I functions. Gulati and Geeta [5] introduced a new class of higher-order (V, α, ρ, d)-invex function and established duality results for Schaible type dual of a nondifferentiable multiobjective fractional programming problem. Gulati and Agarwal [4] focus his study on a nondifferentiable multiobjective programming problem where every component of objective and constraint functions contain a term involving the support function of a compact convex set and established duality theorems for Wolfe and unified higher order dual problems involving higher order (F, α, ρ, d)-type I function.

Motivated by earlier work of Ahmad [1], Gulati and Agarwal [2] and Suneja *et al.* [12], we establish higher order duality results for two types of dual models related to nondifferentiable multiobjective fractional programming problem where the objective and the constraints functions contain support functions of compact convex sets in R^n.

This paper is organized as follows: In Section 2, we have introduced the concept of higher-order (F, α, ρ, d)-V-type I functions. In Sections 3 and 4, the duality results have been established for higher order Mond-Weir and Schaible type duals of a multiobjective nondifferentiable fractional problem. Finally, conclusion and further development are given in Section 5.

2. Preliminaries

Let R^n be n-dimensional Euclidean space and R^n_+ its non-negative orthant. If $x, y \in R^n$ then $x < y \Leftrightarrow x_i < y_i, i = 1, 2, \ldots, n$; $x \leqq y \Leftrightarrow x_i \leqq y_i, i = 1, 2, \ldots, n$ and $x \leq y \Leftrightarrow x_i \leqq y_i, i = 1, 2, \ldots, n$ and $x \neq y$.

Definition 2.1. A functional $F : X \times X \times R^n \to R$ $(X \subseteq R^n)$, is said to be sublinear in its third argument, if $\forall\ x, \bar{x} \in X$,

(i) $F(x, \bar{x}; a_1 + a_2) \leqq F(x, \bar{x}; a_1) + F(x, \bar{x}; a_2)$ $\forall\ a_1, a_2 \in R^n$,

(ii) $F(x, \bar{x}; \alpha a) = \alpha F(x, \bar{x}; a)$ $\forall\ \alpha \in R_+,\ a \in R^n$.

By (ii), it is clear that $F(x, \bar{x}; 0) = 0$.

Definition 2.2. Let C be a compact convex set in R^n. The support function of C is defined by

$$s(x|C) = \max\{x^T y : y \in C\}.$$

A support function, being convex and everywhere finite, has a subdifferential, that is, there exists $z \in R^n$ such that $s(y|C) \geqq s(x|C) + z^T(y - x)$ for all $y \in C$.

The subdifferential of $s(x|C)$ is given by

$$\partial s(x|C) = \{z \in C\ :\ z^T = s(x|C)\}.$$

For any set $D \subset R^n$, the normal cone to D at a point $x \in D$ is defined by

$$N_D(x) = \{y \in R^n \mid y^T(z - x) \leqq 0 \text{ for all } z \in D\}.$$

It is obvious that for a compact convex set C, $y \in N_C(x)$ if and only if $s(y|C) = x^T y$, or equivalently, $x \in \partial s(y|C)$.

Consider the following multiobjective programming problem:

(P) Minimize $f(x)$

subject to $x \in X^0 = \{x \in X : h(x) \leq 0\}$,

where $X \subseteq R^n$ be open, $f : X \to R^k$, $h : X \to R^m$ are continuously differentiable functions.

Definition 2.3. A point $\bar{x} \in X^0$ is an efficient solution of (P) if there exists no $x \in X^0$ such that $f(x) \leq f(\bar{x})$.

Lemma 2.1. $x^0 \in X_0$ is an efficient solution of (P) if and only if x^0 is an optimal solution of $P_r(x^0)$ for each $r = 1, 2, ..., k$,

$$\mathbf{P}_r(x^0) \qquad \text{minimize} \quad f_r(x)$$
$$\text{subject to}$$
$$f_i(x) \leq f_i(x^0), \text{ for all } i = 1, 2, ..., k, \quad i \neq r,$$
$$h(x) \leq 0,$$
$$x \in X.$$

Let $f_i : X \to R, h_j : X \to R, K_i : X \times R^n \to R$ and $H_j : X \times R^n \to R$ be differentiable functions where $i = 1, 2, ..., k$ and $j = 1, 2, ..., m$. Let $d : X \times X \to R$ is a pseudo matric.

Definition 2.4. The pair of functions (f, h) is said to be higher-order $(F, \alpha, \rho, d)-V-$type I at $u \in X$, if there exist vectors $\alpha = (\alpha_1^1, ..., \alpha_k^1, \alpha_1^2, ..., \alpha_m^2)$ and $\rho = (\rho_1^1, ..., \rho_k^1, \rho_1^2, ..., \rho_m^2)$, where $\alpha_i^1, \alpha_j^2 : X \times X \to R_+ \setminus \{0\}$ and $\rho_i^1, \rho_j^2 \in R$ such that for each $x \in X_0$ and for all $p, q \in R^n$, $i = 1, 2, ..., k$ and $j = 1, 2, ..., m$,

$$f_i(x) - f_i(u) \geqq F(x, u; \alpha_i^1(x, u)(\nabla f_i(u) + \nabla_p K_i(u, p)))$$
$$+ K_i(u, p) - p^T \nabla_p K_i(u, p) + \rho_i^1 d^2(x, u),$$

$$- h_j(u) \geqq F(x, u; \alpha_j^2(x, u)(\nabla h_j(u) + \nabla_q H_j(u, q)))$$
$$+ H_j(u, q) - q^T \nabla_q H_j(u, q) + \rho_j^2 d^2(x, u).$$

Remark 2.1.

(i) If $K_i(u, p) = \frac{1}{2} p^t \nabla^2 f_i(u) p$ and $H_j(u, q) = \frac{1}{2} q^t \nabla^2 h_j(u) q$ for $i = 1, 2, ..., k$ and $j = 1, 2, ..., m$, then we obtain the second order (F, α, ρ, d)-V-type I introduced by Gulati and Agarwal [2].

(ii) Let $K_i(u, p) = 0$ and $H_j(u, q) = 0$ for $i = 1, 2, ..., k$ and $j = 1, 2, ..., m$. Then above definition becomes that of (F, α, ρ, d)-V-type I [3].

(iii) If $\alpha_i^1 = \alpha_i^2 = 1$ for $i = 1, 2, ..., k$ and $j = 1, 2, ..., m$, then the higher-order (F, α, ρ, d)-V-type I reduces to the higher order (F, ρ, σ)-type I given by Suneja et al. [12].

We now consider the following the multiobjective nondifferentiable fractional program:

$$\textbf{(FP)} \qquad \text{minimize} \quad \left[\frac{f_1(x) + S(x|C_1)}{g_1(x) - S(x|D_1)}, \frac{f_2(x) + S(x|C_2)}{g_2(x) - S(x|D_2)}, ..., \frac{f_k(x) + S(x|C_k)}{g_k(x) - S(x|D_k)} \right]$$
$$\text{subject to} \quad h_j(x) + S(x|E_j) \leqq 0, \quad j = 1, 2, ..., m,$$

where $x \in X \subseteq R^n$, $f_i, g_i : X \to R$ $(i = 1, 2, ..., k)$ and $h_j : X \to R$ $(j = 1, 2, ..., m)$ are continuously differentiable functions.

$f_i(.) + S(.|C_i) \geqq 0$ and $g_i(.) - S(.|D_i) > 0$; C_i, D_i and E_j are compact convex sets in R^n and $S(x|C_i), S(x|D_i)$ and $S(x|E_j)$ denote the support functions of compact convex sets, C_i, D_i and E_j, respectively.

Lemma 2.2. If u is an efficient solution of (FP), we have the following results.

(FP$\bar{\epsilon}$) minimize $\frac{f_r(x)+S(x|C_r)}{g_r(x)-S(x|D_r)}$

subject to

$$\frac{f_i(x)+S(x|C_i)}{g_i(x)-S(x|D_i)} \leq \bar{\epsilon}_i, \quad i = 12, ..., k, \quad i \neq r,$$

$$h_j(x) + S(x|E_j) \leqq 0, \quad j = 1, 2, ..., m,$$

where $\bar{\epsilon}_i = \frac{f_i(u)+S(u|C_i)}{g_i(u)-S(u|D_i)}$.

Since $g_i(x) - S(x|D_i) > 0$, for each $i = 1, 2, ..., k$, therefore $(FP\bar{\epsilon})$ can be rewritten as

(FP$^1\bar{\epsilon}$) minimize $\frac{f_r(x)+S(x|C_r)}{g_r(x)-S(x|D_r)}$

subject to

$$f_i(x) + S(x|C_i) - \bar{\epsilon}_i(g_i(x) - S(x|D_i)) \leq 0, \quad i = 12, ..., k, \quad i \neq r,$$

$$h_j(x) + S(x|E_j) \leqq 0, \quad j = 1, 2, ..., m.$$

Lemma 2.3. u is an efficient solution of (FP) if and only if u solves (FP$^1\bar{\epsilon}$) for each $r = 1, 2, ..., k$, where $\bar{\epsilon}_i = \frac{f_i(u)+S(u|C_i)}{g_i(u)-S(u|D_i)}$.

3. Higher order Mond-Weir type dual

In connection to **(FP)** we now consider the following higher order Mond-Weir type dual problem [12]:

(MFD) maximize $\left[\frac{f_1(u)+u^Tz_1}{g_1(u)-u^Tv_1}, \frac{f_2(u)+u^Tz_2}{g_2(u)-u^Tv_2}, ..., \frac{f_k(u)+u^Tz_k}{g_k(u)-u^Tv_k}\right]$

subject to

$$\nabla\left[\sum_{i=1}^{k}\lambda_i\Big(\frac{f_i(u)+u^Tz_i}{g_i(u)-u^Tv_i}\Big) + \sum_{j=1}^{m}\mu_j(h_j(u)+u^Tw_j)\right]$$

$$+\sum_{i=1}^{k}\lambda_i\nabla_pG_i(u,p) + \sum_{j=1}^{m}\mu_j\nabla_qH_j(u,q) = 0, \tag{1}$$

$$\sum_{j=1}^{m}\mu_j\{(h_j(u)+u^Tw_j) + H_j(u,q) - q^T\nabla_qH_j(u,q)\} \geqq 0, \tag{2}$$

$$\sum_{i=1}^{k}\lambda_i\big(G_i(u,p) - p^T\nabla_qG_i(u,p)\big) \geqq 0, \tag{3}$$

$$z_i \in C_i, \quad v_i \in D_i, \quad i = 1, 2, ..., k, \quad w_j \in E_j, \quad j = 1, 2, ..., m,$$

$$\mu_j \geqq 0, \quad j = 1, 2, ..., m,$$

$$\lambda_i \geqq 0, \quad i = 1, 2, ..., k, \qquad \sum_{i=1}^{k} \lambda_i = 1.$$

Theorem 3.1 (Weak duality). Let x and $(u, z, v, \mu, \lambda, w, p, q)$ be feasible solutions to (FP) and (MFD), respectively such that

(*i*) $\left[\frac{f_i(.)+(.)^T z_i}{g_i(.)-(.)^T v_i}, h_j(.) + (.)^T w_j\right]$ is higher-order $(F, \alpha, \rho, d) - V-$type I with respect to G_i and H_j, at u for $i = 1, 2, ..., k$ and $j = 1, 2, ..., m$,

(*ii*) $\sum\limits_{i=1}^{k} \frac{\lambda_i}{\alpha_i^1(x,u)} = 1, \alpha_j^2(x, u) = 1, \ j = 1, 2, ..., m,$

(*iii*) $\lambda_i > 0, \ \sum\limits_{i=1}^{k} \frac{\lambda_i \rho_i^1}{\alpha_i^1(x,u)} + \sum_{j=1}^{m} \mu_j \rho_j^2 \geqq 0.$

Then

$$\frac{f_i(x) + S(x|C_i)}{g_i(x) - S(x|D_i)} \leqq \frac{f_i(u) + u^T z_i}{g_i(u) - u^T v_i}, \ i = 1, 2, ..., k, \tag{4}$$

and

$$\frac{f_r(x) + S(x|C_r)}{g_r(x) - S(x|D_r)} < \frac{f_r(u) + u^T z_r}{g_r(u) - u^T v_r}, \ \text{for some } r = 1, 2, ..., k, \tag{5}$$

cannot hold.

Proof. Suppose on the contrary that inequalities (4) and (5) hold. Then as $\lambda_i > 0, x^T z_i \leqq S(x|C_i), x^T v_i \leqq S(x|D_i)$ using hypothesis (ii), we get

$$\sum_{i=1}^{k} \frac{\lambda_i}{\alpha_i^1(x, u)} \Big(\frac{f_i(x) + x^T z_i}{g_i(x) - x^T v_i} - \frac{f_i(u) + u^T z_i}{g_i(u) - u^T v_i} \Big) < 0. \tag{6}$$

Because $\alpha_j^2(x, u) = 1$ for $j \in M$, hypothesis (i) gives

$$\frac{f_i(x) + x^T z_i}{g_i(x) - x^T v_i} - \frac{f_i(u) + u^T z_i}{g_i(u) - u^T v_i} \geqq F\Big(x, u; \alpha_i^1(x, u)(\nabla(\frac{f_i(u) + u^T z_i}{g_i(u) - u^T v_i}) + \nabla_p G_i(u, p))\Big)$$
$$+ G_i(u, p) - p^T \nabla_p G_i(u, p) + \rho_i^1 d^2(x, u). \tag{7}$$

$$- (h_j(u) + u^T w_j) \geqq F\Big(x, u; (\nabla(h_j(u) + u^T w_j) + \nabla_q H_j(u, q)\Big)$$
$$+ H_j(u, q) - q^T \nabla_q H_j(u, q) + \rho_j^2 d^2(x, u). \tag{8}$$

On multiplying the above inequalities (7) and (8) by $\frac{\lambda_i}{\alpha_i^1(x,u)}$ and μ_j, respectively, then summing the two resultant inequalities, we obtain

$$\sum_{i=1}^{k} \frac{\lambda_i}{\alpha_i^1(x, u)} \Big(\frac{f_i(x) + x^T z_i}{g_i(x) - x^T v_i} - \frac{f_i(u) + u^T z_i}{g_i(u) - u^T v_i} \Big)$$

$$\geqq F\Big(x, u; \sum_{i=1}^{k} \lambda_i \big(\nabla\big(\frac{f_i(u) + u^T z_i}{g_i(u) - u^T v_i}\big) + \nabla_p G_i(u, p)\big)\Big)$$

$$+ \sum_{i=1}^{k} \frac{\lambda_i}{\alpha_i^1(x,u)} \big(G_i(u,p) - p^T \nabla_p G_i(u,p)\big) + \sum_{i=1}^{k} \frac{\lambda_i \rho_i^1 d^2(x,u)}{\alpha_i^1(x,u)}, \quad (9)$$

$$-\sum_{j=1}^{m} \mu_j (h_j(u) + u^T w_j) \geqq F\Big(x, u; \sum_{j=1}^{m} \mu_j \big(\nabla(h_j(u) + u^T w_j) + \nabla_q H_j(u,q)\big)\Big)$$

$$+ \sum_{j=1}^{m} \mu_j \big(H_j(u,q) - q^T \nabla_q H_j(u,q)\big) + \sum_{j=1}^{m} \mu_j \rho_j^2 d^2(x,u). \quad (10)$$

Using equation (1) and sublinearity of F, we have

$$0 = F\Big[x, u; \nabla\Big(\sum_{i=1}^{k} \lambda_i \Big(\frac{f_i(u) + u^T z_i}{g_i(u) - u^T v_i}\Big) + \sum_{j=1}^{m} \mu_j (h_j(u) + u^T w_j)\Big)$$

$$+ \sum_{i=1}^{k} \lambda_i \nabla_p G_i(u,p) + \sum_{j=1}^{m} \mu_j \nabla_q H_j(u,q)\Big]$$

$$\leqq F\Big(x, u; \sum_{i=1}^{k} \lambda_i \big(\nabla\big(\frac{f_i(u) + u^T z_i}{g_i(u) - u^T v_i}\big) + \nabla_p G_i(u,p)\big)\Big)$$

$$+ F\Big(x, u; \sum_{j=1}^{m} \mu_j \big(\nabla(h_j(u) + u^T w_j) + \nabla_q H_j(u,q)\big)\Big). \quad (11)$$

The inequalities (9), (10), (11) and hypothesis (iii) give

$$0 \leqq \sum_{i=1}^{k} \frac{\lambda_i}{\alpha_i^1(x,u)} \Big(\frac{f_i(x) + x^T z_i}{g_i(x) - x^T v_i} - \frac{f_i(u) + u^T z_i}{g_i(u) - u^T v_i}\Big) - \sum_{j=1}^{m} \mu_j (h_j(u) + u^T w_j)$$

$$- \sum_{i=1}^{k} \frac{\lambda_i}{\alpha_i^1(x,u)} \big(G_i(u,p) - p^T \nabla_p G_i(u,p)\big) - \sum_{j=1}^{m} \mu_j \big(H_j(u,q) - q^T \nabla_q H_j(u,q)\big).$$

That is,

$$\sum_{i=1}^{k} \frac{\lambda_i}{\alpha_i^1(x,u)} \Big(\frac{f_i(x) + x^T z_i}{g_i(x) - x^T v_i} - \frac{f_i(u) + u^T z_i}{g_i(u) - u^T v_i}\Big)$$

$$\geqq \sum_{j=1}^{m} \mu_j (h_j(u) + u^T w_j + H_j(u,q) - q^T \nabla_q H_j(u,q))$$

$$+ \sum_{i=1}^{k} \frac{\lambda_i}{\alpha_i^1(x,u)} \big(G_i(u,p) - p^T \nabla_p G_i(u,p)\big)$$

From the inequalities (2), (3) and the positivity of $\alpha_i^1(x,u),\ i = 1, 2, ..., k$, we have

$$\sum_{i=1}^{k} \frac{\lambda_i}{\alpha_i^1(x,u)} \Big(\frac{f_i(x) + x^T z_i}{g_i(x) - x^T v_i} - \frac{f_i(u) + u^T z_i}{g_i(u) - u^T v_i}\Big) \geqq 0,$$

which contradicts (6). This completes the proof.

Theorem 3.2 (Strong duality). If u is an efficient solution of (FP), $G_i(u,0)=0, i=1,2,...,k, H_j(u,0)=0, j=1,2,...,m$, and a constraint qualification is satisfied for $(FP\bar{\epsilon})$ for at least one $r=1,2,...,k$, then there exist $\bar{\lambda}\in R^k, \bar{\mu}\in R^m, \bar{z}_i\in R^n, \bar{v}_i\in R^n$ and $\bar{w}_j\in R^n$, $i=1,2,...,k, j=1,2,...,m$, such that $(u,\bar{z},\bar{v},\bar{\mu},\bar{\lambda},\bar{w},p=0,q=0)$ is a feasible solution of (MFD) and the corresponding values of the objective functions are equal. Further if the conditions of Weak duality theorem 3.1 are satisfied for each feasible solution x of (FP) and each feasible solution $(\acute{u},\acute{z},\acute{v},\acute{\mu},\acute{w},p=0,q=0)$ of (MFD) then $(u,\bar{z},\bar{v},\bar{\mu},\bar{\lambda},\bar{w},p=0,q=0)$ is an efficient solution of (MFD).

Proof. The proof follows along the lines of Theorem 3.2 [12] in light of the discussions given above and hence being omitted.

4. Higher order Schaible type dual

Now we consider the following Schaible type higher order dual problem for (FP):

(SFD) maximize $(\gamma_1,\gamma_2,...,\gamma_k)$

subject to

$$\nabla\left[\sum_{i=1}^{k}\lambda_i\Big[(f_i(u)+u^Tz_i)-\gamma_i(g_i(u)-u^Tv_i)\Big]+\sum_{j=1}^{m}\mu_j(h_j(u)+u^Tw_j)\right]$$
$$+\sum_{i=1}^{k}\lambda_i\nabla_p\big(K_i(u,p)-\gamma_iG_i(u,p)\big)+\sum_{j=1}^{m}\mu_j\nabla_qH_j(u,q)=0, \tag{12}$$

$$\sum_{i=1}^{k}\lambda_i\{\Big[(f_i(u)+u^Tz_i)-\gamma_i(g_i(u)-u^Tv_i)\Big]+\big(K_i(u,p)-\gamma_iG_i(u,p)\big)$$
$$-p^T\nabla_p\big(K_i(u,p)-\gamma_iG_i(u,p)\big)\}\geqq 0, \tag{13}$$

$$\sum_{j=1}^{m}\mu_j\{(h_j(u)+u^Tw_j)+H_j(u,q)-q^T\nabla_qH_j(u,q)\}\geqq 0, \tag{14}$$

$$z_i\in C_i,\quad v_i\in D_i,\quad i=1,2,...,k,\quad w_j\in E_j,\quad j=1,2,...,m,$$

$$\mu_j\geqq 0,\quad j=1,2,...,m,$$

$$\lambda_i\geq 0,\quad i=1,2,...,k,\qquad \sum_{i=1}^{k}\lambda_i=1,$$

$$\gamma_i\geqq 0,\quad i=1,2,...,k.$$

Theorem 4.1 (Weak duality). Let x and $(u, \gamma, z, v, w, \mu, \lambda, p, q)$ be feasible solutions of (FP) and (SFD), respectively such that

(i) $(f_i(.) + (.)^T z_i, h_j(.) + (.)^T w_j)$ is higher-order $(F, \alpha, \rho, d) - V-$type I with respect to K_i and H_j and $[-(g_i(.) - (.)^T v_i, h_j(.) + (.)^T w_j]$ is higher-order $(F, \alpha, \rho, d) - V-$type I with respect to $-G_i$ and H_j, at u for $i = 1, 2, ..., k$ and $j = 1, 2, ..., m$,

(ii) $\alpha_i^1(x, u) = \alpha_j^2(x, u) = \hat{\alpha}(x, u),\ i = 1, 2, ..., k,\ j = 1, 2, ..., m,$

(iii) $\sum_{i=1}^{k} \lambda_i \rho_i^3 + \sum_{j=1}^{m} \mu_j \rho_i^2 \geqq 0$, where $\rho_i^3 = \rho_i^1(1 + \gamma_i)$.

Then

$$\frac{f_i(x) + S(x|C_i)}{g_i(x) - S(x|D_i)} \leqq \gamma_i, \quad i = 1, 2, ..., k, \tag{15}$$

and

$$\frac{f_r(x) + S(x|C_r)}{g_r(x) - S(x|D_r)} < \gamma_r, \quad \text{for some } r = 1, 2, ..., k, \tag{16}$$

cannot hold.

Proof. Suppose on the contrary that inequalities (15) and (16) hold. Then as $\lambda_i \geq 0,\ i = 1, 2, ..., k$, using hypothesis (ii), we get

$$\sum_{i=1}^{k} \frac{\lambda_i}{\hat{\alpha}(x, u)} \Big(f_i(x) + x^T z_i - \gamma_i (g_i(x) - x^T v_i) \Big) < 0. \tag{17}$$

Since $(f_i(.) + (.)^T z_i, h_j(.) + (.)^T w_j)$ is higher-order $(F, \alpha, \rho, d) - V-$type I with respect to K_i and H_j and $[-(g_i(.) - (.)^T v_i), h_j(.) + (.)^T w_j]$ is higher-order $(F, \alpha, \rho, d) - V-$type I with respect to $-G_i$ and H_j, at u for $i = 1, 2, ..., k$ and $j = 1, 2, ..., m$, we have

$$\begin{aligned} ((f_i(x) + x^T z_i) - (f_i(u) + u^T z_i)) \geqq F\left(x, u; \alpha_i^1(x, u)\big(\nabla(f_i(u) + u^T z_i) + \nabla_p K_i(u, p)\big)\right) \\ + K_i(u, p) - p^T \nabla_p K_i(u, p) + \rho_i^1 d^2(x, u) \end{aligned} \tag{18}$$

$$\begin{aligned} (-(g_i(x) - x^T v_i) + (g_i(u) - u^T v_i)) \geqq F\left(x, u; -\alpha_i^1(x, u)\big(\nabla(g_i(u) - u^T v_i) - \nabla_p G_i(u, p)\big)\right) \\ - G_i(u, p) + p^T \nabla_p G_i(u, p) + \rho_i^1 d^2(x, u) \end{aligned} \tag{19}$$

and

$$\begin{aligned} -(h_j(u) + u^T w_j) \geqq F\Big(x, u; \alpha_i^2(x, u)\big(\nabla(h_j(u) + u^T w_j) + \nabla_q H_j(u, q)\big)\Big) \\ + H_j(u, q) - q^T \nabla_q H_j(u, q) + \rho_i^2 d^2(x, u) \end{aligned} \tag{20}$$

On multiplying (19) by γ_i and adding with (18),we get

$$[(f_i(x)+x^Tz_i)-\gamma_i(g_i(x)-x^Tv_i)]-[(f_i(u)+u^Tz_i)-\gamma_i(g_i(u)-u^Tv_i)]$$
$$\geqq F\Big[x,u;\alpha_i^1(x,u)\Big(\nabla(f_i(u)+u^Tz_i-\gamma_i(g_i(u)-u^Tv_i))$$
$$+\nabla_p(K_i(u,p)-\gamma_iG_i(u,p))\Big)\Big]+K_i(u,p)-\gamma_iG_i(u,p)$$
$$-p^T\nabla_p(K_i(u,p)-\gamma_iG_i(u,p))+\rho_i^3d^2(x,u), \tag{21}$$

where $\rho_i^3=\rho_i^1(1+\gamma_i)$.

Multiplying (21) by $\lambda_i>0$ and (20) by $\mu_j\geqq 0, i=1,2,...,k, j=i,2,...,m$, and adding, we obtain

$$\sum_{i=1}^k\lambda_i\{[(f_i(x)+x^Tz_i)-\gamma_i(g_i(x)-x^Tv_i)]-[(f_i(u)+u^Tz_i)-\gamma_i(g_i(u)-u^Tv_i)]\}-\sum_{j=1}^m\mu_j(h_j(u)+u^Tw_j)$$
$$\geqq F\Big[x,u;\sum_{i=1}^k\lambda_i\alpha_i^1(x,u)\Big(\nabla(f_i(u)+u^Tz_i-\gamma_i(g_i(u)-u^Tv_i))\Big)+\sum_{j=1}^m\mu_j\alpha_i^2(x,u)\big(\nabla(h_j(u)+u^Tw_j)$$
$$+\sum_{i=1}^k\lambda_i\alpha_i^1(x,u)\Big(\nabla_p(K_i(u,p)-\gamma_iG_i(u,p))\Big)+\sum_{j=1}^m\mu_j\alpha_i^2(x,u)\nabla_qH_j(u,q)\Big]$$
$$+\sum_{i=1}^k\lambda_i\Big(K_i(u,p)-\gamma_iG_i(u,p)\Big)+\sum_{j=1}^m\mu_j\Big(H_j(u,q)-q^T\nabla_qH_j(u,q)\Big)$$
$$-\sum_{i=1}^k\lambda_ip^T\nabla_p(K_i(u,p)-\gamma_iG_i(u,p))+\Big(\sum_{i=1}^k\lambda_i\rho_i^3+\sum_{j=1}^m\mu_j\rho_i^2\Big)d^2(x,u). \tag{22}$$

Using (13), (14) and hypothesis $\sum_{i=1}^k\lambda_i\rho_i^3+\sum_{j=1}^m\mu_j\rho_i^2\geqq 0$, (22) reduces to

$$\sum_{i=1}^k\lambda_i[(f_i(x)+x^Tz_i)-\gamma_i(g_i(x)-x^Tv_i)]$$
$$\geqq F\Big[x,u;\sum_{i=1}^k\lambda_i\alpha_i^1(x,u)\Big(\nabla(f_i(u)+u^Tz_i-\gamma_i(g_i(u)+u^Tv_i))\Big)+\sum_{j=1}^m\mu_j\alpha_i^2(x,u)\big(\nabla(h_j(u)+u^Tw_j)$$
$$+\sum_{i=1}^k\lambda_i\alpha_i^1(x,u)\Big(\nabla_p(K_i(u,p)-\gamma_iG_i(u,p))\Big)+\sum_{j=1}^m\mu_j\alpha_i^2(x,u)\nabla_qH_j(u,q)\Big]\Big) \tag{23}$$

As $\alpha_i^1(x,u)=\alpha_i^2(x,u)=\hat\alpha(x,u)$, using the sublinearity of F, we have

$$\sum_{i=1}^k\frac{\lambda_i}{\hat\alpha(x,u)}[(f_i(x)+x^Tz_i)-\gamma_i(g_i(x)-x^Tv_i)]$$
$$\geqq F\Big[x,u;\sum_{i=1}^k\lambda_i\Big(\nabla(f_i(u)+u^Tz_i-\gamma_i(g_i(u)-u^Tv_i))\Big)+\sum_{j=1}^m\mu_j\big(\nabla(h_j(u)+u^Tw_j)$$
$$+\sum_{i=1}^k\lambda_i\Big(\nabla_p(K_i(u,p)-\gamma_iG_i(u,p))\Big)+\sum_{j=1}^m\mu_j\nabla_qH_j(u,q)\Big]\Big) \tag{24}$$

Now by the feasibility condition (12) and the result $F(x,u;0)=0$, we get

$$\sum_{i=1}^{k} \frac{\lambda_i}{\bar{\alpha}(x,u)}[(f_i(x)+x^T z_i)-\gamma_i(g_i(x)-x^T v_i)] \geqq 0,$$

which contradicts (17). This completes the proof.

Theorem 4.2 (Strong duality). If u is an efficient solution of (FP) and $K_i(u,0)=0, G_i(u,0)=0, i=1,2,...,k, H_j(u,0)=0, j=1,2,...,m$, and a constraint qualification is satisfied for $(FP^1\bar{\epsilon})$ for at least one $r=1,2,...,k$, then there exist $\bar{\lambda}\in R^k, \bar{\mu}\in R^m, \bar{\gamma}\in R^k, \bar{z}_i\in R^n, \bar{v}_i\in R^n$ and $\bar{w}_j\in R^n$, $i=1,2,...,k, j=1,2,...,m$, such that $(u,\bar{\gamma},\bar{z},\bar{v},\bar{w},\bar{\mu},\bar{\lambda},p=0,q=0)$ is a feasible solution of (SFD). Further if the conditions of Weak duality theorem 4.1 are satisfied then $(u,\bar{\alpha},\bar{z},\bar{v},\bar{w},\bar{\mu},\bar{\lambda},p=0,q=0)$ is an efficient solution of (SFD) and the corresponding values of the objective functions are equal.

Proof. The proof follows along the lines of Theorem 4.2 [12] in light of the discussions given above and hence being omitted.

5. Conclusion

In the present analysis, we focus on a Mond-Weir type and Schaible type dual programs of a nondifferentiable multiobjective fractional programming problem in which every component of the objective and constraints functions contains a term involving the support function of a compact convex set and established weak and strong duality theorems under the assumptions of higher order (F,α,ρ,d)-V-type I functions. The question arise whether the duality results developed in this paper still holds for the nondifferentiable minimax fractional programming problem involving the support function of a compact convex set. This will orient the future research of the authors.

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J. COMPUTATIONAL ANALYSIS AND APPLICATIONS, VOL. 15, NO.8, 1456-1466, 2013, COPYRIGHT 2013 EUDOXUS PRESS, LLC

Fuzzy implicative filters of BE-algebras with degrees in the interval $(0, 1]$

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Abstract. In defining a fuzzy filter and a fuzzy implicative filter in BE-algebras, several degrees are provided, and then related properties are investigated.

1. Introduction

In [5], H. S. Kim and Y. H. Kim introduced the notion of a BE-algebra. S. S. Ahn and K. S. So [3,4] introduced the notion of ideals in BE-algebras. S. S. Ahn et al. [1] fuzzified the concept of BE-algebras, investigated some of their properties.

In this paper, we provide several degrees in defining a fuzzy filter and a fuzzy implicative filter. It is a generalization of a fuzzy filter.

2. Preliminaries

We recall some definitions and results discussed in [3,4,5].

An algebra $(X; *, 1)$ of type (2, 0) is called a *BE-algebra* if

(BE1) $x * x = 1$ for all $x \in X$;
(BE2) $x * 1 = 1$ for all $x \in X$;
(BE3) $1 * x = x$ for all $x \in X$;
(BE4) $x * (y * z) = y * (x * z)$ for all $x, y, z \in X$ (*exchange*)

We introduce a relation "$\leq$" on a BE-algebra X by $x \leq y$ if and only if $x * y = 1$. A non-empty subset S of a BE-algebra X is said to be a *subalgebra* of X if it is closed under the operation "$*$". Noticing that $x * x = 1$ for all $x \in X$, it is clear that $1 \in S$. A BE-algebra $(X; *, 1)$ is said to be *self distributive* if $x * (y * z) = (x * y) * (x * z)$ for all $x, y, z \in X$.

Definition 2.1.([5]) Let $(X; *, 1)$ be a BE-algebra and let F be a non-empty subset of X. Then F is called a *filter* of X if

(F1) $1 \in F$;
(F2) $x * y \in F$ and $x \in F$ imply $y \in F$.

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[0]**2010 Mathematics Subject Classification**: 03G25; 06F35; 08A72.

[0]**Keywords**: BE-algebra; enlarged filter; fuzzy enlarged (implicative) filter with degree.

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Example 2.2.([5]) Let $X := \{1, a, b, c, d, 0\}$ be a BE-algebra with the following table:

$*$	1	a	b	c	d	0
1	1	a	b	c	d	0
a	1	1	a	c	c	d
b	1	1	1	c	c	c
c	1	a	b	1	a	b
d	1	1	a	1	1	a
0	1	1	1	1	1	1

Then $F_1 := \{1, a, b\}$ is a filter of X, but $F_2 := \{1, a\}$ is not a filter of X, since $a * b \in F_2$ and $a \in F_2$, but $b \notin F_2$.

Proposition 2.3. *Let $(X; *, 1)$ be a BE-algebra and let F be a filter of X. If $x \leq y$ and $x \in F$ for any $y \in X$, then $y \in F$.*

Proposition 2.4. *Let $(X; *, 1)$ be a self distributive BE-algebra. Then the following hold: for any $x, y, z \in X$,*

(i) *if $x \leq y$, then $z * x \leq z * y$ and $y * z \leq x * z$.*
(ii) *$y * z \leq (z * x) * (y * z)$.*
(iii) *$y * z \leq (x * y) * (x * z)$.*

A BE-algebra $(X; *, 1)$ is said to be *transitive* if it satisfies Proposition 2.4(iii).

3. Fuzzy filters of BE-algebras with degrees in $(0, 1]$

In what follows let X denote a BE-algebra unless specified otherwise.

Definition 3.1. A fuzzy subset μ of a BE-algebra X is called a *fuzzy filter* of X if it satisfies for all $x, y \in X$

(d1) $\mu(1) \geq \mu(x)$,
(d2) $\mu(x) \geq \min\{\mu(y * x), \mu(y)\}$.

Proposition 3.2. *Let μ be a fuzzy filter of a BE-algebra X. Then for any $x, y \in X$, if $x \leq y$, then $\mu(x) \leq \mu(y)$.*

Proof. Straightforward. □

Definition 3.3. Let F be a non-empty subset of a BE-algebra X which is not necessary a filter of X. We say that a subset G of X is an *enlarged filter* of X related to F if it satisfies:

(1) F is a subset of G,
(2) $1 \in G$,
(3) $(\forall y \in X)(\forall x \in F)(x * y \in F \Rightarrow y \in G)$.

Obviously, every filter is an enlarged filter of X related to itself. Note that there exists an enlarged filter of X related to any non-empty subset F of X.

Example 3.4. Let $X = \{1, a, b, c, d, 0\}$ be a BE-algebra which is given in Example 2.2. Note that $F := \{1, a\}$ is not a filter since $a * b = a \in F$, $a \in F$ and $b \notin F$. Then $G := \{1, a, b, c\}$ is an enlarged filter of X related to F and G is not a filter of X since $b * d = c, b \in G$ and $d \notin G$.

In what follows let λ and κ be members of $(0, 1]$, and let n and k denote a natural number and a real number, respectively, such that $k < n$ unless otherwise specified.

Definition 3.5. A fuzzy subset μ of a BE-algebra X is called a *fuzzy filter* of X with degree (λ, κ) if it satisfies:

(1) $(\forall x \in X)(\mu(1) \geq \lambda\mu(x))$,
(2) $(\forall x, y \in X)(\mu(x) \geq \kappa\min\{\mu(y * x), \mu(y)\})$.

Note that if $\lambda \neq \kappa$, then a fuzzy filter with degree (λ, κ) may not be a fuzzy filter with degree (κ, λ), and vice versa. Obviously, every fuzzy filter is a fuzzy filter with degree (λ, κ), but the converse may not be true.

Example 3.6. Let $X := \{1, a, b, c\}$ be a BE-algebra in which the $*$-operation is given by the following table:

$*$	1	a	b	c
1	1	a	b	c
a	1	1	b	b
b	1	a	1	a
c	1	1	1	1

Define a fuzzy subset $\mu : X \to [0, 1]$ by

$$\mu = \begin{pmatrix} 1 & a & b & c \\ 0.4 & 0.3 & 0.7 & 0.7 \end{pmatrix}$$

Then μ is a fuzzy filter of X with degree $(\frac{4}{7}, \frac{4}{7})$, but it is neither a fuzzy filter of X nor a fuzzy filter of X with degree $(\frac{4}{5}, \frac{4}{5})$ since

$$\mu(1) = 0.4 \ngeq \mu(b) = 0.7$$

and

$$\mu(a) = 0.3 \ngeq \frac{4}{5} \times 0.4 = \frac{4}{5} \times \mu(1) = \frac{4}{5} \times \min\{\mu(c * a) = \mu(1), \mu(c)\}.$$

Define a fuzzy subset $\nu : X \to [0, 1]$ by

$$\nu = \begin{pmatrix} 1 & a & b & c \\ 0.6 & 0.4 & 0.7 & 0.7 \end{pmatrix}$$

Then ν is a fuzzy filter of X with degree $(\frac{4}{5}, \frac{3}{5})$, but it is neither a fuzzy filter of X nor a fuzzy filter of X with degree $(\frac{3}{5}, \frac{4}{5})$ since

$$\nu(1) = 0.6 \ngeq \nu(c) = 0.7$$

and

$$\nu(a) = 0.4 \ngeq 0.48 = \frac{4}{5} \times 0.6 = \frac{4}{5} \times \nu(1) = \frac{4}{5} \times \min\{\nu(c * a) = \nu(1), \nu(c)\}.$$

Note that a fuzzy filter with degree (λ, κ) is a fuzzy filter if and only if $(\lambda, \kappa) = (1, 1)$. Let λ_1 and λ_2 be members of $(0, 1]$. If $\lambda_1 > \lambda_2$, then every fuzzy filter with degree λ_2, but the converse is not true(See Example 3.6).

Proposition 3.7. *Every fuzzy filter of a BE-algebra X with degree (λ, κ) satisfies the following assertions.*

(i) $(\forall x, y \in X)(\mu(x * y) \geq \lambda\kappa\mu(y))$.
(ii) $(\forall x, y \in X)(y \leq x \Rightarrow \mu(x) \geq \lambda\kappa\mu(y))$.

Proof. (i) For any $x, y \in X$, we have

$$\begin{aligned}\mu(x * y) \geq& \kappa\min\{\mu(y * (x * y)), \mu(y)\}\\ =&\kappa\min\{\mu(x * (y * y)), \mu(y)\}\\ =&\kappa\min\{\mu(x * 1), \mu(y)\}\\ =&\kappa\min\{\mu(1), \mu(y)\}\\ \geq&\kappa\min\{\lambda\mu(y), \mu(y)\}\\ =&\kappa\lambda\mu(y).\end{aligned}$$

(ii) Let $x, y \in X$ be such that $y \leq x$. Then $y * x = 1$. Hence we have

$$\begin{aligned}\mu(x) \geq& \kappa\min\{\mu(y * x), \mu(y)\}\\ =&\kappa\min\{\mu(1), \mu(y)\}\\ \geq&\kappa\min\{\lambda\mu(y), \mu(y)\}\\ =&\lambda\kappa\mu(y)\end{aligned}$$

for any $x, y \in X$. □

Corollary 3.8. *Let μ be a fuzzy filter of a BE-algebra X with degree (λ, κ). If $\lambda = \kappa$, then*

(i) $(\forall x, y \in X)(\mu(x * y) \geq \lambda^2\mu(y))$.
(ii) $(\forall x, y \in X)(y \leq x \Rightarrow \mu(x) \geq \lambda^2\mu(y))$.

Denote by $\mathcal{F}(X)$ the set of all filters of a BE-algebra X. Note that a fuzzy subset μ of a BE-algebra X is a fuzzy filter of X if and only if

$$(\forall t \in [0, 1])(U(\mu; t) \in \mathcal{F}(X) \cup \{\emptyset\}).$$

But we know that for any fuzzy subset μ of a BE-algebra X there exist $\lambda, \kappa \in (0, 1)$ and $t \in [0, 1]$ such that

(1) μ is a fuzzy filter of X with degree (λ, κ),
(2) $U(\mu; t) \notin \mathcal{F}(X) \cup \{\emptyset\}$.

Example 3.9. Consider the fuzzy subset μ of $X = \{1, a, b, c\}$ which is given Example 3.6. If $t \in (0.4, 0.6]$, then $U(\mu; t) = \{1, b, c\}$ is not a filter of X. But μ is a fuzzy filter of X with degree $(0.4, 0.6)$.

Theorem 3.10. *Let μ be a fuzzy subset of a BE-algebra X. For any $t \in [0,1]$ with $t \leq$ max$\{\lambda,\kappa\}$, if $U(\mu; t)$ is an enlarged filter of X related to $U(\mu; \frac{t}{\max\{\lambda,\kappa\}})$, then μ is a fuzzy filter of X with degree (λ, κ).*

Proof. Assume that $\mu(1) < t \leq \lambda\mu(x)$ for some $x \in X$ and $t \in (0, \lambda]$. Then $\mu(x) \geq \frac{t}{\lambda} \geq \frac{t}{\max\{\lambda,\kappa\}}$ and so $x \in U(\mu; \frac{t}{\max\{\lambda,\kappa\}})$, i.e., $U(\mu; \frac{t}{\max\{\lambda,\kappa\}}) \neq \emptyset$. Since $U(\mu; t)$ is an enlarged filter of X related to $U(\mu; \frac{t}{\max\{\lambda,\kappa\}})$, we have $1 \in U(\mu; t)$, i.e., $\mu(1) \geq t$. This is a contradiction, and thus $\mu(1) \geq \lambda\mu(x)$ for all $x \in X$.

Now suppose that there exist $a, b, c \in X$ such that $\mu(a) < \kappa\min\{\mu(b * a), \mu(b)\}$. If we take $t := \kappa\min\{\mu(b * a), \mu(b)\}$, then $t \in (0, \kappa] \subseteq (0, \max\{\lambda, \kappa\}]$. Hence $b * a \in U(\mu; \frac{t}{\kappa}) \subseteq U(\mu; \frac{t}{\max\{\lambda,\kappa\}})$ and $b \in U(\mu; \frac{t}{\kappa}) \subseteq U(\mu; \frac{t}{\max\{\lambda,\kappa\}})$. It follows from Definition 3.3(3) that $a \in U(\mu; t)$ so that $\mu(a) \geq t$, which is impossible. Therefore

$$\mu(x) \geq \kappa\min\{\mu(y * x), \mu(y)\}$$

for all $x, y \in X$. Thus μ is a fuzzy filter of X with degree (λ, κ). □

Corollary 3.11. *Let μ be a fuzzy subset of a BE-algebra X. For any $t \in [0,1]$ with $t \leq \frac{k}{n}$, if $U(\mu; t)$ is an enlarged filter of X related to $U(\mu; \frac{n}{k}t)$, then μ is a fuzzy filter of X with degree $(\frac{k}{n}, \frac{k}{n})$.*

Theorem 3.12. *Let $t \in [0,1]$ be such that $U(\mu; t)(\neq \emptyset)$ is not necessary a filter of a BE-algebra X. If μ is a fuzzy filter of X with degree (λ, κ), then $U(\mu; t\min\{\lambda, \kappa\})$ is an enlarged filter of X related to $U(\mu; t)$.*

Proof. Since $t\min\{\lambda, \kappa\} \leq t$, we have $U(\mu; t) \subseteq U(\mu; t\min\{\lambda, \kappa\})$. Since $U(\mu; t) \neq \emptyset$, there exists $x \in U(\mu; t)$ and so $\mu(x) \geq t$. By Definition 3.5(1), we obtain $\mu(1) \geq \lambda\mu(x) \geq \lambda t \geq t\min\{\lambda, \kappa\}$. Therefore $1 \in U(\mu; t\min\{\lambda, \kappa\})$.

Let $x, y, z \in X$ be such that $y * x \in U(\mu; t)$ and $y \in U(\mu; t)$. Then $\mu(y * x) \geq t$ and $\mu(y) \geq t$. It follows from Definition 3.5(2) that

$$\begin{aligned}\mu(x) \geq &\kappa\min\{\mu(y * x), \mu(y)\} \\ \geq &\kappa t \geq t\min\{\lambda, \kappa\}.\end{aligned}$$

so that $x \in U(\mu; t\min\{\lambda, \kappa\})$. Thus $U(\mu; t\min\{\lambda, \kappa\})$ is an enlarged filter of X related to $U(\mu; t)$. □

Proposition 3.13. *Let μ be a fuzzy filter of a BE-algebra X with degree (λ, κ). If the inequality $x \leq y * z$ holds for any $x, y, z \in X$, then $\mu(z) \geq \min\{\kappa\mu(y), \lambda\kappa^2\mu(x)\}$.*

Proof. Suppose that $x \leq y * z$ for all $x, y, z \in X$. Then $x * (y * z) = 1$ and hence we have

$$\begin{aligned}\mu(y * z) \geq& \kappa\min\{\mu(x * (y * z)), \mu(x)\}\\ =&\kappa\min\{\mu(1), \mu(x)\}\\ \geq&\kappa\min\{\lambda\mu(x), \mu(x)\}\\ =&\kappa\lambda\mu(x).\end{aligned}$$

It follows that

$$\begin{aligned}\mu(z) \geq&\kappa\min\{\mu(y * z), \mu(y)\}\\ \geq&\kappa\min\{\kappa\lambda\mu(x), \mu(y)\}\\ =&\min\{\kappa\mu(y), \kappa^2\lambda\mu(x)\}\end{aligned}$$

for all $x, y, z \in X$. □

Corollary 3.14. *Let μ be a fuzzy filter of a BE-algebra X with degree (λ, κ). If $\lambda = \kappa$ and the inequality $x \leq y * z$ holds for any $x, y, z \in X$, then*

$$\mu(z) \geq \min\{\lambda\mu(y), \lambda^3\mu(x)\}$$

for all $x, y, z \in X$.

Corollary 3.15. *Let μ be a fuzzy filter of a BE-algebra X. If the inequality $x \leq y * z$ holds for any $x, y, z \in X$, then*

$$\mu(z) \geq \min\{\mu(y), \mu(x)\}$$

for all $x, y, z \in X$.

4. Fuzzy implicative filters of BE-algebras with degrees in $(0, 1]$

Definition 4.1. A non-empty subset F of a BE-algebra X is called an *implicative filter* of X if it satisfies (F1) and

(F3) $x * (y * z) \in F$ and $x * y \in F$ imply $x * z \in F$

for all $x, y, z \in X$.

Example 4.2. Consider a BE-algebra $X = \{1, a, b, c, d, 0\}$ which is given Example 2.2. It is easy to see that the set $F := \{1, a, b\}$ is an implicative filter of X.

Note that every implicative filter of a BE-algebra X is a filter of X.

Definition 4.3. A fuzzy subset μ of a BE-algebra X is called a *fuzzy implicative filter* of X if it satisfies (d1) and

(d3) $\mu(x * z) \geq \min\{\mu(x * (y * z)), \mu(x * y)\}$

for all $x, y, z \in X$.

Definition 4.4. Let F be a non-empty subset of a BE-algebra X which is not necessary an implicative filter of X. We say that a subset G of X is an *enlarged implicative filter* of X related to F if it satisfies:

(1) F is a subset of G,
(2) $1 \in G$,
(3) $(\forall x, y, z \in X)(x * (y * z) \in F$ and $x * y \in F \Rightarrow x * z \in G)$.

Obviously, every implicative filter is an enlarged implicative filter of a BE-algebra X related to itself. Note that there exists an enlarged implicative filter of X related to any non-empty subset F of X.

Example 4.5. Consider a BE-algebra $X = \{1, a, b, c, d, 0\}$ which is given in Example 2.2. Note that $F := \{1, a\}$ is not both a filter and an implicative filter of X. Then $G := \{1, a, b, c\}$ is an enlarged implicative filter of X related to F.

Proposition 4.6. *Let F be a non-empty subset of a BE-algebra X. Every enlarged implicative filter of X related to F is an enlarged filter of X related to F.*

Proof. Let G be an enlarged implicative filter of X related to F. Putting $x = 1$ in Definition 4.4(3) and use (BE3), we have

$$(\forall y, z \in X)(1 * (y * z) = y * z \in F \text{ and } 1 * y = y \in F \Rightarrow 1 * z = z \in G).$$

Hence G is an enlarged filter of X related to F. □

The converse of Proposition 4.6 is not true in general as seen in the following example.

Example 4.7. Let $X := \{1, a, b, c\}$ be a BE-algebra([3]) in which the $*$-operation is given by the following table:

$*$	1	a	b	c
1	1	a	b	c
a	1	1	a	a
b	1	1	1	a
c	1	1	a	1

Let $F := \{1\}$ and $G := \{1, c\}$. Then G is an enlarged filter of F but it is not an enlarged implicative filter of F since $b * (a * c) = 1 \in F, b * a = 1 \in F$ and $b * c = a \notin G$.

Definition 4.8. A fuzzy subset μ of a BE-algebra X is called a *fuzzy implicative filter* of X with degree (λ, κ) if it satisfies Definition 3.5(1)

(2) $(\forall x, y, z \in X)(\mu(x * z) \geq \kappa \min\{\mu(x * (y * z)), \mu(x * y)\})$.

Note that if $\lambda \neq \kappa$, then a fuzzy implicative filter with degree (λ, κ) may not be a fuzzy implicative filter with degree (κ, λ), and vice versa. Obviously, every fuzzy implicative filter is a fuzzy implicative filter with degree (λ, κ), but the converse may not be true.

Example 4.9. Consider a BE-algebra $X = \{1, a, b, c, d, 0\}$ which is given in Example 2.2. Define a fuzzy subset $\mu : X \to [0, 1]$ by

$$\mu = \begin{pmatrix} 1 & a & b & c & d & 0 \\ 0.7 & 0.8 & 0.8 & 0.4 & 0.5 & 0.4 \end{pmatrix}$$

Then μ is a fuzzy implicative filter of X with degree $(\frac{5}{6}, \frac{3}{6})$, but it is neither a fuzzy filter of X nor a fuzzy implicative filter of X with degree $(\frac{3}{6}, \frac{5}{6})$ since

$$\mu(1) = 0.7 \ngeq \mu(a) = 0.8$$

and

$$\begin{aligned}\mu(1 * 0) = \mu(0) = 0.4 \ngeq 0.42 &= \frac{5}{6} \times 0.5 = \frac{5}{6} \times \mu(d) \\ &= \frac{5}{6} \times \min\{\mu(1 * (a * 0) = \mu(d), \mu(1 * a) = \mu(a)\}.\end{aligned}$$

Obviously, every fuzzy implicative filter of a BE-algebra X is a fuzzy implicative filter of X with degree (λ, κ), but the converse may not be true. In fact, the fuzzy implicative filter μ of X with degree $(\frac{3}{6}, \frac{5}{6})$ in Example 4.9 is not a fuzzy implicative filter of X. Note that a fuzzy implicative filter with degree (λ, κ) is a fuzzy implicative filter if and only if $(\lambda, \kappa) = (1, 1)$.

Proposition 4.10. *If μ is a fuzzy implicative filter of a BE-algebra X degree (λ, κ), then μ is a fuzzy filter of X with degree (λ, κ).*

Proof. Putting $x := 1$ in Definition 4.8(2), we have

$$\begin{aligned}\mu(z) = \mu(1 * z) &\geq \kappa \min\{\mu(1 * (y * z)), \mu(1 * y)\} \\ &= \kappa \min\{\mu(y * z), \mu(y)\}\end{aligned}$$

for any $y, z \in X$. Thus μ is a fuzzy filter of X with degree (λ, κ). □

The converse of Proposition 4.10 is not true in general as seen in the following example.

Example 4.11. Consider a BE-algebra $X = \{1, a, b, c\}$ which is given in Example 4.7. Define a fuzzy subset $\mu : X \to [0, 1]$ by

$$\mu = \begin{pmatrix} 1 & a & b & c \\ 0.6 & 0.3 & 0.3 & 0.7 \end{pmatrix}$$

Then μ is a fuzzy filter of X with degree $(\frac{3}{6}, \frac{4}{7})$, but it is neither a fuzzy filter of X nor a fuzzy implicative filter of X with degree $(\frac{3}{6}, \frac{4}{7})$ since

$$\mu(1) = 0.6 \ngeq \mu(c) = 0.7$$

and

$$\begin{aligned}\mu(b * c) = \mu(a) = 0.3 \ngeq 0.34 &= \frac{4}{7} \times 0.6 = \frac{4}{7} \times \mu(1) \\ &= \frac{4}{7} \times \min\{\mu(b * (a * c)) = \mu(1), \mu(b * a) = \mu(1)\}.\end{aligned}$$

Proposition 4.12. *Every fuzzy implicative filter of a BE-algebra X with degree (λ, κ) satisfies the following assertions.*

(i) $(\forall x, y \in X)(\mu(x * y) \geq \lambda\kappa\mu(y))$.
(ii) $(\forall x, y \in X)(x \leq y \Rightarrow \mu(y) \geq \lambda\kappa\mu(x))$.

Proof. It follows from Proposition 3.7 and Proposition 4.10. □

Corollary 4.13. *Let μ be a fuzzy implicative filter of a BE-algebra X with degree (λ,κ). If $\lambda=\kappa$, then*

(i) $(\forall x,y\in X)(\mu(x*y)\geq \lambda^2\mu(y))$.
(ii) $(\forall x,y\in X)(x\leq y\Rightarrow \mu(y)\geq \lambda^2\mu(x))$.

Proposition 4.14. *Let μ be a fuzzy implicative filter of a BE-algebra X with degree (λ,κ). Then the following are hold:*

(i) $\forall x,y\in X)(\mu(x*y)\geq \lambda\kappa\mu(x*(x*y)))$.
(ii) $(\forall x,y,z\in X)(\mu(y*z)\geq \lambda\kappa^2 min\{\mu(x*(y*(y*z))),\mu(x)\})$.

Proof. (i) Assume that μ is a fuzzy implicative filter of a BE-algebra X with degree (λ,κ). Putting $z:=y, y:=x$ in Definition 4.8(2), we have

$$\begin{aligned}\mu(x*y)\geq&\kappa\min\{\mu(x*(x*y)),\mu(x*x)\}\\=&\kappa\min\{\mu(x*(x*y)),\mu(1)\}\\\geq&\kappa\min\{\mu(x*(x*y)),\lambda\mu(x*(x*y))\}\\=&\kappa\lambda\mu(x*(x*y))\end{aligned}$$

for all $x,y\in X$. Thus (i) holds.
(ii) Since μ is a fuzzy filter of X with degree (λ,κ) and using (i), we have

$$\begin{aligned}\mu(y*z)\geq&\lambda\kappa\mu(y*(y*z))\\\geq&\lambda\kappa^2\min\{\mu(x*(x*(y*z))),\mu(x)\}\end{aligned}$$

for any $x,y,z\in X$. Hence (ii) holds.

□

Corollary 4.15. *Let μ be a fuzzy implicative filter of a BE-algebra X with degree (λ,κ). If $\lambda=\kappa$, then*

(i) $(\forall x,y\in X)(\mu(x*y)\geq \lambda^2\mu(x*(x*y)))$.
(ii) $(\forall x,y,z\in X)(\mu(y*z)\geq \kappa^3 min\{\mu(x*(y*(y*z))),\mu(x)\})$.

Proposition 4.16. *Let X be a self distributive BE-algebra X. Then μ is a fuzzy filter of X with degree (λ,κ) if and only if it is a fuzzy implicative filter of X with degree (λ,κ).*

Proof. Proposition 4.10, a fuzzy implicative filter of a BE-algebra X with degree (λ,κ) is a fuzzy filter of X with degree (λ,κ).

Conversely, assume that μ is a fuzzy filter of a BE-algebra X with degree (λ,κ). Since X is a self distributive BE-algebra, we have

$$\begin{aligned}\mu(x*z)\geq&\kappa\min\{\mu((x*y)*(x*z)),\mu(x*y)\}\\=&\kappa\min\{\mu(x*(y*z)),\mu(x*y)\}\end{aligned}$$

for any $x,y,z\in X$. Hence X is a fuzzy implicative filter of X with degree (λ,κ). □

Denote by $\mathcal{F}_I(X)$ the set of all implicative filters of a BE-algebra X. Note that a fuzzy subset μ of a BE-algebra X is a fuzzy implicative filter of X if and only if

$$(\forall t \in [0,1])(U(\mu;t) \in \mathcal{F}_I(X) \cup \{\emptyset\}).$$

But we know that for any fuzzy subset μ of a BE-algebra X there exist $\lambda, \kappa \in (0,1)$ and $t \in [0,1]$ such that

(1) μ is a fuzzy implicative filter of X with degree (λ, κ),
(2) $U(\mu;t) \notin \mathcal{F}_I(X) \cup \{\emptyset\}$.

Example 4.17. Let $X := \{1, a, b, c\}$ be a set in which the $*$-operation is given by the following table:

$*$	1	a	b	c
1	1	a	b	c
a	1	1	b	c
b	1	a	1	c
c	1	1	b	1

Then X is self distributive BE-algebra. Define a fuzzy subset $\mu : X \to [0,1]$ by

$$\mu = \begin{pmatrix} 1 & a & b & c \\ 0.4 & 0.3 & 0.2 & 0.6 \end{pmatrix}$$

If $t \in (0.4, 0.6]$, then $U(\mu;t) = \{1, c\}$ is not an implicative filter of X since $1*(c*a) = 1 \in \{1,c\}$, and $1*c \in \{1,c\}$ but $1*a = a \notin \{1,c\}$. But μ is a fuzzy implicative filter of X with degree $(0.4, 0.6)$.

Theorem 4.18. *Let μ be a fuzzy subset of a BE-algebra X. For any $t \in [0,1]$ with $t \leq \max\{\lambda,\kappa\}$, if $U(\mu;t)$ is an enlarged implicative filter of X related to $U(\mu; \frac{t}{\max\{\lambda,\kappa\}})$, then μ is a fuzzy implicative filter of X with degree (λ,κ).*

Proof. Assume that $\mu(1) < t \leq \lambda\mu(x)$ for some $x \in X$ and $t \in (0,\lambda]$. Then $\mu(x) \geq \frac{t}{\lambda} \geq \frac{t}{\max\{\lambda,\kappa\}}$ and so $x \in U(\mu; \frac{t}{\max\{\lambda,\kappa\}})$, i.e., $U(\mu; \frac{t}{\max\{\lambda,\kappa\}}) \neq \emptyset$. Since $U(\mu;t)$ is an enlarged filter of X related to $U(\mu; \frac{t}{\max\{\lambda,\kappa\}})$, we have $1 \in U(\mu;t)$, i.e., $\mu(1) \geq t$. This is a contradiction, and thus $\mu(1) \geq \lambda\mu(x)$ for all $x \in X$.

Now suppose that there exist $a,b,c \in X$ such that $\mu(a*c) < \kappa\min\{\mu(a*(b*c)), \mu(a*b)\}$. If we take $t := \kappa\min\{\mu(a*(b*c)), \mu(a*b)\}$, then $t \in (0,\kappa] \subseteq (0, \max\{\lambda,\kappa\}]$. Hence $a*(b*c) \in U(\mu; \frac{t}{\kappa}) \subseteq U(\mu; \frac{t}{\max\{\lambda,\kappa\}})$ and $a*b \in U(\mu; \frac{t}{\kappa}) \subseteq U(\mu; \frac{t}{\max\{\lambda,\kappa\}})$. It follows from Definition 4.8(2) that $a*c \in U(\mu;t)$ so that $\mu(a*c) \geq t$, which is impossible. Therefore

$$\mu(x*z) \geq \kappa\min\{\mu(x*(y*z)), \mu(x*y)\}$$

for all $x,y,z \in X$. Thus μ is a fuzzy implicative filter of X with degree (λ,κ). □

Corollary 4.19. *Let μ be a fuzzy subset of a BE-algebra X. For any $t \in [0,1]$ with $t \leq \frac{k}{n}$, if $U(\mu;t)$ is an enlarged implicative filter of X related to $U(\mu; \frac{n}{k}t)$, then μ is a fuzzy implicative filter of X with degree $(\frac{k}{n}, \frac{k}{n})$.*

Theorem 4.20. *Let $t \in [0, 1]$ be such that $U(\mu; t)(\neq \emptyset)$ is not necessary an implicative filter of a BE-algebra X. If μ is a fuzzy implicative filter of X with degree (λ, κ), then $U(\mu; t\min\{\lambda, \kappa\})$ is an enlarged implicative filter of X related to $U(\mu; t)$.*

Proof. Since $t\min\{\lambda, \kappa\} \leq t$, we have $U(\mu; t) \subseteq U(\mu; t\min\{\lambda, \kappa\})$. Since $U(\mu; t) \neq \emptyset$, there exists $x \in U(\mu; t)$ and so $\mu(x) \geq t$. By Definition 4.8(1), we obtain $\mu(1) \geq \lambda\mu(x) \geq \lambda t \geq t\min\{\lambda, \kappa\}$. Therefore $1 \in U(\mu; t\min\{\lambda, \kappa\})$.

Let $x, y, z \in X$ be such that $x * (y * z) \in U(\mu; t)$ and $x * y \in U(\mu; t)$. Then $\mu(x * (y * z)) \geq t$ and $\mu(x * y) \geq t$. It follows from Definition 4.8(2) that

$$\begin{aligned} \mu(x * z) \geq& \kappa\min\{\mu(x * (y * z)), \mu(x * y)\} \\ \geq& \kappa t \geq t\min\{\lambda, \kappa\}. \end{aligned}$$

so that $x * z \in U(\mu; t\min\{\lambda, \kappa\})$. Thus $U(\mu; t\min\{\lambda, \kappa\})$ is an enlarged implicative filter of X related to $U(\mu; t)$. □

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J. COMPUTATIONAL ANALYSIS AND APPLICATIONS, VOL. 15, NO.8, 1467-1475, 2013, COPYRIGHT 2013 EUDOXUS PRESS, LLC

An AQ-functional equation in paranormed spaces

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Abstract. In this paper, we prove the Hyers-Ulam stability of an additive-quadratic functional equation in paranormed spaces.

Keywords: Hyers-Ulam stability, paranormed space, additive-quadratic functional equation.

1. Introduction and preliminaries

The concept of statistical convergence for sequences of real numbers was introduced by Fast [9] and Steinhaus [35] independently and since then several generalizations and applications of this notion have been investigated by various authors (see [10, 19, 22, 23, 33]). This notion was defined in normed spaces by Kolk [20].

We recall some basic facts concerning Fréchet spaces.

Definition 1.1. [37] Let X be a vector space. A paranorm $P : X \to [0, \infty)$ is a function on X such that

(1) $P(0) = 0$;
(2) $P(-x) = P(x)$;
(3) $P(x + y) \leq P(x) + P(y)$ (triangle inequality)
(4) If $\{t_n\}$ is a sequence of scalars with $t_n \to t$ and $\{x_n\} \subset X$ with $P(x_n - x) \to 0$, then $P(t_n x_n - tx) \to 0$ (continuity of multiplication).

The pair (X, P) is called a *paranormed space* if P is a *paranorm* on X.

The paranorm is called *total* if, in addition, we have

(5) $P(x) = 0$ implies $x = 0$.

A *Fréchet space* is a total and complete paranormed space.

The stability problem of functional equations originated from a question of Ulam [36] concerning the stability of group homomorphisms. Hyers [13] gave a first affirmative partial answer to the question of Ulam for Banach spaces. Hyers' Theorem was generalized by Aoki [2] for additive mappings

[0]2010 Mathematics Subject Classification: Primary 35A17; 39B52; 39B72.
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and by Th.M. Rassias [27] for linear mappings by considering an unbounded Cauchy difference. A generalization of the Th.M. Rassias theorem was obtained by Găvruta [12] by replacing the unbounded Cauchy difference by a general control function in the spirit of Th.M. Rassias' approach.

In 1990, Th.M. Rassias [28] during the 27^{th} International Symposium on Functional Equations asked the question whether such a theorem can also be proved for $p \geq 1$. In 1991, Gajda [11] following the same approach as in Th.M. Rassias [27], gave an affirmative solution to this question for $p > 1$. It was shown by Gajda [11], as well as by Th.M. Rassias and Šemrl [32] that one cannot prove a Th.M. Rassias' type theorem when $p = 1$ (cf. the books of P. Czerwik [5], D.H. Hyers, G. Isac and Th.M. Rassias [14]).

In 1982, J.M. Rassias [25] followed the innovative approach of the Th.M. Rassias' theorem [27] in which he replaced the factor $\|x\|^p + \|y\|^p$ by $\|x\|^p \cdot \|y\|^q$ for $p, q \in \mathbb{R}$ with $p + q \neq 1$.

The functional equation

$$f(x+y) + f(x-y) = 2f(x) + 2f(y)$$

is called a *quadratic functional equation.* In particular, every solution of the quadratic functional equation is said to be a *quadratic mapping.* A Hyers-Ulam stability problem for the quadratic functional equation was proved by Skof [34] for mappings $f : X \to Y$, where X is a normed space and Y is a Banach space. Cholewa [3] noticed that the theorem of Skof is still true if the relevant domain X is replaced by an Abelian group. Czerwik [4] proved the Hyers-Ulam stability of the quadratic functional equation. The stability problems of several functional equations have been extensively investigated by a number of authors and there are many interesting results concerning this problem (see [1, 8, 15, 17, 18, 24, 26], [29]–[31]).

Throughout this paper, assume that (X, P) is a Fréchet space and that $(Y, \|\cdot\|)$ is a Banach space.

In this paper, we prove the Hyers-Ulam stability of the following additive-quadratic functional equation

$$2f\left(\frac{x+y}{2}\right) + f\left(\frac{x-y}{2}\right) + f\left(\frac{y-x}{2}\right) = f(x) + f(y) \tag{1.1}$$

in paranormed spaces.

One can easily show that an odd mapping $f : X \to Y$ satisfies (1.1) if and only if the odd mapping mapping $f : X \to Y$ is an additive mapping, i.e.,

$$2f\left(\frac{x+y}{2}\right) = f(x) + f(y).$$

One can easily show that an even mapping $f : X \to Y$ satisfies (1.1) if and only if the even mapping $f : X \to Y$ is a quadratic mapping, i.e.,

$$2f\left(\frac{x+y}{2}\right) + 2f\left(\frac{x-y}{2}\right) = f(x) + f(y).$$

2. Hyers-Ulam stability of the functional equation (1.1): an odd mapping case

For a given mapping f, we define

$$Df(x,y) := \quad 2f\left(\frac{x+y}{2}\right) + f\left(\frac{x-y}{2}\right) + f\left(\frac{y-x}{2}\right) - f(x) - f(y).$$

In this section, we prove the Hyers-Ulam stability of the functional equation $Df(x, y) = 0$ in paranormed spaces: an odd mapping case.

Note that $P(2x) \leq 2P(x)$ for all $x \in Y$.

Theorem 2.1. *Let $\phi : Y \to [0, \infty)$ be a function such that*

$$\pi(x, y) := \sum_{j=0}^{\infty} 2^j \phi\left(\frac{x}{2^j}, \frac{y}{2^j}\right) < +\infty$$

for all $x, y \in Y$. Let $f : Y \to X$ be an odd mapping such that

$$P(Df(x, y)) \leq \phi(x, y) \tag{2.1}$$

for all $x, y \in Y$. Then there exists a unique additive mapping $A : Y \to X$ such that

$$P(f(x) - A(x)) \leq \pi(x, 0) \tag{2.2}$$

for all $x \in Y$.

Proof. Considering f as an odd mapping, we have

$$P\left(2f\left(\frac{x+y}{2}\right) - f(x) - f(y)\right) \leq \phi(x, y) \tag{2.3}$$

for all $x, y \in Y$.

Letting $y = 0$ in (2.3), we get

$$P\left(f(x) - 2f\left(\frac{x}{2}\right)\right) \leq \phi(x, 0)$$

for all $x, y \in Y$.

Hence

$$P\left(2^n f\left(\frac{x}{2^n}\right) - 2^m f\left(\frac{x}{2^m}\right)\right) \leq \sum_{j=n}^{m-1} 2^j \phi\left(\frac{x}{2^j}, 0\right) \tag{2.4}$$

holds for all non-negative integers n and m with $m > n$ and all $x \in Y$. It follows from (2.4) that the sequence $\{2^k f(\frac{x}{2^k})\}$ is a Cauchy sequence for all $x \in Y$. Since X is complete, the sequence $\{2^k f(\frac{x}{2^k})\}$ converges. So the mapping $A : Y \to X$ can be defined as

$$A(x) := \lim_{k\to\infty} 2^k f\left(\frac{x}{2^k}\right)$$

for all $x \in Y$.

By (2.1),

$$P(DA(x, y)) = \lim_{k\to\infty} P\left(2^k Df\left(\frac{x}{2^k}, \frac{y}{2^k}\right)\right) \leq \lim_{k\to\infty} 2^k \phi\left(\frac{x}{2^k}, \frac{y}{2^k}\right) = 0$$

for all $x, y \in Y$. So $DA(x, y) = 0$. Since $f : Y \to X$ is odd, $A : Y \to X$ is odd. So the mapping $A : Y \to X$ is additive. Moreover, letting $n = 0$ and passing the limit $m \to \infty$ in (2.4), we get (2.2). So there exists an additive mapping $A : Y \to X$ satisfying (2.2).

Now, let $T : Y \to X$ be another additive mapping satisfying (2.2). Then we have

$$\begin{aligned}
P(A(x) - T(x)) &= P\left(2^q A\left(\frac{x}{2^q}\right) - 2^q T\left(\frac{x}{2^q}\right)\right) \\
&\leq P\left(2^q \left(A\left(\frac{x}{2^q}\right) - f\left(\frac{x}{2^q}\right)\right)\right) + P\left(2^q \left(T\left(\frac{x}{2^q}\right) - f\left(\frac{x}{2^q}\right)\right)\right) \\
&\leq 2 \times 2^q \pi(\frac{x}{2^q}, 0),
\end{aligned}$$

which tends to zero as $q \to \infty$ for all $x \in Y$. So we can conclude that $A(x) = T(x)$ for all $x \in Y$. This proves the uniqueness of A. Thus the mapping $A : Y \to X$ is the unique additive mapping satisfying (2.2). □

Corollary 2.2. *Let r, θ be positive real numbers with $r > 1$, and let $f : Y \to X$ be an odd mapping such that*

$$P(Df(x,y)) \leq \theta(\|x\|^r + \|y\|^r) \tag{2.5}$$

for all $x, y \in Y$. Then there exists a unique additive mapping $A : Y \to X$ such that

$$P(f(x) - A(x)) \leq \frac{2^r}{2^r - 2}\theta\|x\|^r$$

for all $x \in Y$.

Proof. Letting $\phi(x,y) := \theta(\|x\|^r + \|y\|^r)$ in Theorem 2.1, we obtain the result. $\square$

Theorem 2.3. *Let $\phi : X \to [0,\infty)$ be a function such that*

$$\pi(x,y) := \sum_{j=1}^{\infty} \frac{1}{2^j}\phi\left(2^j x, 2^j y\right) < \infty$$

for all $x, y \in X$. Let $f : X \to Y$ be an odd mapping such that

$$\|Df(x,y)\| \leq \phi(x,y) \tag{2.6}$$

for all $x, y \in X$. Then there exists a unique additive mapping $A : X \to Y$ such that

$$\|f(x) - A(x)\| \leq \pi(x,0) \tag{2.7}$$

for all $x \in X$.

Proof. Considering f as an odd mapping, we have

$$\left\|2f\left(\frac{x+y}{2}\right) - f(x) - f(y)\right\| \leq \phi(x,y) \tag{2.8}$$

for all $x, y \in X$.

Letting $y = 0$ and replacing x by $2x$ in (2.8), we get

$$\|2f(x) - f(2x)\| \leq \phi(2x, 0)$$

for all $x, y \in X$.

Hence

$$\left\|\frac{1}{2^n}f(2^n x) - \frac{1}{2^m}f(2^m x)\right\| \leq \sum_{j=n}^{m-1} \frac{1}{2^{j+1}}\phi(2^{j+1}x, 0) \tag{2.9}$$

holds for all non-negative integers n and m with $m > n$ and all $x \in X$. It follows from (2.9) that the sequence $\{\frac{1}{2^k}f(2^k x)\}$ is a Cauchy sequence for all $x \in X$. Since Y is complete, the sequence $\{\frac{1}{2^k}f(2^k x)\}$ converges. So the mapping $A : X \to Y$ can be defined as

$$A(x) := \lim_{k\to\infty} \frac{1}{2^k} f\left(2^k x\right)$$

for all $x \in X$.

By (2.6),

$$\|DA(x,y)\| = \lim_{k\to\infty}\left\|\frac{1}{2^k}Df\left(2^k x, 2^k y\right)\right\| \leq \lim_{k\to\infty}\frac{1}{2^k}\phi\left(2^k x, 2^k y\right) = 0$$

for all $x, y \in X$, and $DA(x,y) = 0$ follows. Also, since $f : X \to Y$ is odd, $A : X \to Y$ is odd. So the mapping $A : X \to Y$ is additive. Moreover, letting $n = 0$ and passing the limit $m \to \infty$ in (2.9), we get (2.7). So there exists an additive mapping $A : X \to Y$ satisfying (2.7).

Now, let $T : X \to Y$ be another additive mapping satisfying (2.7). Then we have

$$\begin{aligned}\|A(x)-T(x)\| &= \left\|\frac{1}{2^q}A\left(2^q x\right)-\frac{1}{2^q}T\left(2^q x\right)\right\| \\ &\le \left\|\frac{1}{2^q}\left(A\left(2^q x\right)-f\left(2^q x\right)\right)\right\|+\left\|\frac{1}{2^q}\left(T\left(2^q x\right)-f\left(2^q x\right)\right)\right\| \\ &\le 2\times\frac{1}{2^q}\pi(2^q x,0),\end{aligned}$$

which tends to zero as $q \to \infty$ for all $x \in X$. So we have $A(x) = T(x)$ for all $x \in X$. This proves the uniqueness of A. Thus the mapping $A : X \to Y$ is the unique additive mapping satisfying (2.7). □

Corollary 2.4. *Let r, θ be positive real numbers with $r < 1$, and let $f : X \to Y$ be an odd mapping such that*

$$\|Df(x,y)\| \le \theta(P(x)^r + P(y)^r) \tag{2.10}$$

for all $x, y \in X$. Then there exists a unique additive mapping $A : X \to Y$ such that

$$\|f(x)-A(x)\| \le \frac{2^r}{2-2^r}\theta P(x)^r$$

for all $x \in X$.

Proof. Letting $\phi(x,y) := \theta(P(x)^r + P(y)^r)$ in Theorem 2.3, we obtain the result. □

3. Hyers-Ulam stability of the functional equation (1.1): an even mapping case

In this section, we prove the Hyers-Ulam stability of the functional equation $Df(x,y) = 0$ in paranormed spaces: an even mapping case.

Note that $P(2x) \le 2P(x)$ for all $x \in Y$.

Theorem 3.1. *Let $\phi : X \to [0,\infty)$ be a function such that*

$$\pi(x,y) := \sum_{j=1}^{\infty}\frac{1}{4^j}\phi\left(2^j x, 2^j y\right) < +\infty$$

for all $x, y \in X$. Let $f : X \to Y$ be an even mapping such that $f(0) = 0$ and

$$\|Df(x,y)\| \le \phi(x,y) \tag{3.1}$$

for all $x, y \in X$. Then there exists a unique quadratic mapping $Q : X \to Y$ such that

$$\|f(x)-Q(x)\| \le \pi(x,0) \tag{3.2}$$

for all $x \in X$.

Proof. Letting $y = 0$ in (3.1), we get

$$\|4f\left(\frac{x}{2}\right)-f(x)\| \le \phi(x,0) \tag{3.3}$$

for all $x \in X$.

Replacing x by $2^{j+1}x$ in (3.3), we get

$$\|4f\left(2^j x\right)-f\left(2^{j+1}x\right)\| \le \phi\left(2^{j+1}x,0\right)$$

for all $x \in X$. Hence

$$\|\frac{1}{4^m}f\left(2^m x\right)-\frac{1}{4^n}f\left(2^n y\right)\| \le \sum_{j=n+1}^{m}\frac{1}{4^j}\phi\left(2^j x,0\right) \tag{3.4}$$

for all non-negative integers n and m with $m > n$ and all $x \in X$.
It follows from (3.4) that that the sequence $\{\frac{1}{4^k} f\left(2^k x\right)\}$ is a Cauchy sequence for all $x \in X$. Since Y is complete, the sequence $\{\frac{1}{4^k} f\left(2^k x\right)\}$ converges. So one can define the mapping $Q : X \to Y$ by

$$Q(x) := \lim_{k\to\infty} \frac{1}{4^k} f\left(2^k x\right)$$

for all $x \in X$.

By (3.1),

$$\|DQ(x,y))\| = \lim_{k\to\infty} \|\frac{1}{4^k} Df\left(2^k x, 2^k y\right)\| \leq \lim_{k\to\infty} \frac{1}{4^k}\phi\left(2^k x, 2^k y\right) = 0$$

for all $x, y \in X$. So $DQ(x,y) = 0$. Since $f : X \to Y$ is even, $Q : X \to Y$ is even. So the mapping $Q : X \to Y$ is quadratic. Moreover, letting $n = 0$ and passing the limit $m \to \infty$ in (3.4), we get (3.2). So there exists a quadratic mapping $Q : X \to Y$ satisfying (3.2).

Let $T : X \to Y$ be a quadratic mapping satisfying (3.2). Since T satisfies $4T\left(\frac{x}{2}\right) = T(x)$, we have $T(x) = \frac{1}{4^q} T\left(2^q x\right)$ for all integer q. Hence

$$\begin{aligned}
\|Q(x) - T(x)\| &= \|\frac{1}{4^q} Q\left(2^q x\right) - \frac{1}{4^q} T\left(2^q x\right)\| \\
&\leq \|\frac{1}{4^q}\left(Q\left(2^q x\right) - f\left(2^q x\right)\right)\| + \|\frac{1}{4^q}\left(T\left(2^q x\right) - f\left(2^q x\right)\right)\| \\
&\leq 2 \times \frac{1}{4^q}\pi\left(2^q x, 0\right),
\end{aligned}$$

which tends to zero as $q \to \infty$ for all $x \in X$. So $Q(x) = T(x)$ for all $x \in X$. This proves the uniqueness of Q. Thus the mapping $Q : X \to Y$ is the unique quadrative mapping satisfying (3.2). □

Corollary 3.2. *Let r, θ be positive real numbers with $r < 2$, and let $f : X \to Y$ be an even mapping such that $f(0) = 0$ and*

$$\|Df(x,y)\| \leq \theta(P(x)^r + P(y)^r)$$

for all $x, y \in Y$. Then there exists a unique quadrative mapping $Q : X \to Y$ such that

$$\|f(x) - Q(x)\| \leq \frac{2^r}{4 - 2^r}\theta P(x)^r$$

for all $x \in Y$.

Proof. Letting $\phi(x,y) := \theta(P(x)^r + P(y)^r)$ in Theorem 3.1, we obtain the result. □

Theorem 3.3. *Let $\phi : Y \to [0,\infty)$ be a function such that*

$$\pi(x,y) := \sum_{j=0}^{\infty} 4^j \phi\left(\frac{x}{2^j}, \frac{y}{2^j}\right) < +\infty$$

for all $x, y \in Y$. Let $f : Y \to X$ be an even mapping such that $f(0) = 0$ and

$$P\left(Df(x,y)\right) \leq \phi(x,y) \tag{3.5}$$

for all $x, y \in Y$. Then there exists a unique quadratic mapping $Q : Y \to X$ such that

$$P\left(f(x) - Q(x)\right) \leq \pi(x,0) \tag{3.6}$$

for all $x \in Y$.

Proof. Letting $y = 0$ in (3.5), we get

$$P\left(4f\left(\frac{x}{2}\right) - f(x)\right) \leq \phi(x, 0) \tag{3.7}$$

for all $x \in Y$.

Replacing x by $\frac{x}{2^j}$ in (3.7), we get

$$P\left(4f\left(\frac{x}{2^{j+1}}\right) - f\left(\frac{x}{2^j}\right)\right) \leq \phi\left(\frac{x}{2^j}, 0\right)$$

for all $x \in Y$. Hence

$$P\left(4^m f\left(\frac{x}{2^m}\right) - 4^n f\left(\frac{x}{2^n}\right)\right) \leq \sum_{j=n}^{m-1} 4^j \phi\left(\frac{x}{2^j}, 0\right) \tag{3.8}$$

for all non-negative integers n and m with $m > n$ and all $x \in Y$.
It follows from (3.8) that that the sequence $\{4^k f\left(\frac{x}{2^k}\right)\}$ is a Cauchy sequence for all $x \in Y$. Since X is complete, the sequence $\{4^k f\left(\frac{x}{2^k}\right)\}$ converges. So one can define the mapping $Q : Y \to X$ by

$$Q(x) := \lim_{k\to\infty} 4^k f\left(\frac{x}{2^k}\right)$$

for all $x \in Y$.

By (3.5),

$$P\left(DQ(x, y)\right)) = \lim_{k\to\infty} P\left(4^k Df\left(\frac{x}{2^k}, \frac{y}{2^k}\right)\right) \leq \lim_{k\to\infty} 4^k \phi\left(\frac{x}{2^k}, \frac{y}{2^k}\right) = 0$$

for all $x, y \in Y$. So $DQ(x, y) = 0$. Since $f : Y \to X$ is even, $Q : Y \to X$ is even. So the mapping $Q : Y \to X$ is quadratic. Moreover, letting $n = 0$ and passing the limit $m \to \infty$ in (3.8), we get (3.6). So there exists a quadratic mapping $Q : Y \to X$ satisfying (3.6).

Let $T : Y \to X$ be a quadratic mapping satisfying (3.6). Since T satisfies $4T\left(\frac{x}{2}\right) = T(x)$, we have $T(x) = 4^q T\left(\frac{x}{2^q}\right)$ for all integer q. Hence

$$\begin{aligned} P\left(Q(x) - T(x)\right) &= P\left(4^q Q\left(\frac{x}{2^q}\right) - 4^q T\left(\frac{x}{2^q}\right)\right) \\ &\leq P\left(4^q\left(Q\left(\frac{x}{2^q}\right) - f\left(\frac{x}{2^q}\right)\right)\right) + P\left(4^q\left(T\left(\frac{x}{2^q}\right) - f\left(\frac{x}{2^q}\right)\right)\right) \\ &\leq 2 \times 4^q \pi\left(\frac{x}{2^q}, 0\right), \end{aligned}$$

which tends to zero as $q \to \infty$ for all $x \in X$. So $Q(x) = T(x)$ for all $x \in X$. This proves the uniqueness of Q. Thus the mapping $Q : Y \to X$ is the unique quadrative mapping satisfying (3.6). □

Corollary 3.4. *Let r, θ be positive real numbers with $r > 2$, and let $f : X \to Y$ be an even mapping such that $f(0) = 0$ and*

$$P(Df(x, y)) \leq \theta(\|x\|^r + \|y\|^r)$$

for all $x, y \in Y$. Then there exists a unique quadrative mapping $Q : Y \to X$ such that

$$P(f(x) - Q(x)) \leq \frac{2^r}{2^r - 4}\theta\|x\|^r$$

for all $x \in Y$.

Proof. Letting $\phi(x, y) := \theta(\|x\|^r + \|y\|^r)$ in Theorem 3.3, we obtain the result. □

Let $f_o(x) := \frac{f(x)-f(-x)}{2}$ and $f_e(x) := \frac{f(x)+f(-x)}{2}$. Then f_o is odd and f_e is even. f_o, f_e satisfy the functional equation (1.1) if and only if f does.

Theorem 3.5. *Let r, θ be positive real numbers with $r > 2$. Let $f : Y \to X$ be a mapping satisfying $f(0) = 0$ and (2.5). Then there exist an additive mapping $A : Y \to X$ and a quadratic mapping $Q : Y \to X$ such that*

$$P\left(2f(x) - A(x) - Q(x)\right) \leq \left(\frac{2^{r+1}}{2^r - 2} + \frac{2^{r+1}}{2^r - 4}\right)\theta\|x\|^r$$

for all $x \in Y$.

Theorem 3.6. *Let r, θ be positive real numbers with $r < 1$. Let $f : X \to Y$ be a mapping satisfying $f(0) = 0$ and (2.10). Then there exist an additive mapping $A : X \to Y$ and a quadratic mapping $Q : X \to Y$ such that*

$$\|2f(x) - A(x) - Q(x)\| \leq \left(\frac{2^{r+1}}{2 - 2^r} + \frac{2^{r+1}}{4 - 2^r}\right)\theta P(x)^r$$

for all $x \in X$.

Acknowledgments

C. Park was supported by Basic Science Research Program through the National Research Foundation of Korea funded by the Ministry of Education, Science and Technology (NRF-2012R1A1A2004299), and T.M. Kim and S.H. Park were supported by R & E Program in 2012.

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J. COMPUTATIONAL ANALYSIS AND APPLICATIONS, VOL. 15, NO.8, 1476-1483, 2013, COPYRIGHT 2013 EUDOXUS PRESS, LLC

A note on special fuzzy differential subordinations using generalized Sălăgean operator and Ruscheweyh derivative

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Abstract

In the present paper we establish several fuzzy differential subordinations regardind the operator $RD_{\lambda,\alpha}^m$, given by $RD_{\lambda,\alpha}^m : \mathcal{A}_n \to \mathcal{A}_n$, $RD_{\lambda,\alpha}^m f(z) = (1-\alpha)R^m f(z) + \alpha D_\lambda^m f(z)$, where $R^m f(z)$ denote the Ruscheweyh derivative, $D_\lambda^m f(z)$ is the generalized Sălăgean operator and $\mathcal{A}_n = \{f \in \mathcal{H}(U),\ f(z) = z + a_{n+1}z^{n+1} + \ldots,\ z \in U\}$ is the class of normalized analytic functions with $\mathcal{A}_1 = \mathcal{A}$. A certain fuzzy class, denoted by $\mathcal{RD}_m^{\mathcal{F}}(\delta, \lambda, \alpha)$, of analytic functions in the open unit disc is introduced by means of this operator. By making use of the concept of fuzzy differential subordination we will derive various properties and characteristics of the class $\mathcal{RD}_m^{\mathcal{F}}(\delta, \lambda, \alpha)$. Also, several fuzzy differential subordinations are established regarding the operator $RD_{\lambda,\alpha}^m$.

Keywords: fuzzy differential subordination, convex function, fuzzy best dominant, differential operator, generalized Sălăgean operator, Ruscheweyh derivative.
2000 Mathematical Subject Classification: 30C45, 30A20, 34A40.

1 Introduction

One of the most recently study methods in the one complex variable functions theory is the admissible functions method known as "the differential subordination method" introduced by S.S. Miller and P.T. Mocanu in [11], [12] and developed in [13]. The application of this method allows to one obtain some special results and to prove easily some classical results from this domain. More results obtained by the differential subordinations method are differential inequalities. From the development of this method has been written a large number of papers and monographs in the one complex variable functions theory domain.

A justification of the introduction of the differential subordinations theory was presented in [14], "knowing the properties of differential expression for a function we can determine the properties of that function on a given interval." By publication of the papers [14] and [15] the authors wanted to launch a new research direction in mathematics that combines the notions from the complex functions domain with the fuzzy sets theory.

In the same way as mentioned, the author can justify that by knowing the properties of a differential expression on a fuzzy set for a function one can be determined the properties of that function on a given fuzzy set. The author has analyzed the case of one complex functions, leaving as "open problem" the case of real functions.

The author is aware that this new research alternative can be realized only through the joint effort of researchers from both domains. The "open problem" statement leaves open the interpretation of some notions from the fuzzy sets theory such that each one interpret them personally according to their scientific concerns, making this theory more attractive.

The notion of fuzzy subordination was introduced in [14]. In [15] the authors have defined the notion of fuzzy differential subordination. In this paper we will study fuzzy differential subordinations obtained with the differential operator defined in [4].

Denote by U the unit disc of the complex plane, $U = \{z \in \mathbb{C} : |z| < 1\}$ and $\mathcal{H}(U)$ the space of holomorphic functions in U.

Let $\mathcal{A}_n = \{f \in \mathcal{H}(U) : f(z) = z + a_{n+1}z^{n+1} + \ldots,\ z \in U\}$ with $\mathcal{A}_1 = \mathcal{A}$ and $\mathcal{H}[a, n] = \{f \in \mathcal{H}(U) : f(z) = a + a_n z^n + a_{n+1}z^{n+1} + \ldots,\ z \in U\}$ for $a \in \mathbb{C}$ and $n \in \mathbb{N}$.

Denote by $\mathcal{K} = \left\{ f \in \mathcal{A} : \operatorname{Re} \frac{zf''(z)}{f'(z)} + 1 > 0,\ z \in U \right\}$, the class of normalized convex functions in U.

In order to use the concept of fuzzy differential subordination, we remember the following definitions:

Definition 1.1 *[10] A pair (A, F_A), where $F_A : X \to [0,1]$ and $A = \{x \in X : 0 < F_A(x) \leq 1\}$ is called fuzzy subset of X. The set A is called the support of the fuzzy set (A, F_A) and F_A is called the membership function of the fuzzy set (A, F_A). One can also denote $A = \operatorname{supp}(A, F_A)$.*

Remark 1.1 *[8] In the development work we use the following notations for fuzzy sets:*
$F_{f(D)}(f(z)) = supp\big(f(D), F_{f(D)}\cdot\big) = \{z \in D : 0 < F_{f(D)}f(z) \leq 1\}$,
$F_{p(U)}p(z) = supp\big(p(U), F_{p(U)}\cdot\big) = \{z \in U : 0 < F_{p(U)}(p(z)) \leq 1\}$.

We give a new definition of membership function on complex numbers set using the module notion of a complex number $z = x + iy$, $x, y \in \mathbb{R}$, $|z| = \sqrt{x^2 + y^2} \geq 0$.

Example 1.1 *Let $F : \mathbb{C} \to \mathbb{R}_+$ a function such that $F_{\mathbb{C}}(z) = |F(z)|$, $\forall\ z \in \mathbb{C}$. Denote by $F_{\mathbb{C}}(\mathbb{C}) = \{z \in \mathbb{C} : 0 < F(z) \leq 1\} = \{z \in \mathbb{C} : 0 < |F(z)| \leq 1\} = supp(\mathbb{C}, F_{\mathbb{C}})$ the fuzzy subset of the complex numbers set. We call the subset $F_{\mathbb{C}}(\mathbb{C}) = U_{\mathcal{F}}(0,1)$ the fuzzy unit disk.*

Definition 1.2 *([14]) Let $D \subset \mathbb{C}$, $z_0 \in D$ be a fixed point and let the functions $f, g \in \mathcal{H}(D)$. The function f is said to be fuzzy subordinate to g and write $f \prec_{\mathcal{F}} g$ or $f(z) \prec_{\mathcal{F}} g(z)$, if are satisfied the conditions:*
1) $f(z_0) = g(z_0)$,
2) $F_{f(D)}f(z) \leq F_{g(D)}g(z)$, $z \in D$.

Definition 1.3 *([15, Definition 2.2]) Let $\psi : \mathbb{C}^3 \times U \to \mathbb{C}$ and h univalent in U, with $\psi(a, 0; 0) = h(0) = a$. If p is analytic in U, with $p(0) = a$ and satisfies the (second-order) fuzzy differential subordination*

$$F_{\psi(\mathbb{C}^3 \times U)}\psi(p(z), zp'(z), z^2p''(z); z) \leq F_{h(U)}h(z), \quad z \in U, \tag{1.1}$$

then p is called a fuzzy solution of the fuzzy differential subordination. The univalent function q is called a fuzzy dominant of the fuzzy solutions of the fuzzy differential subordination, or more simple a fuzzy dominant, if $F_{p(U)}p(z) \leq F_{q(U)}q(z)$, $z \in U$, for all p satisfying (1.1). A fuzzy dominant $\widetilde{q}$ that satisfies $F_{\widetilde{q}(U)}\widetilde{q}(z) \leq F_{q(U)}q(z)$, $z \in U$, for all fuzzy dominants q of (1.1) is said to be the fuzzy best dominant of (1.1).

Lemma 1.1 *([13, Corollary 2.6g.2, p. 66]) Let $h \in \mathcal{A}$ and $L[f](z) = G(z) = \frac{1}{z}\int_0^z h(t)\,dt$, $z \in U$. If $Re\left(\frac{zh''(z)}{h'(z)} + 1\right) > -\frac{1}{2}$, $z \in U$, then $L(f) = G \in \mathcal{K}$.*

Lemma 1.2 *([16]) Let h be a convex function with $h(0) = a$, and let $\gamma \in \mathbb{C}^*$ be a complex number with $Re\ \gamma \geq 0$. If $p \in \mathcal{H}[a, n]$ with $p(0) = a$, $\psi : \mathbb{C}^2 \times U \to \mathbb{C}$, $\psi(p(z), zp'(z); z) = p(z) + \frac{1}{\gamma}zp'(z)$ an analytic function in U and*

$$F_{\psi(\mathbb{C}^2 \times U)}\left(p(z) + \frac{1}{\gamma}zp'(z)\right) \leq F_{h(U)}h(z), \ \text{i.e. } p(z) + \frac{1}{\gamma}zp'(z) \prec_{\mathcal{F}} h(z), \quad z \in U, \tag{1.2}$$

then $F_{p(U)}p(z) \leq F_{g(U)}g(z) \leq F_{h(U)}h(z)$, i.e. $p(z) \prec_{\mathcal{F}} g(z) \prec_{\mathcal{F}} h(z)$, $z \in U$, where $g(z) = \frac{\gamma}{nz^{\gamma/n}}\int_0^z h(t)t^{\gamma/n-1}dt$, $z \in U$. The function q is convex and is the fuzzy best dominant.

Lemma 1.3 *([16]) Let g be a convex function in U and let $h(z) = g(z) + n\alpha zg'(z)$, $z \in U$, where $\alpha > 0$ and n is a positive integer.*

If $p(z) = g(0) + p_nz^n + p_{n+1}z^{n+1} + \dots$, $z \in U$, is holomorphic in U and $F_{p(U)}(p(z) + \alpha zp'(z)) \leq F_{h(U)}h(z)$, i.e. $p(z) + \alpha zp'(z) \prec_{\mathcal{F}} h(z)$, $z \in U$, then $F_{p(U)}p(z) \leq F_{g(U)}g(z)$, i.e. $p(z) \prec_{\mathcal{F}} g(z)$, $z \in U$, and this result is sharp.

We use the following differential operators.

Definition 1.4 *(Al Oboudi [9]) For $f \in \mathcal{A}_n$, $\lambda \geq 0$ and $n, m \in \mathbb{N}$, the operator D_λ^m is defined by $D_\lambda^m : \mathcal{A}_n \to \mathcal{A}_n$,*

$$\begin{aligned} D_\lambda^0 f(z) &= f(z) \\ D_\lambda^1 f(z) &= (1-\lambda) f(z) + \lambda z f'(z) = D_\lambda f(z), \ \dots \\ D_\lambda^{m+1} f(z) &= (1-\lambda) D_\lambda^m f(z) + \lambda z \left(D_\lambda^m f(z)\right)' = D_\lambda\left(D_\lambda^m f(z)\right), \quad z \in U. \end{aligned}$$

Remark 1.2 *If $f \in \mathcal{A}_n$ and $f(z) = z + \sum_{j=n+1}^{\infty} a_j z^j$, then $D_\lambda^m f(z) = z + \sum_{j=n+1}^{\infty} [1 + (j-1)\lambda]^m a_j z^j$, $z \in U$.*

Remark 1.3 *For $\lambda = 1$ in the above definition we obtain the Sălăgean differential operator [18].*

Definition 1.5 *(Ruscheweyh [17]) For $f \in \mathcal{A}_n$, $n, m \in \mathbb{N}$, the operator R^m is defined by $R^m : \mathcal{A}_n \to \mathcal{A}_n$,*

$$\begin{aligned} R^0 f(z) &= f(z) \\ R^1 f(z) &= z f'(z), \ \ldots \\ (m+1) R^{m+1} f(z) &= z (R^m f(z))' + m R^m f(z), \qquad z \in U. \end{aligned}$$

Remark 1.4 *If $f \in \mathcal{A}_n$, $f(z) = z + \sum_{j=n+1}^{\infty} a_j z^j$, then $R^m f(z) = z + \sum_{j=n+1}^{\infty} C_{m+j-1}^m a_j z^j$, $z \in U$.*

Definition 1.6 *([4]) Let $\alpha, \lambda \geq 0$, $n, m \in \mathbb{N}$. Denote by $RD_{\lambda,\alpha}^m$ the operator given by $RD_{\lambda,\alpha}^m : \mathcal{A}_n \to \mathcal{A}_n$, $RD_{\lambda,\alpha}^m f(z) = (1-\alpha) R^m f(z) + \alpha D_\lambda^m f(z)$, $z \in U$.*

Remark 1.5 *If $f \in \mathcal{A}_n$, $f(z) = z + \sum_{j=n+1}^{\infty} a_j z^j$, then*
$RD_{\lambda,\alpha}^m f(z) = z + \sum_{j=n+1}^{\infty} \left\{\alpha [1 + (j-1)\lambda]^m + (1-\alpha) C_{m+j-1}^m\right\} a_j z^j$, $z \in U$.

Remark 1.6 *For $\alpha = 0$, $RD_{\lambda,0}^m f(z) = R^m f(z)$, $z \in U$, and for $\alpha = 1$, $RD_{\lambda,1}^m f(z) = D_\lambda^m f(z)$, $z \in U$. For $\lambda = 1$, we obtain $RD_{1,\alpha}^m f(z) = L_\alpha^m f(z)$ which was studied in [1], [2], [5]. For $m = 0$, $RD_{\lambda,\alpha}^0 f(z) = (1-\alpha) R^0 f(z) + \alpha D_\lambda^0 f(z) = f(z) = R^0 f(z) = D_\lambda^0 f(z)$, $z \in U$. The operator $RD_{\lambda,\alpha}^m$ was studied in [3], [4], [6], [7].*

2 Main results

Using the operator $RD_{\lambda,\alpha}^m$ defined in Definition 1.6 we define the class $\mathcal{RD}_m^{\mathcal{F}}(\delta, \lambda, \alpha)$ and we study fuzzy subordinations.

Definition 2.1 *[8] Let $f(D) = \mathrm{supp}\left(f(D), F_{f(D)}\right) = \{z \in D : 0 < F_{f(D)} f(z) \leq 1\}$, where $F_{f(D)} \cdot$ is the membership function of the fuzzy set $f(D)$ asociated to the function f. The membership function of the fuzzy set $(\mu f)(D)$ asociated to the function μf coincide with the membership function of the fuzzy set $f(D)$ asociated to the function f, i.e. $F_{(\mu f)(D)}((\mu f)(z)) = F_{f(D)} f(z)$, $z \in D$. The membership function of the fuzzy set $(f+g)(D)$ asociated to the function $f+g$ coincide with the half of the sum of the membership functions of the fuzzy sets $f(D)$, respectively $g(D)$, asociated to the function f, respectively g, i.e. $F_{(f+g)(D)}((f+g)(z)) = \frac{F_{f(D)} f(z) + F_{g(D)} g(z)}{2}$, $z \in D$.*

Remark 2.1 *[8] $F_{(f+g)(D)}((f+g)(z))$ can be defined in other ways. Since $0 < F_{f(D)} f(z) \leq 1$ and $0 < F_{g(D)} g(z) \leq 1$, it is evidently that $0 < F_{(f+g)(D)}((f+g)(z)) \leq 1$, $z \in D$.*

Definition 2.2 *Let $\delta \in [0,1)$, $\alpha, \lambda \geq 0$ and $n, m \in \mathbb{N}$. A function $f \in \mathcal{A}_n$ is said to be in the class $\mathcal{RD}_m^{\mathcal{F}}(\delta, \lambda, \alpha)$ if it satisfies the inequality*

$$F_{\left(RD_{\lambda,\alpha}^m f\right)'(U)} \left(RD_{\lambda,\alpha}^m f(z)\right)' > \delta, \qquad z \in U. \tag{2.1}$$

Theorem 2.1 *The set $\mathcal{RD}_m^{\mathcal{F}}(\delta, \lambda, \alpha)$ is convex.*

Proof. Let the functions $f_j(z) = z + \sum_{j=n+1}^{\infty} a_{jk} z^j$, $k = 1, 2$, $z \in U$, be in the class $\mathcal{RD}_m^{\mathcal{F}}(\delta, \lambda, \alpha)$. It is sufficient to show that the function $h(z) = \eta_1 f_1(z) + \eta_2 f_2(z)$ is in the class $\mathcal{RD}_m^{\mathcal{F}}(\delta, \lambda, \alpha)$, with η_1 and η_2 nonnegative such that $\eta_1 + \eta_2 = 1$.

We have $h'(z) = (\mu_1 f_1 + \mu_2 f_2)'(z) = \mu_1 f_1'(z) + \mu_2 f_2'(z)$, $z \in U$, and
$\left(RD_{\lambda,\alpha}^m h(z)\right)' = \left(RD_{\lambda,\alpha}^m (\mu_1 f_1 + \mu_2 f_2)(z)\right)' = \mu_1 \left(RD_{\lambda,\alpha}^m f_1(z)\right)' + \mu_2 \left(RD_{\lambda,\alpha}^m f_2(z)\right)'$.

From Definition 2.1 we obtain that

$F_{\left(RD_{\lambda,\alpha}^m h\right)'(U)} \left(RD_{\lambda,\alpha}^m h(z)\right)' = F_{\left(RD_{\lambda,\alpha}^m (\mu_1 f_1 + \mu_2 f_2)\right)'(U)} \left(RD_{\lambda,\alpha}^m (\mu_1 f_1 + \mu_2 f_2)(z)\right)' =$

$F_{\left(RD_{\lambda,\alpha}^m (\mu_1 f_1 + \mu_2 f_2)\right)'(U)} \left(\mu_1 \left(RD_{\lambda,\alpha}^m f_1(z)\right)' + \mu_2 \left(RD_{\lambda,\alpha}^m f_2(z)\right)'\right) =$

$\frac{F_{\left(\mu_1 RD_{\lambda,\alpha}^m f_1\right)'(U)} \left(\mu_1 \left(RD_{\lambda,\alpha}^m f_1(z)\right)'\right) + F_{\left(\mu_2 RD_{\lambda,\alpha}^m f_2\right)'(U)} \left(\mu_2 \left(RD_{\lambda,\alpha}^m f_2(z)\right)'\right)}{2} =$

$\frac{F_{\left(RD^m_{\lambda,\alpha}f_1\right)'(U)}\left(RD^m_{\lambda,\alpha}f_1(z)\right)'+F_{\left(RD^m_{\lambda,\alpha}f_2\right)'(U)}\left(RD^m_{\lambda,\alpha}f_2(z)\right)'}{2}$.

Since $f_1, f_2 \in \mathcal{RD}^{\mathcal{F}}_m(\delta,\lambda,\alpha)$ we have $\delta < F_{\left(RD^m_{\lambda,\alpha}f_1\right)'(U)}\left(RD^m_{\lambda,\alpha}f_1(z)\right)' \le 1$ and $\delta < F_{\left(RD^m_{\lambda,\alpha}f_2\right)'(U)}\left(RD^m_{\lambda,\alpha}f_2(z)\right)'$ ≤ 1, $z \in U$. Therefore $\delta < \frac{F_{\left(RD^m_{\lambda,\alpha}f_1\right)'(U)}\left(RD^m_{\lambda,\alpha}f_1(z)\right)'+F_{\left(RD^m_{\lambda,\alpha}f_2\right)'(U)}\left(RD^m_{\lambda,\alpha}f_2(z)\right)'}{2} \le 1$ and we obtain that $\delta < F_{\left(RD^m_{\lambda,\alpha}h\right)'(U)}\left(RD^m_{\lambda,\alpha}h(z)\right)' \le 1$, which means that $h \in \mathcal{RD}^{\mathcal{F}}_m(\delta,\lambda,\alpha)$ and $\mathcal{RD}^{\mathcal{F}}_m(\delta,\lambda,\alpha)$ is convex. ■

We highlight a fuzzy subset obtained using a convex function. Let the function $h(z) = \frac{1+z}{1-z}$, $z \in U$. After a short calculation we obtain that $Re\left(\frac{zh''(z)}{h'(z)}+1\right) = Re\frac{1+z}{1-z} > 0$, so $h \in \mathcal{K}$ and $h(U) = \{z \in \mathbb{C} : Rez > 0\}$. We define the membership function for the set $h(U)$ as $F_{h(U)}(h(z)) = Reh(z)$, $z \in U$ and we have $F_{h(U)}h(z) =$supp$\left(h(U), F_{h(u)}\right) = \{z \in \mathbb{C} : 0 < F_{h(U)}(h(z)) \le 1\} = \{z \in U : 0 < Rez \le 1\}$.

Theorem 2.2 *Let g be a convex function in U and let $h(z) = g(z) + \frac{1}{c+2}zg'(z)$, where $z \in U$, $c > 0$. If $f \in \mathcal{RD}^{\mathcal{F}}_m(\delta,\lambda,\alpha)$ and $G(z) = I_c(f)(z) = \frac{c+2}{z^{c+1}}\int_0^z t^c f(t)\,dt$, $z \in U$, then*

$$F_{\left(RD^m_{\lambda,\alpha}f\right)'(U)}\left(RD^m_{\lambda,\alpha}f(z)\right)' \le F_{h(U)}h(z),\quad i.e.\ \left(RD^m_{\lambda,\alpha}f(z)\right)' \prec_{\mathcal{F}} h(z),\quad z \in U, \tag{2.2}$$

implies $F_{\left(RD^m_{\lambda,\alpha}G\right)'(U)}\left(RD^m_{\lambda,\alpha}G(z)\right)' \le F_{g(U)}g(z)$, i.e. $\left(RD^m_{\lambda,\alpha}G(z)\right)' \prec_{\mathcal{F}} g(z)$, $z \in U$, and this result is sharp.

Proof. We obtain that

$$z^{c+1}G(z) = (c+2)\int_0^z t^c f(t)\,dt. \tag{2.3}$$

Differentiating (2.3), with respect to z, we have $(c+1)G(z) + zG'(z) = (c+2)f(z)$ and

$$(c+1)RD^m_{\lambda,\alpha}G(z) + z\left(RD^m_{\lambda,\alpha}G(z)\right)' = (c+2)RD^m_{\lambda,\alpha}f(z),\quad z \in U. \tag{2.4}$$

Differentiating (2.4) we have

$$\left(RD^m_{\lambda,\alpha}G(z)\right)' + \frac{1}{c+2}z\left(RD^m_{\lambda,\alpha}G(z)\right)'' = \left(RD^m_{\lambda,\alpha}f(z)\right)',\quad z \in U. \tag{2.5}$$

Using (2.5), the fuzzy differential subordination (2.2) becomes

$$F_{RD^m_{\lambda,\alpha}G(U)}\left(\left(RD^m_{\lambda,\alpha}G(z)\right)' + \frac{1}{c+2}z\left(RD^m_{\lambda,\alpha}G(z)\right)''\right) \le F_{g(U)}\left(g(z) + \frac{1}{c+2}zg'(z)\right). \tag{2.6}$$

If we denote

$$p(z) = \left(RD^m_{\lambda,\alpha}G(z)\right)',\quad z \in U, \tag{2.7}$$

then $p \in \mathcal{H}[1,n]$.

Replacing (2.7) in (2.6) we obtain $F_{p(U)}\left(p(z) + \frac{1}{c+2}zp'(z)\right) \le F_{g(U)}\left(g(z) + \frac{1}{c+2}zg'(z)\right)$, $z \in U$. Using Lemma 1.3 we have $F_{p(U)}p(z) \le F_{g(U)}g(z)$, $z \in U$, i.e. $F_{\left(RD^m_{\lambda,\alpha}G\right)'(U)}\left(RD^m_{\lambda,\alpha}G(z)\right)' \le F_{g(U)}g(z)$, $z \in U$, and g is the best dominant. We have obtained that $\left(RD^m_{\lambda,\alpha}G(z)\right)' \prec_{\mathcal{F}} g(z)$, $z \in U$. ■

Theorem 2.3 *Let $h(z) = \frac{1+(2\delta-1)z}{1+z}$, $\delta \in [0,1)$ and $c > 0$. If $\alpha, \lambda \ge 0$, $m \in \mathbb{N}$ and $I_c(f)(z) = \frac{c+2}{z^{c+1}}\int_0^z t^c f(t)\,dt$, $z \in U$, then*

$$I_c\left[\mathcal{RD}^{\mathcal{F}}_m(\delta,\lambda,\alpha)\right] \subset \mathcal{RD}^{\mathcal{F}}_m(\delta^*,\lambda,\alpha), \tag{2.8}$$

where $\delta^* = 2\delta - 1 + \frac{(c+2)(2-2\delta)}{n}\beta\left(\frac{c+2}{n} - 2\right)$ and $\beta(x) = \int_0^1 \frac{t^{x+1}}{t+1}dt$.

Proof. The function h is convex and using the same steps as in the proof of Theorem 2.2 we get from the hypothesis of Theorem 2.3 that $F_{p(U)}\left(p(z) + \frac{1}{c+2}zp'(z)\right) \le F_{h(U)}h(z)$, where $p(z)$ is defined in (2.7).

Using Lemma 1.2 we deduce that $F_{p(U)}p(z) \le F_{g(U)}g(z) \le F_{h(U)}h(z)$, i.e. $F_{\left(RD^m_{\lambda,\alpha}G\right)'(U)}\left(RD^m_{\lambda,\alpha}G(z)\right)' \le F_{g(U)}g(z) \le F_{h(U)}h(z)$, where $g(z) = \frac{c+2}{nz^{\frac{c+2}{n}}}\int_0^z t^{\frac{c+2}{n}-1}\frac{1+(2\delta-1)t}{1+t}dt = (2\delta-1) + \frac{(c+2)(2-2\delta)}{nz^{\frac{c+2}{n}}}\int_0^z \frac{t^{\frac{c+2}{n}-1}}{1+t}dt$. Since g is convex and $g(U)$ is symmetric with respect to the real axis, we deduce

$$F_{\left(RD^m_{\lambda,\alpha}G\right)(U)}\left(RD^m_{\lambda,\alpha}G(z)\right)' \ge \min_{|z|=1} F_{g(U)}g(z) = F_{g(U)}g(1) \tag{2.9}$$

and $\delta^* = g(1) = 2\delta - 1 + \frac{(c+2)(2-2\delta)}{n}\beta\left(\frac{c+2}{n}-2\right)$. From (2.9) we deduce inclusion (2.8). ■

Theorem 2.4 *Let g be a convex function, $g(0) = 1$ and let h be the function $h(z) = g(z) + zg'(z)$, $z \in U$. If $\alpha, \lambda \ge 0$, $n, m \in \mathbb{N}$, $f \in \mathcal{A}_n$ and satisfies the fuzzy differential subordination*

$$F_{\left(RD^m_{\lambda,\alpha}f\right)'(U)}\left(RD^m_{\lambda,\alpha}f(z)\right)' \le F_{h(U)}h(z), \quad i.e. \ \left(RD^m_{\lambda,\alpha}f(z)\right)' \prec_{\mathcal{F}} h(z), \quad z \in U, \tag{2.10}$$

then $F_{RD^m_{\lambda,\alpha}f(U)}\frac{RD^m_{\lambda,\alpha}f(z)}{z} \le F_{g(U)}g(z)$, i.e. $\frac{RD^m_{\lambda,\alpha}f(z)}{z} \prec_{\mathcal{F}} g(z)$, $z \in U$, and this result is sharp.

Proof. By using the properties of operator $RD^m_{\lambda,\alpha}$, we have
$RD^m_{\lambda,\alpha}f(z) = z + \sum_{j=n+1}^{\infty}\left\{\alpha\left[1+(j-1)\lambda\right]^m + (1-\alpha)C^m_{m+j-1}\right\}a_j z^j$, $z \in U$.

Consider $p(z) = \frac{RD^m_{\lambda,\alpha}f(z)}{z} = \frac{z+\sum_{j=n+1}^{\infty}\left\{\alpha[1+(j-1)\lambda]^m+(1-\alpha)C^m_{m+j-1}\right\}a_j z^j}{z} = 1 + p_n z^n + p_{n+1}z^{n+1} + \ldots$, $z \in U$. We deduce that $p \in \mathcal{H}[1,n]$.

Let $RD^m_{\lambda,\alpha}f(z) = zp(z)$, $z \in U$. Differentiating we obtain $\left(RD^m_{\lambda,\alpha}f(z)\right)' = p(z) + zp'(z)$, $z \in U$. Then (2.10) becomes $F_{p(U)}\left(p(z) + zp'(z)\right) \le F_{h(U)}h(z) = F_{g(U)}\left(g(z) + zg'(z)\right)$, $z \in U$.

By using Lemma 1.3, we have $F_{p(U)}p(z) \le F_{g(U)}g(z)$, $z \in U$, i.e. $F_{RD^m_{\lambda,\alpha}f(U)}\frac{RD^m_{\lambda,\alpha}f(z)}{z} \le F_{g(U)}g(z)$, $z \in U$. We obtained that $\left(RD^m_{\lambda,\alpha}f(z)\right)' \prec_{\mathcal{F}} h(z)$, $z \in U$, and this results is sharp. ■

Theorem 2.5 *Let h be an holomorphic function which satisfies the inequality* Re $\left(1 + \frac{zh''(z)}{h'(z)}\right) > -\frac{1}{2}$, $z \in U$, *and $h(0) = 1$. If $\alpha, \lambda \ge 0$, $n, m \in \mathbb{N}$, $f \in \mathcal{A}_n$ and satisfies the fuzzy differential subordination*

$$F_{\left(RD^m_{\lambda,\alpha}f\right)'(U)}\left(RD^m_{\lambda,\alpha}f(z)\right)' \le F_{h(U)}h(z), \ i.e. \ \left(RD^m_{\lambda,\alpha}f(z)\right)' \prec_{\mathcal{F}} h(z), \ z \in U, \tag{2.11}$$

then $F_{RD^m_{\lambda,\alpha}f(U)}\frac{RD^m_{\lambda,\alpha}f(z)}{z} \le F_{q(U)}q(z)$, i.e. $\frac{RD^m_{\lambda,\alpha}f(z)}{z} \prec_{\mathcal{F}} q(z)$, $z \in U$, where $q(z) = \frac{1}{nz^{\frac{1}{n}}}\int_0^z h(t)t^{\frac{1}{n}-1}dt$. The function q is convex and it is the fuzzy best dominant.

Proof. Let $p(z) = \frac{RD^m_{\lambda,\alpha}f(z)}{z} = \frac{z+\sum_{j=n+1}^{\infty}\left\{\alpha[1+(j-1)\lambda]^m+(1-\alpha)C^m_{m+j-1}\right\}a_j z^j}{z} =$
$1 + \sum_{j=n+1}^{\infty}\left\{\alpha\left[1+(j-1)\lambda\right]^m + (1-\alpha)C^m_{m+j-1}\right\}a_j z^{j-1} = 1 + \sum_{j=n+1}^{\infty}p_j z^{j-1}$, $z \in U$, $p \in \mathcal{H}[1,n]$.

Since Re $\left(1 + \frac{zh''(z)}{h'(z)}\right) > -\frac{1}{2}$, $z \in U$, from Lemma 1.1, we obtain that $q(z) = \frac{1}{nz^{\frac{1}{n}}}\int_0^z h(t)t^{\frac{1}{n}-1}dt$ is a convex function and verifies the differential equation asscociated to the fuzzy differential subordination (2.11) $q(z) + zq'(z) = h(z)$, therefore it is the fuzzy best dominant.

Differentiating, we obtain $\left(RD^m_{\lambda,\alpha}f(z)\right)' = p(z)+zp'(z)$, for $z \in U$ and (2.11) becomes $F_{p(U)}\left(p(z) + zp'(z)\right) \le F_{h(U)}h(z)$, $z \in U$. Using Lemma 1.2, we have $F_{p(U)}p(z) \le F_{q(U)}q(z)$, $z \in U$, i.e. $F_{RD^m_{\lambda,\alpha}f(U)}\frac{RD^m_{\lambda,\alpha}f(z)}{z} \le F_{q(U)}q(z)$, $z \in U$. We have obtained that $\frac{RD^m_{\lambda,\alpha}f(z)}{z} \prec_{\mathcal{F}} q(z)$, $z \in U$. ■

Corollary 2.6 *Let $h(z) = \frac{1+(2\beta-1)z}{1+z}$ a convex function in U, $0 \le \beta < 1$. If $\alpha, \lambda \ge 0$, $n, m \in \mathbb{N}$, $f \in \mathcal{A}_n$ and satisfies the fuzzy differential subordination*

$$F_{\left(RD^m_{\lambda,\alpha}f\right)'(U)}\left(RD^m_{\lambda,\alpha}f(z)\right)' \le F_{h(U)}h(z), \ i.e. \ \left(RD^m_{\lambda,\alpha}f(z)\right)' \prec_{\mathcal{F}} h(z), \ z \in U, \tag{2.12}$$

then $F_{RD^m_{\lambda,\alpha}f(U)}\frac{RD^m_{\lambda,\alpha}f(z)}{z} \le F_{q(U)}q(z)$, i.e. $\frac{RD^m_{\lambda,\alpha}f(z)}{z} \prec_{\mathcal{F}} q(z)$, $z \in U$, where q is given by $q(z) = 2\beta - 1 + \frac{2(1-\beta)}{nz^{\frac{1}{n}}}\int_0^z \frac{t^{\frac{1}{n}-1}}{1+t}dt$, $z \in U$. The function q is convex and it is the fuzzy best dominant.

Proof. We have $h(z) = \frac{1+(2\beta-1)z}{1+z}$ with $h(0) = 1$, $h'(z) = \frac{-2(1-\beta)}{(1+z)^2}$ and $h''(z) = \frac{4(1-\beta)}{(1+z)^3}$, therefore $Re\left(\frac{zh''(z)}{h'(z)} + 1\right) = Re\left(\frac{1-z}{1+z}\right) = Re\left(\frac{1-\rho\cos\theta - i\rho\sin\theta}{1+\rho\cos\theta+i\rho\sin\theta}\right) = \frac{1-\rho^2}{1+2\rho\cos\theta+\rho^2} > 0 > -\frac{1}{2}$.

Following the same steps as in the proof of Theorem 2.5 and considering $p(z) = \frac{RD^m_{\lambda,\alpha}f(z)}{z}$, the differential subordination (2.12) becomes $F_{RD^m_{\lambda,\alpha}f(U)}(p(z) + zp'(z)) \leq F_{h(U)}h(z)$, $z \in U$. By using Lemma 1.2 for $\gamma = 1$, we have $F_{p(U)}p(z) \leq F_{q(U)}q(z)$, i.e. $F_{RD^m_{\lambda,\alpha}f(U)}\frac{RD^m_{\lambda,\alpha}f(z)}{z} \leq F_{q(U)}q(z)$ and $q(z) = \frac{1}{nz^{\frac{1}{n}}}\int_0^z h(t)t^{\frac{1}{n}-1}dt = \frac{1}{nz^{\frac{1}{n}}}\int_0^z t^{\frac{1}{n}-1}\frac{1+(2\beta-1)t}{1+t}dt = 2\beta - 1 + \frac{2(1-\beta)}{nz^{\frac{1}{n}}}\int_0^z \frac{t^{\frac{1}{n}-1}}{1+t}dt$, $z \in U$. ■

Example 2.1 *Let* $h(z) = \frac{1-z}{1+z}$ *a convex function in* U *with* $h(0) = 1$ *and* $Re\left(\frac{zh''(z)}{h'(z)} + 1\right) > -\frac{1}{2}$.

Let $f(z) = z + z^2$, $z \in U$. *For* $n = 1$, $m = 1$, $\lambda = \frac{1}{2}$, $\alpha = 2$, *we obtain* $RD^1_{\frac{1}{2},2}f(z) = -R^1f(z) + 2D^1_{\frac{1}{2}}f(z) = -zf'(z) + 2\left(\frac{1}{2}f(z) + \frac{1}{2}zf'(z)\right) = f(z) = z + z^2$, $z \in U$. *Then* $\left(RD^1_{\frac{1}{2},2}f(z)\right)' = f'(z) = 1 + 2z$ *and* $\frac{RD^1_{\frac{1}{2},2}f(z)}{z} = 1 + z$. *We have* $q(z) = \frac{1}{z}\int_0^z \frac{1-t}{1+t}dt = -1 + \frac{2\ln(1+z)}{z}$.

Using Theorem 2.5 we obtain $1 + 2z \prec_{\mathcal{F}} \frac{1-z}{1+z}$, $z \in U$, *induce* $1 + z \prec_{\mathcal{F}} -1 + \frac{2\ln(1+z)}{z}$, $z \in U$.

Theorem 2.7 *Let* g *be a convex function such that* $g(0) = 1$ *and let* h *be the function* $h(z) = g(z) + zg'(z)$, $z \in U$. *If* $\alpha, \lambda \geq 0$, $n, m \in \mathbb{N}$, $f \in \mathcal{A}_n$ *and the fuzzy differential subordination* $F_{RD^m_{\lambda,\alpha}f(U)}(\frac{(m+1)(m+2)}{z}RD^{m+2}_{\lambda,\alpha}f(z) - \frac{(m+1)(2m+1)}{z}RD^{m+1}_{\lambda,\alpha}f(z) + \frac{m^2}{z}RD^m_{\lambda,\alpha}f(z) - \frac{\alpha\left[(m+1)(m+2)-\frac{1}{\lambda^2}\right]}{z}D^{m+2}_\lambda f(z) + \frac{\alpha\left[(m+1)(2m+1)-\frac{2(1-\lambda)}{\lambda^2}\right]}{z}D^{m+1}_\lambda f(z) - \frac{\alpha\left[m^2-\frac{(1-\lambda)^2}{\lambda^2}\right]}{z}D^m_\lambda f(z)) \leq F_{h(U)}h(z)$, *i.e.*

$$\frac{(m+1)(m+2)}{z}RD^{m+2}_{\lambda,\alpha}f(z) - \frac{(m+1)(2m+1)}{z}RD^{m+1}_{\lambda,\alpha}f(z) + \frac{m^2}{z}RD^m_{\lambda,\alpha}f(z) -$$
$$\frac{\alpha\left[(m+1)(m+2)-\frac{1}{\lambda^2}\right]}{z}D^{m+2}_\lambda f(z) + \frac{\alpha\left[(m+1)(2m+1)-\frac{2(1-\lambda)}{\lambda^2}\right]}{z}D^{m+1}_\lambda f(z) -$$
$$\frac{\alpha\left[m^2-\frac{(1-\lambda)^2}{\lambda^2}\right]}{z}D^m_\lambda f(z) \prec_{\mathcal{F}} h(z), \quad z \in U, \tag{2.13}$$

holds, then $F_{\left(RD^m_{\lambda,\alpha}f\right)'(U)}[RD^m_{\lambda,\alpha}f(z)]' \leq F_{g(U)}g(z)$, *i.e.* $[RD^m_{\lambda,\alpha}f(z)]' \prec_{\mathcal{F}} g(z)$, $z \in U$. *This result is sharp.*

Proof. Let

$$p(z) = \left(RD^m_{\lambda,\alpha}f(z)\right)' = (1-\alpha)(R^mf(z))' + \alpha\left(D^m_\lambda f(z)\right)' \tag{2.14}$$

$= 1 + \sum_{j=n+1}^{\infty}\left\{\alpha[1+(j-1)\lambda]^m + (1-\alpha)C^m_{m+j-1}\right\}ja_jz^{j-1} = 1 + p_nz^n + p_{n+1}z^{n+1} + \ldots$. We deduce that $p \in \mathcal{H}[1,n]$.

By using the properties of operators $RD^m_{\lambda,\alpha}$, R^m and D^m_λ, after a short calculation, we obtain

$p(z) + zp'(z) = \frac{(m+1)(m+2)}{z}RD^{m+2}_{\lambda,\alpha}f(z) - \frac{(m+1)(2m+1)}{z}RD^{m+1}_{\lambda,\alpha}f(z) + \frac{m^2}{z}RD^m_{\lambda,\alpha}f(z) - \frac{\alpha\left[(m+1)(m+2)-\frac{1}{\lambda^2}\right]}{z}D^{m+2}_\lambda f(z) + \frac{\alpha\left[(m+1)(2m+1)-\frac{2(1-\lambda)}{\lambda^2}\right]}{z}D^{m+1}_\lambda f(z) - \frac{\alpha\left[m^2-\frac{(1-\lambda)^2}{\lambda^2}\right]}{z}D^m_\lambda f(z)$.

Using the notation in (2.14), the fuzzy differential subordination becomes $F_{p(U)}(p(z) + zp'(z)) \leq F_{h(U)}h(z) = F_{g(U)}(g(z) + zg'(z))$. By using Lemma 1.3, we have $F_{p(U)}p(z) \leq F_{g(U)}g(z)$, $z \in U$, i.e. $F_{RD^m_{\lambda,\alpha}f(U)}\left(RD^m_{\lambda,\alpha}f(z)\right)' \leq F_{g(U)}g(z)$, $z \in U$, and this result is sharp. ■

Theorem 2.8 *Let* h *be an holomorphic function which satisfies the inequality* $\mathrm{Re}\left[1 + \frac{zh''(z)}{h'(z)}\right] > -\frac{1}{2}$, $z \in U$, *and* $h(0) = 1$. *If* $\alpha, \lambda \geq 0$, $n, m \in \mathbb{N}$, $f \in \mathcal{A}_n$ *and satisfies the fuzzy differential subordination*
$F_{RD^m_{\lambda,\alpha}f(U)}(\frac{(m+1)(m+2)}{z}RD^{m+2}_{\lambda,\alpha}f(z) - \frac{(m+1)(2m+1)}{z}RD^{m+1}_{\lambda,\alpha}f(z) + \frac{m^2}{z}RD^m_{\lambda,\alpha}f(z) - \frac{\alpha\left[(m+1)(m+2)-\frac{1}{\lambda^2}\right]}{z}D^{m+2}_\lambda f(z) + \frac{\alpha\left[(m+1)(2m+1)-\frac{2(1-\lambda)}{\lambda^2}\right]}{z}D^{m+1}_\lambda f(z) - \frac{\alpha\left[m^2-\frac{(1-\lambda)^2}{\lambda^2}\right]}{z}D^m_\lambda f(z)) \leq F_{h(U)}h(z)$, *i.e.*

$$\frac{(m+1)(m+2)}{z}RD^{m+2}_{\lambda,\alpha}f(z) - \frac{(m+1)(2m+1)}{z}RD^{m+1}_{\lambda,\alpha}f(z) + \frac{m^2}{z}RD^m_{\lambda,\alpha}f(z) -$$
$$\frac{\alpha\left[(m+1)(m+2)-\frac{1}{\lambda^2}\right]}{z}D^{m+2}_\lambda f(z) + \frac{\alpha\left[(m+1)(2m+1)-\frac{2(1-\lambda)}{\lambda^2}\right]}{z}D^{m+1}_\lambda f(z) -$$

$$\frac{\alpha\left[m^2-\frac{(1-\lambda)^2}{\lambda^2}\right]}{z}D_\lambda^m f(z) \prec_{\mathcal{F}} h(z), \quad z\in U, \tag{2.15}$$

then $F_{RD^m_{\lambda,\alpha}f(U)}\left(RD^m_{\lambda,\alpha}f(z)\right)' \leq F_{q(U)}q(z)$. *i.e.* $\left(RD^m_{\lambda,\alpha}f(z)\right)' \prec_{\mathcal{F}} q(z)$, $z\in U$, *where* q *is given by* $q(z)=\frac{1}{nz^{\frac{1}{n}}}\int_0^z h(t)t^{\frac{1}{n}-1}dt$. *The function* q *is convex and it is the fuzzy best dominant.*

Proof. Since Re $\left(1+\frac{zh''(z)}{h'(z)}\right)>-\frac{1}{2}$, $z\in U$, from Lemma 1.1, we obtain that $q(z)=\frac{1}{nz^{\frac{1}{n}}}\int_0^z h(t)t^{\frac{1}{n}-1}dt$ is a convex function and verifies the differential equation asscociated to the fuzzy differential subordination (2.11) $q(z)+zq'(z)=h(z)$, therefore it is the fuzzy best dominant.

Using the properties of operator $RD^m_{\lambda,\alpha}$ and considering $p(z)=\left(RD^m_{\lambda,\alpha}f(z)\right)'$, we obtain $p(z)+zp'(z)=\frac{(m+1)(m+2)}{z}RD^{m+2}_{\lambda,\alpha}f(z)-\frac{(m+1)(2m+1)}{z}RD^{m+1}_{\lambda,\alpha}f(z)+\frac{m^2}{z}RD^n_{\lambda,\alpha}f(z)-\frac{\alpha\left[(m+1)(m+2)-\frac{1}{\lambda^2}\right]}{z}D_\lambda^{m+2}f(z)+\frac{\alpha\left[(m+1)(2m+1)-\frac{2(1-\lambda)}{\lambda^2}\right]}{z}D_\lambda^{m+1}f(z)-\frac{\alpha\left[m^2-\frac{(1-\lambda)^2}{\lambda^2}\right]}{z}D_\lambda^m f(z)$, $z\in U$.

Then (2.15) becomes $F_{p(U)}\left(p(z)+zp'(z)\right)\leq F_{h(U)}h(z)$, $z\in U$. Since $p\in\mathcal{H}[1,n]$, using Lemma 1.2, we deduce $F_{p(U)}p(z)\leq F_{q(U)}q(z)$, $z\in U$, i.e. $F_{RD^m_{\lambda,\alpha}f(U)}\left(RD^m_{\lambda,\alpha}f(z)\right)'\leq F_{q(U)}q(z)$, $z\in U$. ■

Corollary 2.9 *Let* $h(z)=\frac{1+(2\beta-1)z}{1+z}$ *be a convex function in* U, *where* $0\leq\beta<1$. *If* $\alpha,\lambda\geq 0$, $n,m\in\mathbb{N}$, $f\in\mathcal{A}_n$ *and satisfies the fuzzy differential subordination* $F_{RD^m_{\lambda,\alpha}f(U)}\Big(\frac{(m+1)(m+2)}{z}RD^{m+2}_{\lambda,\alpha}f(z)-\frac{(m+1)(2m+1)}{z}RD^{m+1}_{\lambda,\alpha}f(z)+\frac{m^2}{z}RD^m_{\lambda,\alpha}f(z)-\frac{\alpha\left[(m+1)(m+2)-\frac{1}{\lambda^2}\right]}{z}D_\lambda^{m+2}f(z)+\frac{\alpha\left[(m+1)(2m+1)-\frac{2(1-\lambda)}{\lambda^2}\right]}{z}D_\lambda^{m+1}f(z)-\frac{\alpha\left[m^2-\frac{(1-\lambda)^2}{\lambda^2}\right]}{z}D_\lambda^m f(z)\Big)\leq F_{h(U)}h(z)$, *i.e.*

$$\frac{(m+1)(m+2)}{z}RD^{m+2}_{\lambda,\alpha}f(z)-\frac{(m+1)(2m+1)}{z}RD^{m+1}_{\lambda,\alpha}f(z)+\frac{m^2}{z}RD^m_{\lambda,\alpha}f(z)-$$
$$\frac{\alpha\left[(m+1)(m+2)-\frac{1}{\lambda^2}\right]}{z}D_\lambda^{m+2}f(z)+\frac{\alpha\left[(m+1)(2m+1)-\frac{2(1-\lambda)}{\lambda^2}\right]}{z}D_\lambda^{m+1}f(z)-$$
$$\frac{\alpha\left[m^2-\frac{(1-\lambda)^2}{\lambda^2}\right]}{z}D_\lambda^m f(z)\prec_{\mathcal{F}} h(z), \quad z\in U, \tag{2.16}$$

then $F_{RD^m_{\lambda,\alpha}f(U)}\left(RD^m_{\lambda,\alpha}f(z)\right)'\leq F_{q(U)}q(z)$, *i.e.* $\left(RD^m_{\lambda,\alpha}f(z)\right)'\prec_{\mathcal{F}} q(z)$, $z\in U$, *where* q *is given by* $q(z)=2\beta-1+\frac{2(1-\beta)}{nz^{\frac{1}{n}}}\int_0^z\frac{t^{\frac{1}{n}-1}}{1+t}dt$, $z\in U$. *The function* q *is convex and it is the fuzzy best dominant.*

Proof. Following the same steps as in the proof of Theorem 2.7 and considering $p(z)=\left(RD^m_{\lambda,\alpha}f(z)\right)'$, the differential subordination (2.16) becomes $F_{p(U)}\left(p(z)+zp'(z)\right)\leq F_{h(U)}h(z)$, $z\in U$. By using Lemma 1.2 for $\gamma=1$, we have $F_{p(U)}p(z)\leq F_{q(U)}q(z)$, i.e. $F_{\left(RD^m_{\lambda,\alpha}f\right)'(U)}\left(RD^m_{\lambda,\alpha}f(z)\right)'\leq F_{q(U)}q(z)$, i.e. $\left(RD^m_{\lambda,\alpha}f(z)\right)'\prec_{\mathcal{F}} q(z)$, and $q(z)=\frac{1}{nz^{\frac{1}{n}}}\int_0^z h(t)t^{\frac{1}{n}-1}dt=\frac{1}{nz^{\frac{1}{n}}}\int_0^z t^{\frac{1}{n}-1}\frac{1+(2\beta-1)t}{1+t}dt=2\beta-1+\frac{2(1-\beta)}{nz^{\frac{1}{n}}}\int_0^z\frac{t^{\frac{1}{n}-1}}{1+t}dt$, $z\in U$. ■

Example 2.2 *Let* $h(z)=\frac{1-z}{1+z}$ *a convex function in* U *with* $h(0)=1$ *and Re* $\left(\frac{zh''(z)}{h'(z)}+1\right)>-\frac{1}{2}$.

Let $f(z)=z+z^2$, $z\in U$. *For* $n=1$, $m=1$, $\lambda=\frac{1}{2}$, $\alpha=2$, *we obtain* $RD^1_{\frac{1}{2},2}f(z)=-R^1f(z)+2D^1_{\frac{1}{2}}f(z)=-zf'(z)+2\left(\frac{1}{2}f(z)+\frac{1}{2}zf'(z)\right)=f(z)=z+z^2$ *and* $(n+1)RD^{n+1}_{\lambda,\alpha}f(z)-(n-1)RD^n_{\lambda,\alpha}f(z)-\alpha\left(n+1-\frac{1}{\lambda}\right)\left[D_\lambda^{n+1}f(z)-D_\lambda^n f(z)\right]=2RD^2_{\frac{1}{2},2}f(z)=-2+2z$, *where* $RD^2_{\frac{1}{2},2}f(z)=-R^2f(z)+2D^2_{\frac{1}{2}}f(z)=-\left(1+3z^2\right)+2\left(\frac{1}{2}z+\frac{3}{2}z^2\right)=-1+z$. *We have* $q(z)=\frac{1}{z}\int_0^z\frac{1-t}{1+t}dt=-1+\frac{2\ln(1+z)}{z}$.

Using Theorem 2.8 we obtain $-2+2z\prec_{\mathcal{F}}\frac{1-z}{1+z}$, $z\in U$, *induce* $z+z^2\prec_{\mathcal{F}}-1+\frac{2\ln(1+z)}{z}$, $z\in U$.

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J. COMPUTATIONAL ANALYSIS AND APPLICATIONS, VOL. 15, NO.8, 1484-1489, 2013, COPYRIGHT 2013 EUDOXUS PRESS, LLC

On special fuzzy differential subordinations using convolution product of Sălăgean operator and Ruscheweyh derivative

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Abstract

In this paper we establish several fuzzy differential subordinations regardind the operator defined as Hadamard product of Sălăgean operator S^m and Ruscheweyh derivative R^m, denoted SR^m, given by $SR^m : \mathcal{A} \to \mathcal{A}$, $SR^m f(z) = (S^m * R^m) f(z)$ and $\mathcal{A}_n = \{f \in \mathcal{H}(U),\ f(z) = z + a_{n+1}z^{n+1} + \ldots,\ z \in U\}$ is the class of normalized analytic functions with $\mathcal{A}_1 = \mathcal{A}$. A certain fuzzy class, denoted by $\mathcal{SR}_m^{\mathcal{F}}(\delta)$, of analytic functions in the open unit disc is introduced by means of this operator. By making use of the concept of fuzzy differential subordination we will derive various properties and characteristics of the class $\mathcal{SR}_m^{\mathcal{F}}(\delta)$. Also, several fuzzy differential subordinations are established regarding the operator SR^m.

Keywords: fuzzy differential subordination, convex function, fuzzy best dominant, differential operator, convolution product, Sălăgean operator, Ruscheweyh derivative.
2000 Mathematical Subject Classification: 30C45, 30A20, 34A40.

1 Introduction

In [10] and [11] the authors wanted to launch a new research direction in mathematics that combines the notions from the complex functions domain with the fuzzy sets theory. Also the author can justify that by knowing the properties of a differential expression on a fuzzy set for a function one can be determined the properties of that function on a given fuzzy set. The author has analyzed the case of one complex functions, leaving as "open problem" the case of real functions. The "open problem" statement leaves open the interpretation of some notions from the fuzzy sets theory such that each one interpret them personally according to their scientific concerns, making this theory more attractive.

The notion of fuzzy subordination was introduced in [10]. In [11] the authors have defined the notion of fuzzy differential subordination. In this paper we will study fuzzy differential subordinations obtained with the differential operator defined in [1].

Denote by U the unit disc of the complex plane, $U = \{z \in \mathbb{C} : |z| < 1\}$ and $\mathcal{H}(U)$ the space of holomorphic functions in U.

Let $\mathcal{A}_n = \{f \in \mathcal{H}(U) : f(z) = z + a_{n+1}z^{n+1} + \ldots,\ z \in U\}$ with $\mathcal{A}_1 = \mathcal{A}$ and $\mathcal{H}[a,n] = \{f \in \mathcal{H}(U) : f(z) = a + a_n z^n + a_{n+1}z^{n+1} + \ldots,\ z \in U\}$ for $a \in \mathbb{C}$ and $n \in \mathbb{N}$.

Denote by $\mathcal{K} = \left\{f \in \mathcal{A} : \operatorname{Re} \frac{zf''(z)}{f'(z)} + 1 > 0,\ z \in U\right\}$, the class of normalized convex functions in U.

In order to use the concept of fuzzy differential subordination, we remember the following definitions:

Definition 1.1 *[6] A pair (A, F_A), where $F_A : X \to [0,1]$ and $A = \{x \in X : 0 < F_A(x) \leq 1\}$ is called fuzzy subset of X. The set A is called the support of the fuzzy set (A, F_A) and F_A is called the membership function of the fuzzy set (A, F_A). One can also denote $A = \operatorname{supp}(A, F_A)$.*

Remark 1.1 *[5] In the development work we use the following notations for fuzzy sets:*
$F_{f(D)}(f(z)) = supp\big(f(D), F_{f(D)}\cdot\big) = \{z \in D : 0 < F_{f(D)} f(z) \leq 1\}$,
$p(U) = supp\big(p(U), F_{p(U)}\cdot\big) = \{z \in U : 0 < F_{p(U)}(p(z)) \leq 1\}$.

We give a new definition of membership function on complex numbers set using the module notion of a complex number $z = x + iy$, $x, y \in \mathbb{R}$, $|z| = \sqrt{x^2 + y^2} \geq 0$.

Example 1.1 *Let $F:\mathbb{C}\to\mathbb{R}_+$ a function such that $F_{\mathbb{C}}(z)=|F(z)|$, $\forall\ z\in\mathbb{C}$. Denote by $F_{\mathbb{C}}(\mathbb{C})=\{z\in\mathbb{C}: 0<F(z)\le 1\}=\{z\in\mathbb{C}:0<|F(z)|\le 1\}=supp(\mathbb{C},F_{\mathbb{C}})$ the fuzzy subset of the complex numbers set. We call the subset $F_{\mathbb{C}}(\mathbb{C})=U_{\mathcal{F}}(0,1)$ the fuzzy unit disk.*

Definition 1.2 *([10]) Let $D\subset\mathbb{C}$, $z_0\in D$ be a fixed point and let the functions $f,g\in\mathcal{H}(D)$. The function f is said to be fuzzy subordinate to g and write $f\prec_{\mathcal{F}} g$ or $f(z)\prec_{\mathcal{F}} g(z)$, if are satisfied the conditions:*
1) $f(z_0)=g(z_0)$,
2) $F_{f(D)}f(z)\le F_{g(D)}g(z)$, $z\in D$.

Definition 1.3 *([11, Definition 2.2]) Let $\psi:\mathbb{C}^3\times U\to\mathbb{C}$ and h univalent in U, with $\psi(a,0;0)=h(0)=a$. If p is analytic in U, with $p(0)=a$ and satisfies the (second-order) fuzzy differential subordination*

$$F_{\psi(\mathbb{C}^3\times U)}\psi(p(z),zp'(z),z^2p''(z);z)\le F_{h(U)}h(z),\quad z\in U,\tag{1.1}$$

then p is called a fuzzy solution of the fuzzy differential subordination. The univalent function q is called a fuzzy dominant of the fuzzy solutions of the fuzzy differential subordination, or more simple a fuzzy dominant, if $F_{p(U)}p(z)\le F_{q(U)}q(z)$, $z\in U$, for all p satisfying (1.1). A fuzzy dominant $\widetilde{q}$ that satisfies $F_{\widetilde{q}(U)}\widetilde{q}(z)\le F_{q(U)}q(z)$, $z\in U$, for all fuzzy dominants q of (1.1) is said to be the fuzzy best dominant of (1.1).

Lemma 1.1 *([9, Corollary 2.6g.2, p. 66]) Let $h\in\mathcal{A}$ and $L[f](z)=G(z)=\frac{1}{z}\int_0^z h(t)\,dt$, $z\in U$. If $Re\left(\frac{zh''(z)}{h'(z)}+1\right)>-\frac{1}{2}$, $z\in U$, then $L(f)=G\in\mathcal{K}$.*

Lemma 1.2 *([12]) Let h be a convex function with $h(0)=a$, and let $\gamma\in\mathbb{C}^*$ be a complex number with $Re\ \gamma\ge 0$. If $p\in\mathcal{H}[a,n]$ with $p(0)=a$, $\psi:\mathbb{C}^2\times U\to\mathbb{C}$, $\psi(p(z),zp'(z);z)=p(z)+\frac{1}{\gamma}zp'(z)$ an analytic function in U and*

$$F_{\psi(\mathbb{C}^2\times U)}\left(p(z)+\frac{1}{\gamma}zp'(z)\right)\le F_{h(U)}h(z),\ \text{i.e. } p(z)+\frac{1}{\gamma}zp'(z)\prec_{\mathcal{F}} h(z),\quad z\in U,\tag{1.2}$$

then $F_{p(U)}p(z)\le F_{g(U)}g(z)\le F_{h(U)}h(z)$, i.e. $p(z)\prec_{\mathcal{F}} g(z)\prec_{\mathcal{F}} h(z)$, $z\in U$, where $g(z)=\frac{\gamma}{nz^{\gamma/n}}\int_0^z h(t)t^{\gamma/n-1}dt$, $z\in U$. The function q is convex and is the fuzzy best dominant.

Lemma 1.3 *([12]) Let g be a convex function in U and let $h(z)=g(z)+n\alpha zg'(z)$, $z\in U$, where $\alpha>0$ and n is a positive integer.*

If $p(z)=g(0)+p_nz^n+p_{n+1}z^{n+1}+\dots$, $z\in U$, is holomorphic in U and $F_{p(U)}(p(z)+\alpha zp'(z))\le F_{h(U)}h(z)$, i.e. $p(z)+\alpha zp'(z)\prec_{\mathcal{F}} h(z)$, $z\in U$, then $F_{p(U)}p(z)\le F_{g(U)}g(z)$, i.e. $p(z)\prec_{\mathcal{F}} g(z)$, $z\in U$, and this result is sharp.

We use the following differential operators.

Definition 1.4 *(Sălăgean [14]) For $f\in\mathcal{A}$, $m\in\mathbb{N}$, the operator S^m is defined by $S^m:\mathcal{A}\to\mathcal{A}$,*

$$\begin{aligned}S^0f(z)&=f(z)\\S^1f(z)&=zf'(z),\ ...\\S^{m+1}f(z)&=z\left(S^mf(z)\right)',\ \ z\in U.\end{aligned}$$

Remark 1.2 *If $f\in\mathcal{A}$, $f(z)=z+\sum_{j=2}^{\infty}a_jz^j$, then $S^mf(z)=z+\sum_{j=2}^{\infty}j^ma_jz^j$, $z\in U$.*

Definition 1.5 *(Ruscheweyh [13]) For $f\in\mathcal{A}$, $m\in\mathbb{N}$, the operator R^m is defined by $R^m:\mathcal{A}\to\mathcal{A}$,*

$$\begin{aligned}R^0f(z)&=f(z)\\R^1f(z)&=zf'(z),\ ...\\(m+1)R^{m+1}f(z)&=z\left(R^mf(z)\right)'+mR^mf(z),\qquad z\in U.\end{aligned}$$

Remark 1.3 *If $f\in\mathcal{A}$, $f(z)=z+\sum_{j=2}^{\infty}a_jz^j$, then $R^mf(z)=z+\sum_{j=n+1}^{\infty}C_{m+j-1}^{m}a_jz^j$, $z\in U$.*

Definition 1.6 *[1] Let $m\in\mathbb{N}\cup\{0\}$. Denote by SR^m the operator given by the Hadamard product (the convolution product) of the Sălăgean operator S^m and the Ruscheweyh operator R^m, $SR^m:\mathcal{A}\to\mathcal{A}$, $SR^mf(z)=(S^m*R^m)f(z)$.*

Remark 1.4 *[1] If $f\in\mathcal{A}$, $f(z)=z+\sum_{j=2}^{\infty}a_jz^j$, then $SR^mf(z)=z+\sum_{j=2}^{\infty}C_{m+j-1}^{m}j^ma_j^2z^j$, $z\in U$.*

Remark 1.5 *The operator SR^m was studied in [1], [2], [3], [4].*

2 Main results

Using the operator $RD^m_{\lambda,\alpha}$ defined in Definition 1.6 we define the class $\mathcal{SR}^{\mathcal{F}}_m(\delta)$ and we study fuzzy subordinations.

Definition 2.1 *[5] Let* $f(D) = supp\big(f(D), F_{f(D)}\big) = \{z \in D : 0 < F_{f(D)}f(z) \le 1\}$, *where* $F_{f(D)}\cdot$ *is the membership function of the fuzzy set* $f(D)$ *asociated to the function* f. *The membership function of the fuzzy set* $(\mu f)(D)$ *asociated to the function* μf *coincide with the membership function of the fuzzy set* $f(D)$ *asociated to the function* f, *i.e.* $F_{(\mu f)(D)}((\mu f)(z)) = F_{f(D)}f(z)$, $z \in D$. *The membership function of the fuzzy set* $(f+g)(D)$ *asociated to the function* $f+g$ *coincide with the half of the sum of the membership functions of the fuzzy sets* $f(D)$, *respectively* $g(D)$, *asociated to the function* f, *respectively* g, *i.e.* $F_{(f+g)(D)}((f+g)(z)) = \frac{F_{f(D)}f(z)+F_{g(D)}g(z)}{2}$, $z \in D$.

Remark 2.1 *[5]* $F_{(f+g)(D)}((f+g)(z))$ *can be defined in other ways. Since* $0 < F_{f(D)}f(z) \le 1$ *and* $0 < F_{g(D)}g(z) \le 1$, *it is evidently that* $0 < F_{(f+g)(D)}((f+g)(z)) \le 1$, $z \in D$.

Definition 2.2 *Let* $\delta \in [0,1)$ *and* $m \in \mathbb{N}$. *A function* $f \in \mathcal{A}$ *is said to be in the class* $\mathcal{SR}^{\mathcal{F}}_m(\delta)$ *if it satisfies the inequality*

$$F_{(SR^m f)'(U)}(SR^m f(z))' > \delta, \qquad z \in U. \tag{2.1}$$

Theorem 2.1 *Let* g *be a convex function in* U *and let* $h(z) = g(z) + \frac{1}{c+2}zg'(z)$, $z \in U$, *where* $c > 0$. *If* $f \in \mathcal{SR}^{\mathcal{F}}_m(\delta)$ *and* $G(z) = I_c(f)(z) = \frac{c+2}{z^{c+1}}\int_0^z t^c f(t)\,dt$, $z \in U$, *then*

$$F_{(SR^m f)'(U)}(SR^m f(z))' \le F_{h(U)}h(z), \text{ i.e. } (SR^m f(z))' \prec_{\mathcal{F}} h(z), \quad z \in U, \tag{2.2}$$

implies $F_{(SR^m G)'(U)}(SR^m G(z))' \le F_{g(U)}g(z)$, *i.e.* $(SR^m G(z))' \prec_{\mathcal{F}} g(z)$, $z \in U$, *and this result is sharp.*

Proof. We have $z^{c+1}G(z) = (c+2)\int_0^z t^c f(t)\,dt$. Differentiating, with respect to z, we obtain $(c+1)G(z) + zG'(z) = (c+2)f(z)$ and

$$(c+1)SR^m G(z) + z(SR^m G(z))' = (c+2)SR^m f(z), \; z \in U. \tag{2.3}$$

Differentiating (2.3) we have

$$(SR^m G(z))' + \frac{1}{c+2}z(SR^m G(z))'' = (SR^m f(z))', \; z \in U. \tag{2.4}$$

Using (2.4), the fuzzy differential subordination (2.2) becomes

$$F_{(SR^m G)'(U)}\left((SR^m G(z))' + \frac{1}{c+2}z(SR^m G(z))''\right) \le F_{g(U)}\left(g(z) + \frac{1}{c+2}zg'(z)\right). \tag{2.5}$$

If we denote

$$p(z) = (SR^m G(z))' \tag{2.6}$$

then $p \in \mathcal{H}[1,n]$.

Replacing (2.6) in (2.5) we obtain

$$F_{p(U)}\left(p(z) + \frac{1}{c+2}zp'(z)\right) \le F_{g(U)}\left(g(z) + \frac{1}{c+2}zg'(z)\right), \quad z \in U.$$

Using Lemma 1.3 we have $F_{p(U)}p(z) \le F_{g(U)}g(z)$, $z \in U$, i.e. $F_{(SR^m G)'(U)}(SR^m G(z))' \le F_{g(U)}g(z)$, $z \in U$, and g is the fuzzy best dominant. We have obtained that $(SR^m G(z))' \prec_{\mathcal{F}} g(z)$, $z \in U$. ∎

Theorem 2.2 *Let* $h(z) = \frac{1+(2\beta-1)z}{1+z}$, $\beta \in [0,1)$ *and* $c > 0$. *If* $m \in \mathbb{N}$ *and* I_c *is given by Theorem 2.1, then*

$$I_c\left[\mathcal{SR}^{\mathcal{F}}_m(\delta)\right] \subset \mathcal{SR}^{\mathcal{F}}_m(\delta^*), \tag{2.7}$$

where $\beta^* = 2\beta - 1 + (c+2)(2-2\beta)\int_0^1 \frac{t^{c+1}}{t+1}dt$.

Proof. The function h is convex and using the same steps as in the proof of Theorem 2.1 we get from the hypothesis of Theorem 2.2 that $F_{p(U)}\left(p(z)+\frac{1}{c+2}zp'(z)\right)\leq F_{h(U)}h(z)$, where $p(z)$ is defined in (2.6).

Using Lemma 1.2 we deduce that $F_{p(U)}p(z)\leq F_{g(U)}g(z)\leq F_{h(U)}h(z)$, that is $F_{(SR^mG)'(U)}(SR^mG(z))'\leq F_{g(U)}g(z)\leq F_{h(U)}h(z)$, where $g(z)=\frac{c+2}{z^{c+2}}\int_0^z t^{c+1}\frac{1+(2\beta-1)t}{1+t}dt=2\beta-1+\frac{(c+2)(2-2\beta)}{z^{c+2}}\int_0^z\frac{t^{c+1}}{t+1}dt$. Since g is convex and $g(U)$ is symmetric with respect to the real axis, we deduce

$$F_{(SR^mG)'(U)}(SR^mG(z))'\geq\min_{|z|=1}F_{g(U)}g(z)=F_{g(U)}g(1)\tag{2.8}$$

and $\beta^*=g(1)=2\beta-1+(c+2)(2-2\beta)\int_0^1\frac{t^{c+1}}{t+1}dt$. From (2.8) we deduce inclusion (2.7). ■

Theorem 2.3 *Let g be a convex function, $g(0)=1$, and let h be the function $h(z)=g(z)+zg'(z)$, $z\in U$. If $m\in\mathbb{N}\cup\{0\}$, $f\in\mathcal{A}$ and verifies the fuzzy differential subordination*

$$F_{(SR^mf)'(U)}(SR^mf(z))'\leq F_{h(U)}h(z),\ i.e.\ (SR^mf(z))'\prec_{\mathcal{F}}h(z),\quad z\in U,\tag{2.9}$$

then $F_{SR^mf(U)}\frac{SR^mf(z)}{z}\leq F_{g(U)}g(z)$, i.e. $\frac{SR^mf(z)}{z}\prec_{\mathcal{F}}g(z)$, $z\in U$, and this result is sharp.

Proof. Consider $p(z)=\frac{SR^mf(z)}{z}=\frac{z+\sum_{j=2}^{\infty}C_{m+j-1}^{m}j^ma_j^2z^j}{z}=1+\sum_{j=2}^{\infty}C_{m+j-1}^{m}j^ma_j^2z^{j-1}$. We have $p(z)+zp'(z)=(SR^mf(z))'$, $z\in U$. Then $F_{(SR^mf)'(U)}(SR^mf(z))'\leq F_{h(U)}h(z)$, $z\in U$, becomes $F_{p(U)}(p(z)+zp'(z))\leq F_{h(U)}h(z)=F_{g(U)}(g(z)+zg'(z))$, $z\in U$. By using Lemma 1.3, we obtain $F_{p(U)}p(z)\leq F_{g(U)}g(z)$, $z\in U$, i.e. $F_{SR^mf(U)}\frac{SR^mf(z)}{z}\leq F_{g(U)}g(z)$, $z\in U$. We obtain that $\frac{SR^mf(z)}{z}\prec_{\mathcal{F}}g(z)$, $z\in U$, and this result is sharp. ■

Theorem 2.4 *Let $h\in\mathcal{H}(U)$, with $h(0)=1$, which verifies the inequality* Re $\left(1+\frac{zh''(z)}{h'(z)}\right)>-\frac{1}{2}$, *$z\in U$. If $m\in\mathbb{N}$, $f\in\mathcal{A}$ and verifies the fuzzy differential subordination*

$$F_{(SR^mf)'(U)}(SR^mf(z))'\leq F_{h(U)}h(z),\ i.e.\ (SR^mf(z))'\prec_{\mathcal{F}}h(z),\quad z\in U,\tag{2.10}$$

then $F_{SR^mf(U)}\frac{SR^mf(z)}{z}\leq F_{q(U)}q(z)$, i.e. $\frac{SR^mf(z)}{z}\prec_{\mathcal{F}}q(z)$, $z\in U$, where $q(z)=\frac{1}{z}\int_0^z h(t)dt$. The function q is convex and it is the fuzzy best dominant.

Proof. Let $p(z)=\frac{SR^mf(z)}{z}=1+\sum_{j=2}^{\infty}C_{m+j-1}^{m}j^ma_j^2z^{j-1}=1+\sum_{j=2}^{\infty}p_jz^{j-1}$, $z\in U$, $p\in\mathcal{H}[1,1]$.

Since Re $\left(1+\frac{zh''(z)}{h'(z)}\right)>-\frac{1}{2}$, $z\in U$, from Lemma 1.1, we obtain that $q(z)=\frac{1}{z}\int_0^z h(t)dt$ is a convex function and verifies the differential equation asscociated to the fuzzy differential subordination (2.10) $q(z)+zq'(z)=h(z)$, therefore it is the fuzzy best dominant.

Differentiating, we obtain $(SR^mf(z))'=p(z)+zp'(z)$, $z\in U$, and (2.10) becomes $F_{p(U)}(p(z)+zp'(z))\leq F_{h(U)}h(z)$, $z\in U$.

Using Lemma 1.3, we have $F_{p(U)}p(z)\leq F_{q(U)}q(z)$, $z\in U$, i.e. $F_{SR^mf(U)}\frac{SR^mf(z)}{z}\leq F_{q(U)}q(z)$, $z\in U$. We have obtained that $\frac{SR^mf(z)}{z}\prec_{\mathcal{F}}q(z)$, $z\in U$. ■

Corollary 2.5 *Let $h(z)=\frac{1+(2\beta-1)z}{1+z}$ a convex function in U, $0\leq\beta<1$. If $m\in\mathbb{N}$, $f\in\mathcal{A}$ and verifies the fuzzy differential subordination*

$$F_{(SR^mf)'(U)}(SR^mf(z))'\leq F_{h(U)}h(z),\ i.e.\ (SR^mf(z))'\prec_{\mathcal{F}}h(z),\quad z\in U,\tag{2.11}$$

then $F_{SR^mf(U)}\frac{SR^mf(z)}{z}\leq F_{q(U)}q(z)$, i.e. $\frac{SR^mf(z)}{z}\prec_{\mathcal{F}}q(z)$, $z\in U$, where q is given by $q(z)=2\beta-1+\frac{2(1-\beta)}{z}\ln(1+z)$, $z\in U$. The function q is convex and it is the fuzzy best dominant.

Proof. We have $h(z)=\frac{1+(2\beta-1)z}{1+z}$ with $h(0)=1$, $h'(z)=\frac{-2(1-\beta)}{(1+z)^2}$ and $h''(z)=\frac{4(1-\beta)}{(1+z)^3}$, therefore $Re\left(\frac{zh''(z)}{h'(z)}+1\right)=Re\left(\frac{1-z}{1+z}\right)=Re\left(\frac{1-\rho\cos\theta-i\rho\sin\theta}{1+\rho\cos\theta+i\rho\sin\theta}\right)=\frac{1-\rho^2}{1+2\rho\cos\theta+\rho^2}>0>-\frac{1}{2}$.

Following the same steps as in the proof of Theorem 2.4 and considering $p(z)=\frac{SR^mf(z)}{z}$, the fuzzy differential subordination (2.11) becomes $F_{p(U)}(p(z)+zp'(z))\leq F_{h(U)}h(z)$, $z\in U$.

By using Lemma 1.2 for $\gamma=1$ and $n=1$, we have $F_{p(U)}p(z)\leq F_{q(U)}q(z)$, i.e. $F_{SR^mf(U)}\frac{SR^mf(z)}{z}\leq F_{q(U)}q(z)$ and $q(z)=\frac{1}{z}\int_0^z h(t)\,dt=\frac{1}{z}\int_0^z\frac{1+(2\beta-1)t}{1+t}dt=2\beta-1+\frac{2(1-\beta)}{z}\ln(1+z)$, $z\in U$. ■

Example 2.1 *Let $h(z)=\frac{1-z}{1+z}$ a convex function in U with $h(0)=1$ and $Re\left(\frac{zh''(z)}{h'(z)}+1\right)>-\frac{1}{2}$.*

Let $f(z)=z+z^2$, $z\in U$. For $n=1$, $m=1$, we obtain $SR^1f(z)=z+C_2^1\cdot 2\cdot 1^1\cdot z^2=z+4z^2$. Then $\left(SR^1f(z)\right)'=1+8z$ and $\frac{SR^1f(z)}{z}=1+4z$. We have $q(z)=\frac{1}{z}\int_0^z\frac{1-t}{1+t}dt=-1+\frac{2\ln(1+z)}{z}$.

Using Theorem 2.4 we obtain $1+8z\prec_{\mathcal{F}}\frac{1-z}{1+z}$, $z\in U$, induce $1+4z\prec_{\mathcal{F}}-1+\frac{2\ln(1+z)}{z}$, $z\in U$.

Theorem 2.6 *Let g be a convex function such that $g(0)=1$ and let h be the function $h(z)=g(z)+zg'(z)$, $z\in U$. If $m\in\mathbb{N}\cup\{0\}$, $f\in\mathcal{A}$ and verifies the fuzzy differential subordination*

$$F_{SR^mf(U)}\left(\frac{zSR^{m+1}f(z)}{SR^mf(z)}\right)'\leq F_{h(U)}h(z)\text{, i.e. }\left(\frac{zSR^{m+1}f(z)}{SR^mf(z)}\right)'\prec_{\mathcal{F}}h(z),\quad z\in U,\tag{2.12}$$

then $F_{SR^mf(U)}\frac{SR^{m+1}f(z)}{SR^mf(z)}\leq F_{g(U)}g(z)$, i.e. $\frac{SR^{m+1}f(z)}{SR^mf(z)}\prec_{\mathcal{F}}g(z)$, $z\in U$, and this result is sharp.

Proof. Consider $p(z)=\frac{SR^{m+1}f(z)}{SR^mf(z)}=\frac{z+\sum_{j=n+1}^{\infty}C_{m+j}^{m+1}j^{m+1}a_j^2z^j}{z+\sum_{j=n+1}^{\infty}C_{m+j-1}^{m}j^ma_j^2z^j}=\frac{1+\sum_{j=n+1}^{\infty}C_{m+j}^{m+}j^{m+1}a_j^2z^{j-1}}{1+\sum_{j=n+1}^{\infty}C_{m+j-1}^{m}j^ma_j^2z^{j-1}}$. We have $p'(z)=\frac{\left(SR^{m+1}f(z)\right)'}{SR^mf(z)}-p(z)\cdot\frac{(SR^mf(z))'}{SR^mf(z)}$. Then $p(z)+zp'(z)=\left(\frac{zSR^{m+1}f(z)}{SR^mf(z)}\right)'$. Relation (2.12) becomes $F_{p(U)}(p(z)+zp'(z))\leq F_{h(U)}h(z)=F_{g(U)}(g(z)+zg'(z))$, $z\in U$, and by using Lemma 1.3, we obtain $F_{p(U)}p(z)\leq F_{g(U)}g(z)$, $z\in U$, i.e. $F_{SR^mf(U)}\frac{SR^{m+1}f(z)}{SR^mf(z)}\leq F_{g(U)}g(z)$, $z\in U$. We obtained that $\frac{SR^{m+1}f(z)}{SR^mf(z)}\prec_{\mathcal{F}}g(z)$, $z\in U$. ■

Theorem 2.7 *Let g be a convex function such that $g(0)=1$ and let h be the function $h(z)=g(z)+\frac{1}{m+1}zg'(z)$, $z\in U$, $m\in\mathbb{N}$. If $f\in\mathcal{A}$ and the fuzzy differential subordination*

$$F_{SR^mf(U)}\left(\frac{1}{z}SR^{m+1}f(z)\right)\leq F_{h(U)}h(z)\text{, i.e. }\frac{1}{z}SR^{m+1}f(z)\prec_{\mathcal{F}}h(z),\quad z\in U,\tag{2.13}$$

holds, then $F_{(SR^mf)'(U)}(SR^mf(z))'\leq F_{g(U)}g(z)$, i.e. $(SR^mf(z))'\prec_{\mathcal{F}}g(z)$, $z\in U$, and this result is sharp.

Proof. With notation $p(z)=(SR^mf(z))'=1+\sum_{j=2}^{\infty}C_{m+j-1}^{m}j^{m+1}a_j^2z^{j-1}$ and $p(0)=1$, we obtain for $f(z)=z+\sum_{j=2}^{\infty}a_jz^j$, $p(z)+zp'(z)=\frac{1}{z}SR^{m+1}f(z)+z\frac{m}{m+1}(SR^mf(z))''$.

We have $F_{p(U)}\left(p(z)+\frac{1}{m+1}zp'(z)\right)\leq F_{h(U)}h(z)=F_{g(U)}\left(g(z)+\frac{1}{m+1}zg'(z)\right)$, $z\in U$. By using Lemma 1.3, we obtain $F_{p(U)}p(z)\leq F_{g(U)}g(z)$, $z\in U$, i.e. $F_{(SR^mf)'(U)}(SR^mf(z))'\leq F_{g(U)}g(z)$, $z\in U$, and this result is sharp. We obtained that $(SR^mf(z))'\prec_{\mathcal{F}}g(z)$, $z\in U$. ■

Theorem 2.8 *Let $h\in\mathcal{H}(U)$ with $h(0)=1$, which verifies the inequality* Re $\left[1+\frac{zh''(z)}{h'(z)}\right]>-\frac{1}{2}$*, $z\in U$. If $m\in\mathbb{N}$, $f\in\mathcal{A}$ and satisfies the fuzzy differential subordination*

$$F_{SR^mf(U)}\left(\frac{1}{z}SR^{m+1}f(z)\right)\leq F_{h(U)}h(z)\text{, i.e. }\frac{1}{z}SR^{m+1}f(z)\prec_{\mathcal{F}}h(z),\quad z\in U,\tag{2.14}$$

then $F_{(SR^mf)'(U)}(SR^mf(z))'\leq F_{q(U)}q(z)$, i.e. $(SR^mf(z))'\prec_{\mathcal{F}}q(z)$, $z\in U$, where q is given by $q(z)=\frac{m+1}{z^{m+1}}\int_0^zh(t)t^mdt$. The function q is convex and it is the fuzzy best dominant.

Proof. Since Re $\left(1+\frac{zh''(z)}{h'(z)}\right)>-\frac{1}{2}$, $z\in U$, from Lemma 1.1, we obtain that $q(z)=\frac{m+1}{z^{m+1}}\int_0^zh(t)t^mdt$ is a convex function and verifies the differential equation associated to the fuzzy differential subordination (2.14) $q(z)+\frac{1}{m+1}zq'(z)=h(z)$, therefore it is the fuzzy best dominant.

Using the properties of operator SR^m and considering $p(z)=(SR^mf(z))'$, we obtain $F_{SR^mf(U)}SR^mf(U)=F_{p(U)}\left(p(z)+\frac{1}{m+1}zp'(z)\right)$, $z\in U$. Then (2.14) becomes $F_{p(U)}\left(p(z)+\frac{1}{m+1}zp'(z)\right)\leq F_{h(U)}h(z)$, $z\in U$. Since $p\in\mathcal{H}[1,1]$, using Lemma 1.3 for $\gamma=m+1$, we deduce $F_{p(U)}p(z)\leq F_{q(U)}q(z)$, $z\in U$, where $q(z)=\frac{m+1}{z^{m+1}}\int_0^zh(t)t^mdt$, $z\in U$, i.e. $F_{(SR^mf)'(U)}(SR^mf(z))'\leq F_{q(U)}q(z)$, $z\in U$. We have obtained that $(SR^mf(z))'\prec_{\mathcal{F}}q(z)$, $z\in U$. ■

Corollary 2.9 *Let $h(z)=\frac{1+(2\beta-1)z}{1+z}$ a convex function in U, $0\leq\beta<1$. If $m\in\mathbb{N}$, $f\in\mathcal{A}$ and verifies the fuzzy differential subordination*

$$F_{SR^mf(U)}\left(\frac{1}{z}SR^{m+1}f(z)\right)\leq F_{h(U)}h(z)\text{, i.e. }\frac{1}{z}SR^{m+1}f(z)\prec_{\mathcal{F}}h(z),\quad z\in U,\tag{2.15}$$

then $F_{(SR^mf)'(U)}(SR^mf(z))'\leq F_{q(U)}q(z)$, i.e. $(SR^mf(z))'\prec_{\mathcal{F}}q(z)$, $z\in U$, where q is given by $q(z)=2\beta-1+\frac{2(1-\beta)(m+1)}{z^{m+1}}\int_0^z\frac{t^m}{1+t}dt$, $z\in U$. The function q is convex and it is the fuzzy best dominant.

Proof. Following the same steps as in the proof of Theorem 2.7 and considering $p(z) = (SR^m f(z))'$, the fuzzy differential subordination (2.15) becomes $F_{p(U)}\left(p(z) + \frac{1}{m+1} zp'(z)\right) \leq F_{h(U)}h(z)$, $z \in U$.

By using Lemma 1.2 for $\gamma = m+1$ and $n = 1$, we have $F_{p(U)}p(z) \leq F_{q(U)}q(z)$, i.e. $F_{(SR^m f)'(U)}\,(SR^m f(z))' \leq F_{q(U)}q(z)$ and $q(z) = \frac{m+1}{z^{m+1}} \int_0^z h(t)t^m dt = \frac{m+1}{z^{m+1}} \int_0^z t^m \frac{1+(2\beta-1)t}{1+t} dt = 2\beta - 1 + \frac{2(1-\beta)(m+1)}{z^{m+1}} \int_0^z \frac{t^m}{1+t} dt$, $z \in U$. ■

Example 2.2 *Let* $h(z) = \frac{1-z}{1+z}$ *a convex function in* U *with* $h(0) = 1$ *and* $Re\left(\frac{zh''(z)}{h'(z)} + 1\right) > -\frac{1}{2}$.

Let $f(z) = z + z^2$, $z \in U$. *For* $n = 1$, $m = 1$,*we obtain* $SR^1 f(z) = z + 4z^2$. *Then* $\left(SR^1 f(z)\right)' = 1 + 8z$. *We obtain also* $\frac{1}{z}SR^{m+1}f(z) = \frac{1}{z}SR^2 f(z) = 1 + 12z$, *where* $SR^2 f(z) = z + C_3^2 \cdot 2^2 \cdot 1^2 \cdot z^2 + C_4^2 \cdot 3^2 \cdot 0 \cdot z^3 = z + 12z^2$. *We have* $q(z) = \frac{2}{z^2} \int_0^z \frac{1-t}{1+t} t dt = -1 + \frac{4}{z} - \frac{4\ln(1+z)}{z^2}$.

Using Theorem 2.8 we obtain $1 + 12z \prec_{\mathcal{F}} \frac{1-z}{1+z}$, $z \in U$, *induce* $1 + 8z \prec_{\mathcal{F}} -1 + \frac{4}{z} - \frac{4\ln(1+z)}{z^2}$, $z \in U$.

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J. COMPUTATIONAL ANALYSIS AND APPLICATIONS, VOL. 15, NO.8, 1490-1495, 2013, COPYRIGHT 2013 EUDOXUS PRESS, LLC

Strong differential superordination and sandwich theorem

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Abstract

In this paper we study certain strong differential superordinations and give a sandwich theorem, obtained by using a new integral operator introduced in [21].

Keywords. Analytic function, univalent function, starlike function, convex function, strong differential superordination, best dominant, best subordinant.
2000 Mathematical Subject Classification: 30C80, 30C20, 30C40, 34C40.

1 Introduction and preliminaries

The concept of differential subordination was introduced in [11], [12] and developed in [13], by S.S. Miller and P.T. Mocanu. The concept of differential superordination was introduced in [14], like a dual problem of the differential superordination by S.S. Miller and P.T. Mocanu. The concept of strong differential subordination was introduced in [10] by J.A. Antonino and S. Romaguera and developed in [1], [2], [3], [4], [5], [16], [18], [19], [20], [22], [24]. The concept of strong differential superordination was introduced in [17], like a dual concept of the strong differential subordination and developed in [6], [7], [8], [9], [21], [23].

Let $\mathcal{H}(U\times\overline{U})$ denote the class of analytic function in $U\times\overline{U}$, $U=\{z\in\mathbb{C}:|z|<1\}$, $\overline{U}=\{z\in\mathbb{C}:|z|\le 1\}$, $\partial U=\{z\in\mathbb{C}:|z|=1\}$.

For $a\in\mathbb{C}$ and $n\in\mathbb{N}^*$, let $\mathcal{H}\zeta[a,n]=\{f(z,\zeta)\in\mathcal{H}(U\times\overline{U}):f(z,\zeta)=a+a_n(\zeta)z^n+\ldots+a_{n+1}(\zeta)z^{n+1}+\ldots\}$ with $z\in U$, $\zeta\in\overline{U}$, $a_k(\zeta)$ holomorphic functions in $\overline{U}$, $k\ge n$, $A\zeta_n=\{f(z,\zeta)\in\mathcal{H}(U\times\overline{U}):\ f(z,\zeta)=z+a_{n+1}(\zeta)z^{n+1}+a_{n+2}(\zeta)z^{n+2}+\ldots\}$ with $z\in U$, $\zeta\in\overline{U}$, $a_k(\zeta)$ holomorphic functions in $\overline{U}$, $k\ge n+1$, so $A\zeta_1=A\zeta$, $\mathcal{H}\zeta_u(U)=\{f(z,\zeta)\in\mathcal{H}\zeta[a,n]:\ f(z,\zeta)$ univalent in U, for all $\zeta\in\overline{U}\}$, $S\zeta=\{f(z,\zeta)\in A\zeta,\ f(z,\zeta)$ univalent in U, for all $\zeta\in\overline{U}\}$, denote the class of univalent functions in $U\times\overline{U}$, $S^*\zeta=\{f(z,\zeta)\in A\zeta:\ \mathrm{Re}\,\frac{zf'(z,\zeta)}{f(z,\zeta)}>0,\ z\in U$, for all $\zeta\in\overline{U}\}$, denote the class of normalized starlike functions in $U\times\overline{U}$, $K\zeta=\{f(z,\zeta)\in A\zeta:\ \mathrm{Re}\left[\frac{zf''(z,\zeta)}{f'(z,\zeta)}+1\right]>0,\ z\in U$, for all $\zeta\in\overline{U}\}$, denote the class of normalized convex functions in $U\times\overline{U}$.

For $r\in\mathbb{N}$, $A(r)\zeta$ denote the subclass of the functions $f(z,\zeta)\in(U\times\overline{U})$ of the form $f(z,\zeta)=z^r+\sum_{k=r+1}^{\infty}a_k(\zeta)z^k,\ r\in\mathbb{N},\ z\in U,\ \zeta\in\overline{U}$ and set $A(1)\zeta=A\zeta$.

To prove our main results, we need the following definitions and lemmas:

Definition 1.1 *[16], [18] Let $f(z,\zeta)$ and $F(z,\zeta)$ analytic functions from $\mathcal{H}(U\times\overline{U})$. The function $f(z,\zeta)$ is said to be strongly subordinated to $F(z,\zeta)$, or $F(z,\zeta)$ is said to be strongly superordinated to $f(z,\zeta)$, if there exists a function w analytic in $\overline{U}$ with $w(0)=0$ and $|w(z)|<1$, such that $f(z,\zeta)=F(w(z),\zeta)$. In such a case we write $f(z,\zeta)\prec\prec F(z,\zeta)$.*

If $F(z,\zeta)$ is univalent then $f(z,\zeta)\prec\prec F(z,\zeta)$ if and only if $f(0,\zeta)=F(0,\zeta)$ and $f(U\times\overline{U})\subset F(U\times\overline{U})$.

Remark 1.1 *If $f(z,\zeta)\equiv f(z)$ and $F(z,\zeta)\equiv F(z)$, then the strong differential subordination or strong differential superordination becomes the usual notion of differential subordination or differential superordination.*

Definition 1.2 *[14], [16] We denote by Q_ζ the set of functions $q(z,\zeta)$ that are analytic and injective with respect to z on $\overline{U}\setminus E(q(z,\zeta))$, where $E(q(z,\zeta))=\left\{\xi\in\partial U:\ \lim\limits_{z\to\xi}q(z,\zeta)=\infty\right\}$ and $q'(\xi,\zeta)\neq 0$, for $\xi\in\partial U\setminus E(q(z,\zeta))$. The class of Q_ζ for which $q(0,\zeta)=a$, is denoted by $Q_\zeta(a)$.*

We mention that all the derivatives which appear in this paper are considered with respect to variable z.

Let $\psi : \mathbb{C}^3 \times U \times \overline{U} \to \mathbb{C}$ and let $h(z,\zeta)$ be univalent in U, for all $\zeta \in \overline{U}$. If $p(z,\zeta)$ is analytic in $U \times \overline{U}$ and satisfies the (second-order) strong differential subordination

$$\psi(p(z,\zeta), z'(z,\zeta), z^2p''(z,\zeta); z,\zeta) \prec\prec h(z,\zeta),\ z \in U,\ \zeta \in \overline{U} \tag{1.1}$$

then $p(z,\zeta)$ is called a solution of the strong differential subordination.

The univalent function $q(z,\zeta)$ is called a dominant of the solutions of the strong differential subordination or simply a dominant, if $p(z,\zeta) \prec\prec q(z,\zeta)$ for all $p(z,\zeta)$ satisfying (1.1).

A dominant $\widetilde{q}(z,\zeta)$ that satisfies $\widetilde{q}(z,\zeta) \prec\prec q(z,\zeta)$ for all dominants $q(z,\zeta)$ of (1.1) is said to be the best dominant of (1.1). (Note that the best dominant is unique up to a rotation of U).

Let $\varphi : \mathbb{C}^3 \times U \times \overline{U} \to \mathbb{C}$ and let $h(z,\zeta)$ be analytic in $U \times \overline{U}$.

If $p(z,\zeta)$ and $\varphi(p(z,\zeta), zp'(z,\zeta), z^2p''(z,\zeta); z,\zeta)$ are univalent in U, for all $\zeta \in \overline{U}$ and satisfy the (second-order) strong differential superordination

$$h(z,\zeta) \prec\prec \varphi(p(z,\zeta), zp'(z,\zeta), z^2p''(z,\zeta); z,\zeta) \tag{1.2}$$

then $p(z,\zeta)$ is called a solution of the strong differential superordination. An analytic function $q(z,\zeta)$ is called a subordinant of the solutions of the differential superordination, or more simply a subordinant, if $q(z,\zeta) \prec\prec p(z,\zeta)$ for all $p(z,\zeta)$ satisfying (1.2). A univalent subordinant $\widetilde{q}(z,\zeta)$ that satisfies $q(z,\zeta) \prec\prec \widetilde{q}(z,\zeta)$ for all subordinants of (1.2) is said to be the best subordinant. (Note that the best subordinant is unique up to a rotation of U).

In order to prove the original results of this paper, we need the following definitions and lemmas.

Definition 1.3 *[11] For $f(z,\zeta) \in A\zeta_n$, $n \in \mathbb{N}^*$, $m \in \mathbb{N}$, $\gamma \in \mathbb{C}$, let L_γ be the integral operator given by $L_\gamma : A\zeta_n \to A\zeta_n$*

$$\begin{aligned}
L^0_\gamma f(z,\zeta) &= f(z,\zeta)\\
L^1_\gamma f(z,\zeta) &= \frac{\gamma+1}{z^\gamma}\int_0^z L^0_\gamma f(z,\zeta)t^{\gamma-1}dt\\
L^2_\gamma f(z,\zeta) &= \frac{\gamma+1}{z^\gamma}\int_0^z L^1_\gamma f(z,\zeta)t^{\gamma-1}dt, ...\\
L^m_\gamma f(z,\zeta) &= \frac{\gamma+1}{z^\gamma}\int_0^z L^{m-1}_\gamma f(z,\zeta)t^{\gamma-1}dt.
\end{aligned}$$

By using Definition 1.3, we can prove the following properties for this integral operator:

For $f(z,\zeta) \in A\zeta_n$, $n \in \mathbb{N}^*$, $m \in \mathbb{N}$, $\gamma \in \mathbb{C}$, we have

$$L^m_\gamma f(z,\zeta) = z + \sum_{k=n+1}^{\infty} \frac{(\gamma+1)^m}{(\gamma+k)^m} a_k(\zeta) z^k,\ z \in U,\ \zeta \in \overline{U} \tag{1.3}$$

and

$$z[L^m_\gamma f(z,\zeta)]'_z = (\gamma+1)L^{m-1}_\gamma f(z,\zeta) - \gamma L^m_\lambda f(z,\zeta),\ z \in U,\ \zeta \in \overline{U}. \tag{1.4}$$

Definition 1.4 *[20] For $r \in \mathbb{N}$, $f(z,\zeta) \in A(r)\zeta$, let H be the integral operator given by $H : A(r)\zeta \to A(r)\zeta$*

$$\begin{aligned}
H^0 f(z,\zeta) &= f(z,\zeta)\\
H^1 f(z,\zeta) &= \frac{r+1}{z}\int_0^z H^0 f(t,\zeta)dt\\
H^2 f(z,\zeta) &= \frac{r+1}{z}\int_0^z H^1 f(t,\zeta)dt, ...\\
H^m f(z,\zeta) &= \frac{r+1}{z}\int_0^z H^{m-1} f(t,\zeta)dt,\ z \in U,\ \zeta \in \overline{U}.
\end{aligned}$$

From Definition 1.4 we have

$$H^m f(z,\zeta) = z^t + \sum_{k=r+1}^{\infty} \frac{(r+1)^m}{(r+k)^m} a_k(\zeta) z^k \tag{1.5}$$

and

$$z[H^m f(z,\zeta)]'_z = (r+1)H^{m-1} f(z,\zeta) - H^m f(z,\zeta),\ z \in U,\ \zeta \in \overline{U}. \tag{1.6}$$

Lemma 1.1 *[14, Corollary 9.1] Let $h_1(z,\zeta)$ and $h_2(z,\zeta)$ be starlike in $U\times\overline{U}$, with $h_1(0,\zeta)=h_2(0,\zeta)=0$ and the functions $q_i(z,\zeta)$ defined by $q_i(z,\zeta)=\int_0^z h_i(t,\zeta)t^{-1}dt$, for $i=1,2$. If $p(z,\zeta)\in[0,1]\cap Q_\zeta$ and $zp'(z,\zeta)$ is univalent in $U\times\overline{U}$, then $h_1(z,\zeta)\prec\prec zp'(z,\zeta)\prec\prec h_2(z,\zeta)$ implies $q_1(z,\zeta)\prec\prec p(z,\zeta)\prec\prec q_2(z,\zeta)$.*

The functions $q_1(z,\zeta)$ and $q_2(z,\zeta)$ are convex and they are respectively the best subordinant and best dominant.

Lemma 1.2 *[15, Theorem 3] Let θ and ϕ be analytic in a domain D, and let $q(z,\zeta)$ be univalent in U, for all $\zeta\in\overline{U}$, with $q(0,\zeta)=a$ and $q(U\times\overline{U})\subset D$. Let $Q(z,\zeta)=zq'(z,\zeta)\cdot\phi(q(z))$, $h(z,\zeta)=\theta(q(z,\zeta))+Q(z,\zeta)$ and suppose that*

(i) Re $\left[\frac{\theta'(q(z,\zeta))}{\phi(q(z,\zeta))}\right]>0$, and

(ii) $Q(z,\zeta)$ is starlike in U, for all $\zeta\in\overline{U}$.

If $p(z,\zeta)\in[a,1]\cap Q_\zeta$, $p(U\times\overline{U})\subset D$ and $\theta(p(z,\zeta))+zp'(z,\zeta)\cdot\phi(z,\zeta)$ is univalent in U, for all $\zeta\in\overline{U}$, then $h(z,\zeta)\prec\prec\theta(p(z,\zeta))+zp'(z,\zeta)\cdot\phi(p(z,\zeta))$ implies $q(z,\zeta)\prec\prec p(z,\zeta)$, $z\in U$, $\zeta\in\overline{U}$ and $q(z,\zeta)$ is the best subordinant.

2 Main results

Theorem 2.1 *Let $h_1(z,\zeta)=\frac{\zeta z}{\zeta-z}$ and $h_2(z,\zeta)=\frac{z}{\zeta+z}$, be starlike in U, for all $\zeta\in\overline{U}$, with $h_1(0,\zeta)=h_2(0,\zeta)=0$, and $q_1(z,\zeta)=\int_0^z\frac{\zeta}{\zeta-t}dt=\zeta\ln\frac{\zeta}{\zeta-z}$ and $q_2(z,\zeta)=\int_0^z\frac{1}{\zeta+t}dt=\ln\frac{\zeta+z}{\zeta}$. For $m\in\mathbb{N}$, $\gamma\in\mathbb{C}$, $f(z,\zeta)\in A\zeta$, if $\frac{z^2[L_\gamma^m f(z,\zeta)]'}{L_\gamma^m f(z,\zeta)}\in[0,1]\cap Q_\zeta$ and $\frac{2z^2[L_\gamma^m f(z,\zeta)]'L_\gamma^m f(z,\zeta)+z^3[L_\gamma^m f(z,\zeta)]''L_\gamma^m f(z,\zeta)-z^3[(L_\gamma^m f(z,\zeta))']^2}{[L_\gamma^m f(z,\zeta)]^2}$ is univalent in U, for all $\zeta\in\overline{U}$, then*

$$\frac{\zeta z}{\zeta-z}\prec\prec\frac{2z^2[L_\gamma^m f(z,\zeta)]'L_\gamma^m f(z,\zeta)+z^3[L_\gamma^m f(z,\zeta)]''L_\gamma^m f(z,\zeta)-z^3[(L_\gamma^m f(z,\zeta))']^2}{[L_\gamma^m f(z,\zeta)]^2}\prec\prec\frac{z}{\zeta+z}\tag{2.1}$$

implies $\zeta\ln\frac{\zeta}{\zeta-z}\prec\prec\frac{z^2[L_\gamma^m f(z,\zeta)]'}{L_\gamma^m f(z,\zeta)}\prec\prec\ln\frac{\zeta+z}{\zeta}$, $z\in U$, $\zeta\in\overline{U}$.

The functions $q_1(z,\zeta)=\zeta\ln\frac{\zeta}{\zeta-n}$ and $q_2(z,\zeta)=\ln\frac{\zeta+z}{z}$ are convex and they are respectively the best subordinant and best dominant.

Proof. In order to prove the theorem, we shall use Lemma 1.1.

We have $\mathrm{Re}\frac{zh_1'(z,\zeta)z}{h_1(z,\zeta)}=\mathrm{Re}\,\frac{\zeta}{\zeta-z}=\frac{1}{2}>0$, $z\in U$, $\zeta\in U$ and $\mathrm{Re}\,\frac{zh_2'(z,\zeta)z}{h_2(z,\zeta)}=\mathrm{Re}\,\frac{\zeta}{\zeta+z}=\frac{1}{2}>0$, $z\in U$, $\zeta\in U$ hence $h_1(z,\zeta)$ and $h_2(z,\zeta)$ are starlike in U, for all $\zeta\in\overline{U}$.

We consider

$$p(z,\zeta)=\frac{z^2[L_\gamma^m f(z,\zeta)]'}{L_\gamma^m f(z,\zeta)},\ z\in U,\ \zeta\in\overline{U}.\tag{2.2}$$

Using (1.3) in (2.2), we have $p(z,\zeta)=\frac{z^2\left(z+\sum_{k=2}^\infty\frac{(\gamma+1)^m}{(\gamma+k)^m}a_k(\zeta)z^k\right)}{z+\sum_{k=2}^\infty\frac{(\gamma+1)^m}{(\gamma+k)^m}a_k(\zeta)z^k}=\frac{z\left(1+\sum_{k=2}^\infty\frac{(\gamma+1)^m}{(\gamma+k)^m}a_k\cdot\zeta\cdot k\cdot z^{k-1}\right)}{1+\sum_{k=2}^\infty\frac{(\gamma+1)^m}{(\gamma+k)^m}a_k(\zeta)z^k}$. Since $p(0,\zeta)=0$, we have $p(z,\zeta)\in[0,1]\zeta\cap Q_\zeta$.

Differentiating (2.2), and after a short calculus we obtain

$$zp'(z,\zeta)=\frac{2z^2[L_\gamma^m f(z,\zeta)]'L_\gamma^m f(z,\zeta)+z^3[L_\gamma^m f(z,\zeta)]''L_\gamma^m f(z,\zeta)}{[L_\gamma^m f(z,\zeta)]^2}-\frac{z^3[L_\gamma^m f(z,\zeta)]'^2}{[L_\gamma^m f(z,\zeta)]^2}.\tag{2.3}$$

Using (2.3) in (2.1), we have

$$\frac{\zeta z}{\zeta-z}\prec\prec zp'(z,\zeta)\prec\prec\frac{z}{\zeta+z},\ z\in U,\ \zeta\in\overline{U}.\tag{2.4}$$

Using Lemma 1.1, we obtain $\zeta\ln\frac{\zeta}{\zeta-z}\prec\prec\frac{z^2[L_\gamma^m f(z,\zeta)]'}{L_\gamma^m f(z,\zeta)}\prec\prec\ln\frac{\zeta+z}{\zeta}$, $z\in U$, $\zeta\in\overline{U}$. ■

Theorem 2.2 *Let $m\in\mathbb{N}$, $\gamma\in\mathbb{C}$, $\lambda\in\mathbb{C}$, $q(z,\zeta)=e^{\lambda z\zeta}$ starlike (univalent) function in U, for all $\zeta\in\overline{U}$, with $q(0,\zeta)=1$, and suppose that*

(j) $\mathrm{Re}\,\lambda z\zeta>-\frac{1}{2}$,

(jj) $\mathrm{Re}\,\lambda\zeta>0$.

Let $Q(z,\zeta) = \lambda z\zeta e^{2\lambda z\zeta}$ *and* $h(z,\zeta) = \lambda z\zeta e^{2\lambda z\zeta} + e^{\lambda z\zeta}$, $z \in U$, $\zeta \in \overline{U}$. *If* $f(z,\zeta) \in A\zeta$, $[L_\gamma^m f(z,\zeta)]' \in [1,1] \cap Q_\zeta$ *and* $[L_\gamma^m f(z,\zeta)]' + z[L_\gamma^m f(z,\zeta)]'[L_\gamma^m f(z,\zeta)]''$ *is univalent in* U, *for all* $\zeta \in \overline{U}$, *then*

$$\lambda z\zeta e^{2\lambda z\zeta} + e^{\lambda z\zeta} \prec\prec [L_\gamma^m f(z,\zeta)]' + z[L_\gamma^m f(z,\zeta)]'[L_\gamma^m f(z,\zeta)]'' \tag{2.5}$$

implies $e^{\lambda z\zeta} \prec\prec [L_\gamma^m f(z,\zeta)]'$, $z \in U$, $\zeta \in \overline{U}$ *and* $q(z,\zeta) = e^{\lambda z\zeta}$ *is the best subordinant.*

Proof. In order to prove the theorem, we shall use Lemma 1.2. For that, we show that the necessary conditions are satisfied.

Let the functions $\theta : \mathbb{C} \to \mathbb{C}$, $\varphi : \mathbb{C} \to \mathbb{C}$, with

$$Q(w) = w \tag{2.6}$$

and

$$\varphi(w) = w. \tag{2.7}$$

We check the conditions from the hypothesis of Lemma 1.2. Using (2.6), (2.7), (i) and (ii) we have

$$\mathrm{Re}\,\frac{\theta'(q(z,\zeta))}{\varphi(q(z,\zeta))} = \mathrm{Re}\,\frac{\lambda\zeta e^{\lambda z\zeta}}{e^{\lambda z\zeta}} = \mathrm{Re}\,\lambda\zeta > 0, \tag{2.8}$$

and

$$\mathrm{Re}\,\frac{zQ'(z,\zeta)}{Q(z,\zeta)} = \mathrm{Re}\,(1 + 2\lambda z\zeta) > 0. \tag{2.9}$$

We consider

$$p(z,\zeta) = [L_\gamma^m f(z,\zeta)]', \ z \in U, \ \zeta \in \overline{U}. \tag{2.10}$$

Using (1.3) in (2.10), we have $p(z,\zeta) = \left[z + \sum_{k=2}^{\infty} \frac{(\gamma+1)^m}{(\gamma+k)^m} a_k(\zeta) z^k\right]' = 1 + \sum_{k=2}^{\infty} \frac{(\gamma+1)^m}{(\gamma+k)^m} a_k(\zeta) k z^{k-1}$. Since $p(0,\zeta) = 1$, we have $p(z,\zeta) \in [1,1] \cap Q_\zeta$. Differentiating (2.10) and after a short calculus we obtained

$$p(z,\zeta) + zp'(z,\zeta)p(z,\zeta) = [L_\gamma^m f(z,\zeta)]' + z[L_\gamma^m f(z,\zeta)]''[L_\gamma^m f(z,\zeta)]'. \tag{2.11}$$

Using (2.6) and (2.7), we have

$$\theta(p(z,\zeta)) = p(z,\zeta) \text{ and } \varphi(p(z,\zeta)) = p(z,\zeta) \tag{2.12}$$

and (2.11) becomes

$$\theta(p(z,\zeta)) + zp'(z,\zeta)\varphi(p(z,\zeta)) = [L_\gamma^m f(z,\zeta)]' + z[L_\gamma^m f(z,\zeta)]''[L_\gamma^m f(z,\zeta)]'. \tag{2.13}$$

Using (2.6) and (2.7), we have $\theta(q(z,\zeta)) = q(z,\zeta)$ and $\varphi(q(z,\zeta)) = q(z,\zeta)$,

$$h(z,\zeta) = q(z,\zeta) + zq'(z,\zeta)q(z,\zeta) = e^{\lambda z\zeta} + \lambda z\zeta e^{2\lambda z\zeta}. \tag{2.14}$$

Using (2.13) and (2.14), the strong superordination (2.5) becomes

$$h(z,\zeta) \prec\prec \theta(p(z,\zeta)) + zp'(z,\zeta)\varphi(p(z,\zeta)), \ z \in U, \ \zeta \in \overline{U}. \tag{2.15}$$

Since (2.8) and (2.9) give the conditions from the hypothesis of Lemma 1.2 and using (2.15) by applying Lemma 1.2 we obtain $q(z,\zeta) = e^{\lambda z\zeta} \prec\prec [L_\gamma^m f(z,\zeta)]'$, $z \in U$, $\zeta \in \overline{U}$ and $q(z,\zeta) = e^{\lambda z\zeta}$ is the best dominant. ■

Theorem 2.3 *Let* $p \in \mathbb{N}$, $m \in \mathbb{N}$, $h_1(z,\zeta) = \frac{z\zeta}{1+z\zeta}$, $h_2(z,\zeta) = \frac{z}{1-z\zeta}$ *be starlike in* U, *for all* $\zeta \in \overline{U}$, *with* $h_1(0,\zeta) = h_2(0,\zeta) = 0$, *and* $q_1(z,\zeta) = \int_0^z \frac{h_1(t,\zeta)}{t}dt = \int_0^z \frac{\zeta}{1+t\zeta}dt = \ln(1+\zeta z)$, $q_2(z,\zeta) = \int_0^z \frac{h_2(t,\zeta)}{t}dt = \int_0^z \frac{1}{1-t\zeta}dt = -\frac{\ln(1-z\zeta)}{\zeta}$. *If* $\frac{H^m f(z,\zeta)}{z^{r-1}} \in [0,1] \cap Q_\zeta$ *and* $\frac{z[H^m f(z,\zeta)]' - (r-1)H^m f(z,\zeta)}{z^{r-1}}$ *is univalent in* U, *for all* $\zeta \in \overline{U}$, *then*

$$\frac{z\zeta}{1+z\zeta} \prec\prec \frac{z[H^m f(z,\zeta)]' - (r-1)H^m f(z,\zeta)}{z^{r-1}} \prec\prec \frac{z}{1-z\zeta} \tag{2.16}$$

implies $\ln(1+z\zeta) \prec\prec \frac{H^m f(z,\zeta)}{z^{r-1}} \prec\prec -\frac{\ln(1-z\zeta)}{\zeta}$, $z \in U$, $\zeta \in \overline{U}$.

The functions $q_1(z,\zeta) = \frac{\ln(1+z\zeta)}{\zeta}$ *and* $q_2(z,\zeta) = -\frac{\ln(1-z\zeta)}{\zeta}$ *are convex and they are respectively the best subordinant and best dominant.*

Proof. In order to prove the theorem, we shall use Lemma 1.1.

We have $\text{Re}\,\frac{z[h_1(z,\zeta)]'}{h_1(z,\zeta)} = \text{Re}\,\frac{1}{1+z\zeta} = \frac{1}{2} > 0,\ z \in U,\ \zeta \in \overline{U}$ and $\text{Re}\frac{z[h_2(z,\zeta)]'}{h_2(z,\zeta)} = \text{Re}\,\frac{1}{1-z\zeta} = \frac{1}{2} > 0,\ z \in U,\ \zeta \in \overline{U}$. Hence $h_1(z,\zeta)$ and $h_2(z,\zeta)$ are starlike in U, for $\zeta \in \overline{U}$.

We consider

$$p(z,\zeta) = \frac{H^m f(z,\zeta)}{z^{r-1}},\ z \in U,\ \zeta \in \overline{U}. \tag{2.17}$$

Using (1.3), we have $p(z,\zeta) = \frac{z^r + \sum_{k=r+1}^{\infty} \frac{(r+1)^m}{(r+\gamma)^m} a_k(\zeta) z^k}{z^{r-1}} = z + \sum_{k=r+1}^{\infty} \frac{(r+1)^m}{(r+k)^m} a_k(\zeta) z^{k-r+1}$. Since $p(0,\zeta) = 0$, we have $p(z,\zeta) \in A\zeta$. Differentiating (2.17) and after a short calculus, we obtain

$$zp'(z,\zeta) = \frac{z[H^m f(z,\zeta)]' - (r-1)H^m f(z,\zeta)}{z^{r-1}},\ z \in U,\ \zeta \in \overline{U}. \tag{2.18}$$

Using (2.18) in (2.16), we have

$$\frac{z\zeta}{1+z\zeta} \prec\prec zp'(z,\zeta) \prec\prec \frac{z}{1-z\zeta},\ z \in U,\ \zeta \in \overline{U}. \tag{2.19}$$

From Lemma 1.1, we obtain $\ln(1+z\zeta) \prec\prec \frac{H^m f(z,\zeta)}{z^{r-1}} \prec\prec -\frac{\ln(1-z\zeta)}{\zeta},\ z \in U,\ \zeta \in \overline{U}$. The functions $q_1(z,\zeta) = \ln(1+z\zeta)$ and $q_2(z,\zeta) = -\frac{\ln(1-z\zeta)}{\zeta}$ are convex and they are respectively the best subordinant and best dominant. ■

Example 2.1 *Let* $\gamma = 2,\ m = 1,\ f(z,\zeta) = z + 5\zeta z^3,\ L_2^1 f(z,\zeta) = \frac{2}{3}z + 2\zeta z^3,\ p(z,\zeta) = \frac{z+9\zeta z^2}{1+6\zeta z^2},\ zp'(z,\zeta) = \frac{z+18\zeta z^3 - 36\zeta^2 z^4}{(1+3\zeta z^2)^2}$*. From Theorem 2.1, we have* $\frac{\zeta z}{\zeta - z} \prec\prec \frac{z^3+18\zeta z^5-36\zeta^2 z^7}{(z+3\zeta z^3)^2} \prec\prec \frac{z}{\zeta+z}$ *implies* $\zeta \ln\frac{\zeta}{\zeta - z} \prec\prec \frac{z+9\zeta z^2}{1+6\zeta z^2} \prec\prec \frac{\zeta+z}{\zeta},\ z \in U,\ \zeta \in \overline{U}$.

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J. COMPUTATIONAL ANALYSIS AND APPLICATIONS, VOL. 15, NO.8, 1496-1501, 2013, COPYRIGHT 2013 EUDOXUS PRESS, LLC

Strong differential subordinations and superordinations and sandwich theorem

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Abstract

In this paper we study certain strong differential subordinations and strong differential superordinations, obtained by using a new integral operator introduced in [21]. We also give some results as a sandwich theorem.

Keywords. Analytic function, univalent function, starlike function, convex function, strong differential subordination, strong differential superordination, best dominant, best subordinant.
2000 Mathematical Subject Classification: 30C80, 30C20, 30C40, 34C40.

1 Introduction and preliminaries

The concept of differential subordination was introduced in [11], [12] and developed in [13], by S.S. Miller and P.T. Mocanu. The concept of differential superordination was introduced in [14], [15] like a dual problem of the differential superordination by S.S. Miller and P.T. Mocanu. The concept of strong differential subordination was introduced in [10] by J.A. Antonino and S. Romaguera and developed in [1], [2], [3], [4], [5], [16], [18], [19], [20], [22], [24]. The concept of strong differential superordination was introduced in [17], like a dual concept of the strong differential subordination and developed in [6], [7], [8], [9], [21], [23].

In [16] the author defines the following classes:

Let $\mathcal{H}(U\times\overline{U})$ denote the class of analytic function in $U\times\overline{U}$, $U=\{z\in\mathbb{C}:|z|<1\}$, $\overline{U}=\{z\in\mathbb{C}:|z|\leq 1\}$, $\partial U=\{z\in\mathbb{C}:|z|=1\}$.

For $a\in\mathbb{C}$ and $n\in\mathbb{N}^*$, let $\mathcal{H}\zeta[a,n]=\{f(z,\zeta)\in\mathcal{H}(U\times\overline{U}):f(z,\zeta)=a+a_n(\zeta)\,z^n+\ldots+a_{n+1}(\zeta)\,z^{n+1}+\ldots\}$ with $z\in U$, $\zeta\in\overline{U}$, $a_k(\zeta)$ holomorphic functions in $\overline{U}$, $k\geq n$, $A\zeta_n=\{f(z,\zeta)\in\mathcal{H}(U\times\overline{U}):\ f(z,\zeta)=z+a_{n+1}(\zeta)\,z^{n+1}+a_{n+2}(\zeta)\,z^{n+2}+\ldots\}$ with $z\in U$, $\zeta\in\overline{U}$, $a_k(\zeta)$ holomorphic functions in $\overline{U}$, $k\geq n+1$, so $A\zeta_1=A\zeta$, $\mathcal{H}\zeta_u(U)=\{f(z,\zeta)\in\mathcal{H}\zeta[a,n]:\ f(z,\zeta)$ univalent in U, for all $\zeta\in\overline{U}\}$, $S\zeta=\{f(z,\zeta)\in A\zeta,\ f(z,\zeta)$ univalent in U, for all $\zeta\in\overline{U}\}$, denote the class of univalent functions in $U\times\overline{U}$, $S^*\zeta=\{f(z,\zeta)\in A\zeta:\ \mathrm{Re}\,\frac{zf'(z,\zeta)}{f(z,\zeta)}>0,\ z\in U$, for all $\zeta\in\overline{U}\}$, denote the class of normalized starlike functions in $U\times\overline{U}$, $K\zeta=\{f(z,\zeta)\in A\zeta:\ \mathrm{Re}\left[\frac{zf''(z,\zeta)}{f'(z,\zeta)}+1\right]>0,\ z\in U$, for all $\zeta\in\overline{U}\}$, denote the class of normalized convex functions in $U\times\overline{U}$.

For $r\in\mathbb{N}$, $A(r)\zeta$ denote the subclass of the functions $f(z,\zeta)\in(U\times\overline{U})$ of the form $f(z,\zeta)=z^r+\sum_{k=r+1}^{\infty}a_k(\zeta)z^k,\ r\in\mathbb{N},\ z\in U,\ \zeta\in\overline{U}$ and set $A(1)\zeta=A\zeta$. To prove our main results, we need the following definitions and lemmas:

Definition 1.1 *[16], [18] Let $f(z,\zeta)$ and $F(z,\zeta)$ analytic functions from $\mathcal{H}(U\times\overline{U})$. The function $f(z,\zeta)$ is said to be strongly subordinated to $F(z,\zeta)$, or $F(z,\zeta)$ is said to be strongly superordinated to $f(z,\zeta)$, if there exists a function w analytic in $\overline{U}$ with $w(0)=0$ and $|w(z)|<1$, such that $f(z,\zeta)=F(w(z),\zeta)$. In such a case we write $f(z,\zeta)\prec\prec F(z,\zeta)$.*

If $F(z,\zeta)$ is univalent then $f(z,\zeta)\prec\prec F(z,\zeta)$ if and only if $f(0,\zeta)=F(0,\zeta)$ and $f(U\times\overline{U})\subset F(U\times\overline{U})$.

If $f(z,\zeta)\equiv f(z)$ and $F(z,\zeta)\equiv F(z)$, then the strong differential subordination, or strong differential superordination, becomes the usual notion of differential subordination or differential superordination.

Definition 1.2 *[14], [16] We denote by Q_ζ the set of functions $q(z,\zeta)$ that are analytic and injective, with respect to z on $\overline{U}\setminus E(q(z,\zeta))$, where $E(q(z,\zeta))=\left\{\xi\in\partial U:\ \lim_{z\to\xi} q(z,\zeta)=\infty\right\}$ and are such that $q'(\xi,\zeta)\neq 0$, for $\xi\in\partial U\setminus E(q(z,\zeta))$. The class of Q_ζ for which $q(0,\zeta)=a$, is denoted by $Q_\zeta(a)$.*

We mention that all the derivatives which appear in this paper are considered with respect to variable z. We shall not indicate that in the paper due to the complexity of the writing.

Let $\psi:\mathbb{C}^3\times U\times\overline{U}\to\mathbb{C}$ and let $h(z,\zeta)$ be univalent in U, for all $\zeta\in\overline{U}$. If $p(z,\zeta)$ is analytic in $U\times\overline{U}$ and satisfies the (second-order) strong differential subordination

$$\psi(p(z,\zeta),z'(z,\zeta),z^2p''(z,\zeta);z,\zeta)\prec\prec h(z,\zeta),\ z\in U,\ \zeta\in\overline{U} \tag{1.1}$$

then $p(z,\zeta)$ is called a solution of the strong differential subordination.

The univalent function $q(z,\zeta)$ is called a dominant of the solutions of the strong differential subordination or simply a dominant, if $p(z,\zeta)\prec\prec q(z,\zeta)$ for all $p(z,\zeta)$ satisfying (1.1).

A dominant $\widetilde{q}(z,\zeta)$ that satisfies $\widetilde{q}(z,\zeta)\prec\prec q(z,\zeta)$ for all dominants $q(z,\zeta)$ of (1.1) is said to be the best dominant of (1.1). (Note that the best dominant is unique up to a rotation of U).

Let $\varphi:\mathbb{C}^3\times U\times\overline{U}\to\mathbb{C}$ and let $h(z,\zeta)$ be analytic in $U\times\overline{U}$. If $p(z,\zeta)$ and $\varphi(p(z,\zeta),zp'(z,\zeta),z^2p''(z,\zeta);z,\zeta)$ are univalent in U, for all $\zeta\in\overline{U}$ and satisfy the (second-order) strong differential superordination

$$h(z,\zeta)\prec\prec\varphi(p(z,\zeta),zp'(z,\zeta),z^2p''(z,\zeta);z,\zeta) \tag{1.2}$$

then $p(z,\zeta)$ is called a solution of the strong differential superordination. An analytic function $q(z,\zeta)$ is called a subordinant of the solutions of the differential superordination, or more simply a subordinant, if $q(z,\zeta)\prec\prec p(z,\zeta)$ for all $p(z,\zeta)$ satisfying (1.2). A univalent subordinant $\widetilde{q}(z,\zeta)$ that satisfies $q(z,\zeta)\prec\prec\widetilde{q}(z,\zeta)$ for all subordinants of (1.2) is said to be the best subordinant. (Note that the best subordinant is unique up to a rotation of U).

Definition 1.3 *[20] For $f(z,\zeta)\in A\zeta_n$, $n\in\mathbb{N}^*$, $m\in\mathbb{N}$, $\gamma\in\mathbb{C}$, let L_γ be the integral operator given by $L_\gamma: A\zeta_n\to A\zeta_n$*

$$\begin{aligned}
L_\gamma^0 f(z,\zeta)&=f(z,\zeta)\\
L_\gamma^1 f(z,\zeta)&=\frac{\gamma+1}{z^\gamma}\int_0^z L_\gamma^0 f(z,\zeta)t^{\gamma-1}dt\\
L_\gamma^2 f(z,\zeta)&=\frac{\gamma+1}{z^\gamma}\int_0^z L_\gamma^1 f(z,\zeta)t^{\gamma-1}dt,...\\
L_\gamma^m f(z,\zeta)&=\frac{\gamma+1}{z^\gamma}\int_0^z L_\gamma^{m-1} f(z,\zeta)t^{\gamma-1}dt.
\end{aligned}$$

By using Definition 1.3, we can prove the following properties for this integral operator:

For $f(z,\zeta)\in A\zeta_n$, $n\in\mathbb{N}^*$, $m\in\mathbb{N}$, $\gamma\in\mathbb{C}$, we have

$$L_\gamma^m f(z,\zeta)=z+\sum_{k=n+1}^{\infty}\frac{(\gamma+1)^m}{(\gamma+k)^m}a_k(\zeta)z^k,\ z\in U,\ \zeta\in\overline{U}, \tag{1.3}$$

and

$$z[L_\gamma^m f(z,\zeta)]'_z=(\gamma+1)L_\gamma^{m-1}f(z,\zeta)-\gamma L_\lambda^m f(z,\zeta),\ z\in U,\ \zeta\in\overline{U}. \tag{1.4}$$

Definition 1.4 *[20] For $r\in\mathbb{N}$, $f(z,\zeta)\in A(r)\zeta$, let H be the integral operator given by $H: A(r)\zeta\to A(r)\zeta$*

$$\begin{aligned}
H^0 f(z,\zeta)&=f(z,\zeta)\\
H^1 f(z,\zeta)&=\frac{r+1}{z}\int_0^z H^0 f(t,\zeta)dt\\
H^2 f(z,\zeta)&=\frac{r+1}{z}\int_0^z H^1 f(t,\zeta)dt,...\\
H^m f(z,\zeta)&=\frac{r+1}{z}\int_0^z H^{m-1} f(t,\zeta)dt,\ z\in U,\ \zeta\in\overline{U}.
\end{aligned}$$

From Definition 1.4 we have

$$H^m f(z,\zeta) = z^r + \sum_{k=p+1}^{\infty} \frac{(r+1)^m}{(r+k)^m} a_k(\zeta) z^k \tag{1.5}$$

and

$$z[H^m f(z,\zeta)]'_z = (r+1)H^{m-1} f(z,\zeta) - H^m f(z,\zeta),\ z \in U,\ \zeta \in \overline{U}. \tag{1.6}$$

Lemma 1.1 *[15, Corollary 3.1] Let $\beta, \gamma \in \mathbb{C}$, and $q(z,\zeta)$ univalent in U, for all $\zeta \in \overline{U}$, with $q(0,\zeta) = a$. Let $h(z,\zeta) = q(z,\zeta) + \frac{zq'(z,\zeta)}{\beta q(z,\zeta)+\gamma}$ and suppose that*

(i) Re $[\beta q(z,\zeta) + \gamma] > 0$, and

(ii) $\frac{zq'(z,\zeta)}{\beta q(z,\zeta)+\gamma}$ is starlike.

If $p(z,\zeta) \in \zeta[a,1] \cap Q_\zeta$ and $p(z,\zeta) + \frac{zp'(z,\zeta)}{p(z,\zeta)+1}$ is univalent in U, for all $\zeta \in \overline{U}$, then $h(z,\zeta) \prec\prec p(z,\zeta) + \frac{zp'(z,\zeta)}{\beta p(z,\zeta)+\gamma}$ implies $q(z,\zeta) \prec\prec p(z,\zeta)$ and $q(z,\zeta)$ is the best subordinant.

Lemma 1.2 *[13, Theorem 3.2b, p.83] Let $h(z,\zeta)$ be convex in U, for all $\zeta \in \overline{U}$, and n a positive integer. Suppose that the differential equation $q(z,\zeta) + \frac{nzq'(z,\zeta)}{\beta q(z,\zeta)+\gamma} = h(z,\zeta)$ has an univalent solution $q(z,\zeta)$ that satisfies $q(z,\zeta) \prec\prec h(z,\zeta)$.*

If $p(z,\zeta) \in \zeta[a,n]$ satisfies $p(z,\zeta) + \frac{zp'(z,\zeta)}{\beta p(z,\zeta)+\gamma} \prec\prec h(z,\zeta)$, then $p(z,\zeta) \prec\prec q(z,\zeta)$ and $q(z,\zeta)$ is the best dominant.

Lemma 1.3 *[14, Corollary 6.1] Let $h_1(z,\zeta)$ and $h_2(z,\zeta)$ be convex in U, for all $\zeta \in \overline{U}$, with $h_1(0,\zeta) = h_2(0,\zeta) = a$. Let $\gamma \in \mathbb{C}$, $\gamma \neq 0$, with Re $\gamma \geq 0$, and the functions $q_i(z,\zeta)$ be defined by $q_i(z,\zeta) = \frac{\gamma}{z^\gamma}\int_0^z h_i(t,\zeta)t^{\gamma-1}dt$ for $i = 1,2$.*

If $p(z,\zeta) \in [a,1] \cap Q_\zeta$ and $p(z,\zeta) + \frac{zp'(z,\zeta)}{\gamma}$ is univalent, then $h_1(z,\zeta) \prec\prec p(z,\zeta) + \frac{zp'(z,\zeta)}{\gamma} \prec\prec h_2(z,\zeta)$ implies $q_1(z,\zeta) \prec\prec p(z,\zeta) \prec\prec q_2(z,\zeta)$, $z \in U$, $\zeta \in \overline{U}$.

The functions $q_1(z,\zeta)$ and $q_2(z,\zeta)$ are convex and they are respectively the best subordinant and best dominant.

2 Main results

Theorem 2.1 *Let $\gamma \in \mathbb{C}$, with $\mathrm{Re}\,\gamma \geq 0$, and $q(z,\zeta) = \frac{1+z\zeta}{1-z\zeta}$ be univalent in U, for all $\zeta \in \overline{U}$, with $q(0,\zeta) = 1$. Let*

$$h(z,\zeta) = \frac{1+z\zeta}{1-z\zeta} + \frac{\frac{z\zeta}{(1-z\zeta)^2}}{\frac{1+z\zeta}{1-z\zeta}+1} = \frac{1+z\zeta}{1-z\zeta} + \frac{z\zeta}{2(1-z\zeta)} = \frac{2+3z\zeta}{2(1-z\zeta)} \tag{2.1}$$

with

$$\mathrm{Re}\left(1 + \frac{1+z\zeta}{1-z\zeta}\right) = \mathrm{Re}\,\frac{2}{1-z\zeta} > 0 \tag{2.2}$$

and

$$r(z,t) = \frac{zq'(z,\zeta)}{q(z,\zeta)+1} = \frac{z\zeta}{1-z\zeta} \tag{2.3}$$

starlike in U, for all $\zeta \in \overline{U}$.

If $\frac{L_\gamma^m f(z,\zeta)}{z[L_\gamma^m f(z,\zeta)]'} \in [1,1] \cap Q_\zeta$ and $\frac{L_\gamma^m f(z,\zeta)}{z[L_\gamma^m f(z,\zeta)]'} + \frac{[L_\gamma^m f(z,\zeta)]'}{L_\gamma^m f(z,\zeta)} - \frac{[L_\gamma^m f(z,\zeta)]''}{[L_\gamma^m f(z,\zeta)]'} - 1$ is univalent in U, for all $\zeta \in \overline{U}$, then

$$\frac{2+3z\zeta}{2(1-z\zeta)} \prec\prec \frac{L_\gamma^m f(z,\zeta)}{z[L_\gamma^m f(z,\zeta)]'} + \frac{[L_\gamma^m f(z,\zeta)]'}{L_\gamma^m f(z,\zeta)} - \frac{[L_\gamma^m f(z,\zeta)]''}{[L_\gamma^m f(z,\zeta)]'} - 1 \tag{2.4}$$

implies $\frac{1+z\zeta}{1-z\zeta} \prec\prec \frac{L_\gamma^m f(z,\zeta)}{z[L_\gamma^m f(z,\zeta)]'}$, $z \in U$, $\zeta \in \overline{U}$ and $q(z,\zeta) = \frac{1+z\zeta}{1-z\zeta}$ is the best dominant.

Proof. In order to prove the theorem, we shall use Lemma 1.1. For that, we show that the necessary conditions are satisfied.

Let the functions $\theta : \mathbb{C} \to \mathbb{C}$ and $\phi : \mathbb{C} \to \mathbb{C}$ with

$$\theta(w) = w \tag{2.5}$$

and

$$\phi(w) = \frac{1}{w+1}, \quad \phi(w) \neq 0. \tag{2.6}$$

We check the conditions from the hypothesis of Lemma 1.1. For $\beta = 1$, $\gamma = 1$, we have $\mathrm{Re}\,[1 \cdot q(z,\zeta) + 1] = \mathrm{Re}\left(\frac{1+z\zeta}{1-z\zeta} + 1\right) = \mathrm{Re}\,\frac{2}{1-z\zeta} > 0$, hence condition (i) is satisfied.

Let $r(z,\zeta) = \frac{zq'(z,\zeta)}{1 \cdot q(z,\zeta)+1} = \frac{\frac{2z\zeta}{(1-z\zeta)^2}}{\frac{2}{1-z\zeta}} = \frac{z\zeta}{1-z\zeta}$. We have $\mathrm{Re}\,\frac{zr'(z,\zeta)}{r(z,\zeta)} = \mathrm{Re}\,\frac{\frac{z\zeta}{(1-z\zeta)^2}}{\frac{z\zeta}{1-z\zeta}} = \mathrm{Re}\,\frac{1}{1-z\zeta} > 0$, hence condition (ii) is satisfied.

We consider

$$p(z,\zeta) = \frac{L_\gamma^m f(z,\zeta)}{z[L_\gamma^m f(z,\zeta)]'}, \; z \in U, \; \zeta \in \overline{U}. \tag{2.7}$$

Using (1.3) in (2.7), we obtain

$$p(z,\zeta) = \frac{z + \sum_{k=n+1}^{\infty} \frac{(\gamma+1)^m}{(\gamma+k)^m} a_k(\zeta) z^k}{z\left(1 + \sum_{k=n+1}^{\infty} \frac{(\gamma+1)^m}{(\gamma+k)^m} a_k(\zeta) k z^{k-1}\right)} = \frac{1 + \sum_{k=n+1}^{\infty} \frac{(\gamma+1)^m}{(\gamma+k)^m} a_k(\zeta) z^{k-1}}{1 + \sum_{k=n+1}^{\infty} \frac{(\gamma+1)^m}{(\gamma+k)^m} a_k(\zeta) k z^{k-1}}. \tag{2.8}$$

Since $p(0,\zeta) = 1$, we have $p(z,\zeta) \in [1,1] \cap Q_\zeta$. Differentiating (2.7) and after a short calculus we obtain

$$p(z,\zeta) + \frac{zp'(z,\zeta)}{p(z,\zeta)+1} = \frac{L_\gamma^m f(z,\zeta)}{z[L_\gamma^m f(z,\zeta)]'} + \frac{[L_\gamma^m f(z,\zeta)]'}{L_\gamma^m f(z,\zeta)} - \frac{[L_\gamma^m f(z,\zeta)]''}{[L_\gamma^m f(z,\zeta)]'} - 1. \tag{2.9}$$

Using (2.9) in (2.4), the strong differential superordination becomes $\frac{2+3z\zeta}{2(1-z\zeta)} \prec\prec p(z,\zeta) + \frac{zp'(z,\zeta)}{p(z,\zeta)+1}$.

From Lemma 1.1, we have $q(z,\zeta) \prec\prec p(z,\zeta)$, i.e., $\frac{1+z\zeta}{1-z\zeta} \prec\prec \frac{L_\gamma^m f(z,\zeta)}{z[L_\gamma^m f(z,\zeta)]'}$, $z \in U$, $\zeta \in \overline{U}$ and $q(z,\zeta) = \frac{1+z\zeta}{1-z\zeta}$ is the best subordinant. ■

Theorem 2.2 *Let $h(z,\zeta) = \frac{\zeta-3z}{\zeta+z}$, be a convex function in U, for all $\zeta \in \overline{U}$, with $h(0) = 1$. Suppose that the Briot-Bouquet differential equation*

$$q(z,\zeta) + \frac{zq'(z,\zeta)}{q(z,\zeta)+1} = \frac{\zeta-3z}{\zeta+z} \tag{2.10}$$

has an univalent solution $q(z,\zeta) = \frac{\zeta-z}{\zeta+z}$, that satisfies $\frac{\zeta-z}{\zeta+z} \prec\prec \frac{\zeta-3z}{\zeta+z}$.

If $p(z,\zeta) = \frac{H^m f(z,\zeta)}{z^r} \in [1,1] \cap Q_\zeta$ satisfies

$$\frac{H^m f(z,\zeta)}{z^r} + \frac{z^{r+1}[H^m f(z,\zeta)]'}{[H^m f(z,\zeta)]^2} - \frac{rz^r}{H^m f(z,\zeta)} \prec\prec \frac{\zeta-3z}{\zeta+z} \tag{2.11}$$

then $\frac{H^m f(z,\zeta)}{z^r} \prec\prec \frac{\zeta-z}{\zeta+z}$, $z \in U$, $\zeta \in \overline{U}$ and $q(z,\zeta) = \frac{\zeta-z}{\zeta+z}$ is the best dominant.

Proof. In order to prove the theorem, we shall use Lemma 1.2. For that, we show that the necessary conditions are satisfied.

After a short calculus we obtain

$$\mathrm{Re}\left[1 + \frac{zh''(z,\zeta)}{h'(z,\zeta)}\right] = \mathrm{Re}\left(\frac{\zeta-z}{\zeta+z}\right) \geq 0, \; z \in U, \; \zeta \in \overline{U}. \tag{2.12}$$

The function $q(z,\zeta) = \frac{\zeta-z}{\zeta+z}$ is the univalent solution of equation (2.10), hence

$$\mathrm{Re}\left[1 + \frac{zq''(z,\zeta)}{q'(z,\zeta)}\right] = \mathrm{Re}\left[1 - \frac{2z}{\zeta+z}\right] \geq 0. \tag{2.13}$$

We consider

$$p(z,\zeta) = \frac{H^m f(z,\zeta)}{z^r}. \tag{2.14}$$

Using (1.5) în (2.14), we obtain $p(z,\zeta) = \frac{z^r + \sum_{k=r+1}^{\infty} \frac{(r+1)^m}{(r+k)^m} a_k(\zeta) z^k}{z^r} = 1 + \sum_{k=r+1}^{\infty} \frac{(r+1)^m}{(r+k)^m} a_k(\zeta) z^{k-r}$. Since $p(0,\zeta) = 1$, we have $p(z,\zeta) \in \zeta[1,1] \cap Q_\zeta$. Differentiating (2.14) and after a short calculus we obtain

$$p(z,\zeta) + \frac{zp'(z,\zeta)}{p(z,\zeta)+1} = \frac{H^m f(z,\zeta)}{z^r} + \frac{z^{r+1}[H^m f(z,\zeta)]'}{[H^m f(z,\zeta)]^2} - \frac{rz^r}{H^m f(z,\zeta)}. \tag{2.15}$$

Using (2.15) in (2.11), the strong differential superordination becomes

$$p(z,\zeta)+\frac{zp'(z,\zeta)}{p(z,\zeta)+1}\prec\prec q(z,\zeta)+\frac{zq'(z,\zeta)}{q(z,\zeta)+1}. \tag{2.16}$$

From Lemma 1.1, we have $\frac{H^m f(z,\zeta)}{z^r}\prec\prec\frac{\zeta-z}{\zeta+z},\ z\in U,\ \zeta\in\overline{U}$ and $q(z,\zeta)=\frac{\zeta-z}{\zeta+z}$ is the best dominant. ■

Theorem 2.3 *Let* $h_1(z,\zeta)=\frac{1-z\zeta}{1+z\zeta}$ *and* $h_2(z,\zeta)=1+\frac{z^2}{\zeta}$ *be convex in* U, *for all* $\zeta\in\overline{U}$, *with* $h_1(0,\zeta)=h_2(0,\zeta)=1$. *Let* $\gamma\in\mathbb{C}$, $\lambda\neq 0$, *with* $\operatorname{Re}\gamma\geq 0$, *and the functions defined by* $q_1(z,\zeta)=-1+\frac{2\gamma\zeta}{z^\gamma}\cdot\sigma_1(z,\zeta)$, *where* $\sigma_1(z,\zeta)$ *is given by*

$$\sigma_1(z,\zeta)=\int_0^z\frac{t^{\gamma-1}}{1+t\zeta}dt \tag{2.17}$$

and $q_2(z,\zeta)=1+\frac{\gamma}{\gamma+2}\cdot\frac{z^2}{\zeta},\ z\in U,\ \zeta\in\overline{U}$.
If $f(z,\zeta)\in A\zeta(r)$, $\frac{H^m f(z,\zeta)[H^m f(z,\zeta)]'}{rz^{2r-1}}\in[1,1]\cap Q_\zeta$, *and* $\frac{H^m f(z,\zeta)(H^m f(z,\zeta))'}{\gamma rz^{2r-1}}+\frac{[(H^m f(z,\zeta))']^2}{\gamma rz^{2r-2}}+\frac{H^m f(z,\zeta)(H^m f(z,\zeta))''}{\gamma rz^{2r-2}}-\frac{(2r-1)H^m f(z,\zeta)(H^m f(z,\zeta))'}{\gamma rz^{2r-1}}$ *is univalent in* U, *for all* $\zeta\in\overline{U}$, *then*

$$\frac{1-z\zeta}{1+z\zeta}\prec\prec\frac{(2-2r)H^m f(z,\zeta)(H^m f(z,\zeta))'}{\gamma rz^{2r-1}}+\frac{[(H^m f(z,\zeta))']^2+H^m f(z,\zeta)(H^m f(z,\zeta))''}{\gamma rz^{2r-2}}\prec\prec 1+\frac{z^2}{\zeta}, \tag{2.18}$$

implies $-1+\frac{2\gamma\zeta}{z^\gamma}\sigma_1(z,\zeta)\prec\prec\frac{H^m f(z,\zeta)(H^m f(z,\zeta))'}{rz^{2r-1}}\prec\prec 1+\frac{\gamma}{\gamma+2}\cdot\frac{z^2}{\zeta}$, *where* $\sigma_1(z,\zeta)$, *given by (2.17),* $z\in U,\ \zeta\in\overline{U}$.

The functions $q_1(z,\zeta)=-1+\frac{2\gamma\zeta}{z^\gamma}\sigma_1(z,\zeta)$ *and* $q_2(z,\zeta)=1+\frac{\gamma}{\gamma+2}\cdot\frac{z^2}{\zeta}$ *are convex and they are respectively the best subordinant and best dominant.*

Proof. In order to prove the theorem, we shall use Lemma 1.3. For that, we show that the necessary conditions are satisfied. $\operatorname{Re}\left[1+\frac{zh_1''(z,\zeta)}{h_1'(z,\zeta)}\right]=\operatorname{Re}\frac{1-z\zeta}{1+z\zeta}\geq 0,\ z\in U,\ \zeta\in\overline{U}$ and $\operatorname{Re}\left[1+\frac{zh_2''(z,\zeta)}{h_1'(z,\zeta)}\right]=\operatorname{Re}2\geq 0,\ z\in U,\ \zeta\in\overline{U}$ we put

$$p(z,\zeta)=\frac{H^m f(z,\zeta)(H^m f(z,\zeta))'}{rz^{2r-1}},\ z\in U,\ \zeta\in\overline{U}. \tag{2.19}$$

Using (1.5) in (2.14), we obtain $p(z,\zeta)=\frac{\left[z^r+\sum_{k=r+1}^\infty\frac{(\gamma+1)^m}{(\gamma+k)^m}a_k(\zeta)z^k\right]\left[rz^{r-1}+\sum_{k=r+1}^\infty\frac{(\gamma+1)^m}{(\gamma+k)^m}a_k(\zeta)kz^{k-1}\right]}{rz^{2r-1}}=\left[1+\sum_{k=r+1}^\infty\frac{(\gamma+1)^m}{(\gamma+k)^m}a_k(\zeta)z^{k-r}\right]\left[r+\sum_{k=r+1}^\infty\frac{(\gamma+1)^m}{(\gamma+k)^m}a_k(\zeta)kz^{k-r}\right]$.

Since $p(0,\zeta)=1$, we have $p(z,\zeta)\in\zeta[1,1]\cap Q_\zeta$. Differentiating (2.14), and after a short calculus we obtain

$$p(z,\zeta)+\frac{zp'(z,\zeta)}{\gamma}=\frac{(2-2r)H^m f(z,\zeta)(H^m f(z,\zeta))'}{\gamma rz^{2r-1}}+\frac{[(H^m f(z,\zeta))']^2+H^m f(z,\zeta)(H^m f(z,\zeta))''}{\gamma rz^{2r-2}}. \tag{2.20}$$

Using (2.20) in (2.18), we have

$$\frac{1-z\zeta}{1+z\zeta}\prec\prec p(z,\zeta)+\frac{zp'(z,\zeta)}{\gamma}\prec\prec 1+\frac{z^2}{\zeta},\ z\in U,\ \zeta\in\overline{U}. \tag{2.21}$$

Using Lemma 1.3, we have $-1+\frac{2\gamma\zeta}{z^\gamma}\sigma_1(z,\zeta)\prec\prec\frac{H^m f(z,\zeta)(H^m f(z,\zeta))'}{rz^{2r-1}}\prec\prec 1+\frac{\gamma}{\gamma+2}\cdot\frac{z^2}{\zeta}$. ■

Example 2.1 *Let* $\gamma=1,\ m=1,\ r=3,\ f(z,\zeta)=x^3+x^4\zeta,\ H^1(z,\zeta)=\frac{2}{z}\int_0^z(t^3+t^4\zeta)dt=\frac{1}{4}z^3+\frac{2\zeta}{5}z^4$, $p(z,\zeta)=\frac{1}{16}+\frac{7\zeta}{30}z+\frac{16\zeta^2}{75}z^2$, $p(z,\zeta)+zp'(z,\zeta)=\frac{1}{6}+\frac{7\zeta}{15}z+\frac{16\zeta^2}{25}z^2$, $q_1(z,\zeta)=-1+\frac{\ln(1+z\zeta)}{z}$, $q_2(z,\zeta)=1+\frac{z^2}{3\zeta}$.

From Theorem 2.3, we have $\frac{1-z\zeta}{1+z\zeta}\prec\prec\frac{1}{6}+\frac{7\zeta}{15}z+\frac{16\zeta^2}{25}z^2\prec\prec 1+\frac{z^2}{3\zeta}$ *implies* $-1+\frac{\ln(1+z\zeta)}{z}\prec\prec\frac{1}{4}z^3+\frac{2\zeta}{5}z^4\prec\prec 1+\frac{z^2}{3\zeta},\ z\in U,\ \zeta\in\overline{U}$.

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