On additive properties of the Drazin inverse of block matrices and representations

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Abstract

In this paper, we give a new additive formula for the Drazin inverse under conditions weaker than those used in some current literature on this subject. Also, we obtain representations for the Drazin inverse of a complex block matrix having generalized Schur complement equal to zero.

2000 Mathematics Subject Classification: 15A09
Key words: Drazin inverse; block matrix; additive formula

1 Introduction

Let \( \mathbb{C}^{n \times n} \) denote the set of all \( n \times n \) complex matrices and let \( A \in \mathbb{C}^{n \times n} \). By \( \mathcal{R}(A), \mathcal{N}(A) \) and \( \text{rank}(A) \) we denote the range, the null space and the rank of matrix \( A \), respectively. The smallest nonnegative integer \( k \) such that \( \text{rank}(A^k) = \text{rank}(A^{k+1}) \), denoted by \( \text{ind}(A) \), is called the index of matrix \( A \). If \( \text{ind}(A) = k \), then there exists the unique matrix \( A^d \in \mathbb{C}^{n \times n} \), which satisfies the following relations:

\[
A^{k+1} A^d = A^k, \quad A^d AA^d = A^d, \quad AA^d = A^d A.
\]

The matrix \( A^d \) is called the Drazin inverse of \( A \) (see [1]). If \( \text{ind}(A) = 1 \), then the Drazin inverse of \( A \) is called the group inverse of \( A \) and it is denoted by \( A^\# \).

Clearly, \( \text{ind}(A) = 0 \) if and only if \( A \) is nonsingular, and in that case \( A^d = A^{-1} \).

In this paper we use notation \( A^\pi = I - AA^d \) to denote the projection on \( \mathcal{N}(A^k) \) along \( \mathcal{R}(A^k) \).

The Drazin inverse of square complex matrices has applications in several areas, such as differential and difference equations, Markov chains and iterative methods (see [2, 3, 4, 5, 6, 7]). For applications of the Drazin inverse of a \( 2 \times 2 \) block matrix we refer readers to [2, 8, 9].

*Supported by Grant No. 174007 of the Ministry of Education, Science and Technological Development, Republic of Serbia
Suppose $P, Q \in \mathbb{C}^{n \times n}$. In 1958, Drazin (see [10]) studied the problem of finding the formula for $(P + Q)^d$ and he offered the formula $(P + Q)^d = P^d + Q^d$, which is valid when $PQ = QP = 0$. In the present, there is no formula for $(P + Q)^d$ without any side condition for matrices $P$ and $Q$, so this problem remains open. However, many authors have considered this problem and provided a formula for $(P + Q)^d$ with some specific conditions for matrices $P$ and $Q$. Some of them are as follows:

(i) $PQ = 0$ [6];
(ii) $Q^2 = 0$ and $P^2Q = 0$ [11];
(iii) $PQ^2 = 0$ and $P^2Q = 0$ [12];
(iv) $PQ^2 = 0$ and $PQP = 0$ [13].

In this paper we derive a formula for $(P + Q)^d$ under conditions $P^2QP = 0$, $P^2Q^2 = 0$, $PQ^2P = 0$ and $PQ^3 = 0$ which are weaker than those from previous list.

Another aim–objective of this paper is to derive a representation of the Drazin inverse of $2 \times 2$ complex block matrix

$$M = \begin{bmatrix} A & B \\ C & D \end{bmatrix},$$

where $A$ and $D$ are square matrices, not necessarily of the same size. This problem was firstly posed in 1979 by Campbell and Meyer [3]. According to current literature, there has been no formula for $M^d$ without any side conditions for blocks of matrix $M$. Special cases of this open problem have been considered, so at present time there are many formulas for $M^d$ under specific conditions for blocks of $M$. In some papers the expression of $M^d$ is given under conditions which concern the generalized Schur complement of matrix $M$ defined by $S = D - CA^\pi B$. Here we list some of them:

(i) $CA^\pi = 0$, $A^\pi B = 0$ and $S = 0$ [14];
(ii) $CA^\pi B = 0$, $AA^\pi B = 0$ and $S = 0$ [9];
(iii) $CA^\pi B = 0$, $CA^\pi A = 0$ and $S = 0$ [9];
(iv) $CA^\pi BC = 0$, $AA^\pi BC = 0$ and $S = 0$ [13];
(v) $BCA^\pi B = 0$, $BCA^\pi A = 0$ and $S = 0$ [13];
(vi) $ABCA^\pi = 0$, $BCA^\pi$ is nilpotent and $S = 0$ [11];
(vii) $A^\pi BCA = 0$, $A^\pi BC$ is nilpotent and $S = 0$ [11];
(viii) $ABCA^\pi = 0$, $A^\pi ABC = 0$ and $S = 0$ [15];
(ix) $ABCA^\pi = 0$, $CBCA^\pi = 0$ and $S = 0$ [15].
In this paper we derive some new representations for $M^d$, which generalizes representations given under conditions from previous list.

Before we give our main results, we state some auxiliary lemmas as follows.

**Lemma 1.1** [1] Let $A \in \mathbb{C}^{m \times n}$, $B \in \mathbb{C}^{n \times m}$. Then $(AB)^d = A((BA)^2)^d B$.

**Lemma 1.2** [6] Let $P,Q \in \mathbb{C}^{n \times n}$ be such that $\text{ind}(P) = r$ and $\text{ind}(Q) = s$. If $PQ = 0$ then

$$(P + Q)^d = \sum_{i=0}^{s-1} Q^{-i} (P^d)^{i+1} + \sum_{i=0}^{r-1} (Q^d)^{i+1} P^i P^s.$$ 

**Lemma 1.3** [14] Let $M$ be a matrix of the form (1.1), such that $S = 0$. If $A^7 B = 0$ and $CA = 0$, then

$$M^d = \left[ \begin{array}{c} I \\ CA^d \end{array} \right] ((AW)^d)^2 A \left[ \begin{array}{c} I \\ A^d B \end{array} \right],$$

where $W = AA^d + A^d BCA^d$.

## 2 Additive results

In this section we investigate the Drazin inverse of the sum of two matrices. The following theorem is the main tool in our sequel development.

**Theorem 2.1** Let $P,Q \in \mathbb{C}^{n \times n}$. If $P^2 Q P = 0$, $P^2 Q^2 = 0$, $P Q^2 P = 0$ and $P Q^3 = 0$ then

$$(P + Q)^d = \sum_{i=0}^{\text{ind}(P+Q)Q-1} (P + Q)Q^i ((P + Q)Q)^i ((P + Q)Q)^{i+1}$$

$$+ \sum_{i=0}^{\text{ind}(P+Q)P-1} (((P + Q)Q)^d)^{i+1} ((P + Q)Q)^i ((P + Q)Q)^{i+1} (P + Q),$$

where for $n \in \mathbb{N}$

$$((P + Q)P)^d)^n = \sum_{i=0}^{\text{ind}(PQ) - 1} (Q)P^i (P)^2(2i+n) + \sum_{i=0}^{\text{ind}(P^2) - 1} (Q)^d)^{i+1} P^2i P^i$$

$$- \sum_{i=1}^{\text{ind}(Q)^2 - 1} (Q)^2i (P)^2(2i-n-i),$$

$$((P + Q)Q)^d)^n = \sum_{i=0}^{\text{ind}(PQ) - 1} Q^2i(Q)^d)^{i+1} + \sum_{i=0}^{\text{ind}(PQ) - 1} (Q)^2(2i+n)(P)^i(Q)^p$$

$$- \sum_{i=1}^{\text{ind}(Q)^2 - 1} (Q)^2i(Q)^d)^{i+1}.$$
Therefore we get

Furthermore, by Lemma 1.1 we have

Proof. Using Lemma 1.1, we have that \((P + Q)^d = (P + Q)((P + Q)^d)^2 = (P + Q)(P(P + Q) + Q(P + Q))^d\). Denote by \(F = P(P + Q)\) and \(G = Q(P + Q)\). Since \(FG = 0\), matrices \(F\) and \(G\) satisfy the condition of Lemma 1.2 and therefore

Furthermore, by Lemma 1.1 we have \(F^d = P(((P + Q)P^d)^2(P + Q)\) and \(G^d = Q(((P + Q)Q^d)^2(P + Q)\). If we denote by \(F_1 = (P + Q)P\) and \(G_1 = (P + Q)Q\), we get \(F^d = P(F_1^d)^2(P + Q)\) and \(G^d = Q(G_1^d)^2(P + Q)\). Moreover,

for every \(n \in \mathbb{N}\). After some computations we get

Notice that \(F_1 = P^2 + QP\). Since \(P^2QP = 0\), matrices \(P^2\) and \(QP\) satisfy condition of Lemma 1.2. After applying Lemma 1.2 we obtain

\[
(F_1^d)^n = \sum_{i=0}^{\text{ind}(Q^P) - 1} (QP)^i (P^d)^{2(i+n)} + \sum_{i=0}^{\text{ind}(P^2) - 1} ((QP)^d)^i P^{2i} P^\pi
- \sum_{i=1}^{n-1} ((QP)^d)^i (P^d)^{2(n-i)},
\]
for every \( n \in \mathbb{N} \). Similarly, from \( PQ^3 = 0 \) and Lemma 1.2, for every \( n \in \mathbb{N} \) we have

\[
(G_1^d)^n = \sum_{i=0}^{\text{ind}(Q^2)^{-1}} Q^\pi Q^{2(i+1)(PQ)^i(PQ)^\pi} + \sum_{i=0}^{\text{ind}(PQ)^{-1}} (Q^d)^{2(i+n+1)(PQ)^i(PQ)^\pi} - \sum_{i=1}^{n-1} (Q^d)^{2i}((PQ)^d)^{n-i}.
\] (2.3)

Substituting (2.2) and (2.3) into (2.1) we get that the statement of the theorem is valid. \( \Box \)

The next theorem is a symmetrical formulation of Theorem 2.1.

**Theorem 2.2** Let \( P, Q \in \mathbb{C}^{n \times n} \). If \( PQP^2 = 0, Q^2P^2 = 0, PQ^2P = 0 \) and \( Q^3P = 0 \) then

\[
(P + Q)^d = (P + Q) \left( \sum_{i=0}^{\text{ind}(Q(P+Q))^{-1}} ((P(P + Q))^{d})^{i+1}(Q(P + Q))^{i}(Q(P + Q))^\pi \right.
\]

\[
+ \sum_{i=0}^{\text{ind}(P(P+Q))^{-1}} (P(P + Q))^\pi(P(P + Q))^i((Q(P + Q))^{d})^{i+1} \right),
\]

where for \( n \in \mathbb{N} \)

\[
((P(P + Q))^{d})^n = \sum_{i=0}^{\text{ind}(PQ)^{-1}} (P^d)^{2(i+n+1)}(PQ)^i(PQ)^\pi + \sum_{i=0}^{\text{ind}(P^2)^{-1}} P^\pi P^{2i}((PQ)^d)^{i+n} - \sum_{i=1}^{n-1} (P^d)^{2(n-i)}((PQ)^d)^{i},
\]

\[
((Q(P + Q))^{d})^n = \sum_{i=0}^{\text{ind}(Q^2)^{-1}} ((Q^P)^d)^{i+n} Q^2 Q^\pi + \sum_{i=0}^{\text{ind}(QP)^{-1}} (Q^P)^\pi(QP)^i(Q^d)^{2(i+n)} - \sum_{i=1}^{n-1} ((QP)^d)^{n-i}(Q^d)^{2i},
\]

and

\[
(P(P + Q))^\pi = P^\pi(PQ)^\pi - \sum_{i=0}^{\text{ind}(P^2)^{-2}} (P^d)^{2(i+1)}(PQ)^i(PQ)^\pi - \sum_{i=0}^{\text{ind}(P^2)^{-2}} P^\pi P^{2(i+1)}((PQ)^d)^{i+1},
\]
\[(Q(P + Q))^\pi = (QP)^\pi Q^\pi - \sum_{i=0}^{\text{ind}(Q^2)-2} ((QP)^d)_{i+1}^1 Q^{2(i+1)}Q^\pi - \sum_{i=0}^{\text{ind}(QP)-2} (QP)^\pi (QP)^{i+1} (Q^d)^{2(i+1)}.\]

Notice that one special case of Theorem 2.1 is when matrices \(P\) and \(Q\) satisfy the conditions \(P^2QP = 0\) and \(PQ^2 = 0\). Similarly, a special case of Theorem 2.2 is when \(PQP^2 = 0\) and \(Q^2P = 0\) is valid. The following additive formulas are corollaries of these cases, respectively, which we will use in section 3 to obtain representations for the Drazin inverse of block matrix.

**Corollary 2.1** Let \(P, Q \in \mathbb{C}^{n \times n}\). If \(P^2QP = 0\) and \(Q^2 = 0\), then

\[(P + Q)^d = \sum_{i=0}^{r-1} (\left(\left((PQ)^d\right)^{i+1} + \left((QP)^d\right)^{i+1}\right) P^{2i} P^\pi + \sum_{i=0}^{s-1} (\left((PQ)^\pi\right)^i (PQ)^{i+1} (P^d)^{2(i+1)} - (P^d)^i) (P + Q),\]

where \(r = \text{ind}(P^2)\) and \(s = \max\{\text{ind}(PQ), \text{ind}(QP)\}\).

**Corollary 2.2** Let \(P, Q \in \mathbb{C}^{n \times n}\). If \(PQP^2 = 0\) and \(Q^2 = 0\), then

\[(P + Q)^d = (P + Q) \left(\sum_{i=0}^{r-1} P^\pi P^{2i} \left(\left((PQ)^d\right)^{i+1} + \left((QP)^d\right)^{i+1}\right) + \sum_{i=0}^{s-1} (P^d)^{2(i+1)} (\left((PQ)^\pi\right)^i (PQ)^{i+1} (P^d)^{i+1} - (P^d)^i)\right),\]

where \(r = \text{ind}(P^2)\) and \(s = \max\{\text{ind}(PQ), \text{ind}(QP)\}\).

### 3 Representations for the Drazin inverse of block matrix

Through this section we assume that matrix \(M\) is defined by (1.1), where \(A\) and \(D\) are square matrices and generalized Schur complement \(S = D - CA^d B\) of matrix \(M\) is equal to zero.

In [14] Miao offered a representation for \(M^d\) under conditions \(CA^\pi = 0\) and \(A^\pi B = 0\). This result was generalized in [9], where authors gave the formula for \(M^d\) under conditions \(CA^\pi A = 0\) and \(CA^\pi B = 0\). Yang and Liu [13] extended this result and derived the representation for \(M^d\) when \(BCA^\pi A = 0\) and \(BCA^\pi B = 0\) holds. The following theorem is a generalization of this result.
Theorem 3.1 Let $M$ be a matrix of the form (1.1) such that $S = 0$. If $ABCA^β A = 0$ and $ABCA^β B = 0$, then

$$M^d = \left[ \begin{array}{cc} (BCA^γ)^{n} & 0 \\ -(CA^β B)^d (CA^β A) & (CA^β B)^{n} \end{array} \right] (P^d)^2 + \sum_{i=0}^{t-1} \left[ \begin{array}{cc} (BCA^γ)^{i+1} & 0 \\ (CA^β B)^i (CA^β A) & (CA^β B)^{i+1} \end{array} \right] (P^d)^{2i+1} + \sum_{i=0}^{r-1} \left[ \begin{array}{cc} (CA^β B)^{i+1} & 0 \\ ((CA^β B)^i + 2CA^γ A) & ((CA^β B)^i + 1) \end{array} \right] P^{2i} P^r \right) M,$$

where

$$P = \left[ \begin{array}{cc} A & B \\ CA^d A & CA^d B \end{array} \right],$$

$$(P^d)^n = \left( I + \sum_{j=0}^{n-1} \left[ \begin{array}{cc} 0 & 0 \\ 0 & 0 \end{array} \right] (P^d)^{j+1} \right) (P^d)^n,$$

$$(P^d)^n = \left[ \begin{array}{cc} I & (AW)^{d+1} \end{array} \right] (AW)^{d+1} A \left[ \begin{array}{cc} I & A^d B \end{array} \right], W = AA^d + A^d BCA^d,$$

for every $n \in \mathbb{N}$, and $r = \text{ind}(P^d), l = \text{ind}(A), t = \max \{ \text{ind}(CA^β B), \text{ind}(BCA^γ) - 1 \}$.

Proof. Consider the splitting of matrix $M$

$$M = \left[ \begin{array}{cc} A & B \\ C & CA^d B \end{array} \right] = \left[ \begin{array}{cc} A & B \\ CA^d A & CA^d B \end{array} \right] + \left[ \begin{array}{cc} 0 & 0 \\ CA^γ & 0 \end{array} \right].$$

If we denote by $P = \left[ \begin{array}{cc} A & B \\ CA^d A & CA^d B \end{array} \right]$ and $Q = \left[ \begin{array}{cc} 0 & 0 \\ CA^γ & 0 \end{array} \right]$, we have that $P^2 Q P = 0$ and $Q^2 = 0$. Hence, the conditions of Corollary 2.1 are satisfied and

$$\begin{align*}
(P + Q)^d &= \sum_{i=0}^{r-1} ((PQ)^{i+1} + (QP)^{i+1}) (P^d)^{2i} P^r \\
&\quad + \sum_{i=0}^{s-1} ((PQ)^r (PQ)^i + (QP)^r (QP)^i) (P^d)^{2i+1} - (P^d)^2 \right) M,
\end{align*} \tag{3.1}$$

where $r = \text{ind}(P^d)$ and $s = \max \{ \text{ind}(PQ), \text{ind}(QP) \}$.

Obviously $Q^d = 0$ and $Q^r = I$. If we split matrix $P$ as

$$P = \left[ \begin{array}{cc} A & B \\ CA^d A & CA^d B \end{array} \right] = \left[ \begin{array}{cc} A^2 A^d & AA^d B \\ CA^d A & CA^d B \end{array} \right] + \left[ \begin{array}{cc} AA^γ & A^γ B \\ 0 & 0 \end{array} \right],$$

and denote by $P_1 = \left[ \begin{array}{cc} A^2 A^d & AA^d B \\ CA^d A & CA^d B \end{array} \right], P_2 = \left[ \begin{array}{cc} AA^γ & A^γ B \\ 0 & 0 \end{array} \right]$, we get $P_1 P_2 = 0$ and $P_2$ is $(l + 1)$-nilpotent. After using Lemma 1.2 we get

$$(P^d)^n = \left( I + \sum_{i=0}^{l-1} P_2^{i+1} (P_1^{i+1}) \right) (P^d)^n,$$
for \( n \in \mathbb{N} \). Notice that matrix \( P_1 \) satisfy conditions of Lemma 1.3, so after applying it we obtain

\[
(P_d^1)^n = \begin{bmatrix}
I & CA^d
\end{bmatrix}((AW)^d)^{n+1}A \begin{bmatrix}
I & A^d B
\end{bmatrix}.
\]

Therefore,

\[
(P_d^n) = \left(I + \sum_{j=0}^{l-1} \begin{bmatrix}
0 & A^i A^\pi B
0 & 0
\end{bmatrix}(P_d^1)^{j+1}\right)(P_d^1)^n.
\]

After computation we get:

\[
(PQ)^n = \begin{cases}
\begin{bmatrix}
BCA^\pi & 0 \\
CA^d BCA^\pi & 0
\end{bmatrix}, & \text{if } n = 1 \\
\begin{bmatrix}
(BCA^\pi)^{j+1}CA^\pi B & (CA^\pi)^{j+1}
0 & 0
\end{bmatrix}, & \text{if } n \geq 2
\end{cases},
\]

\[
((PQ)^d)^n = \begin{bmatrix}
((BCA^\pi)^{j+1}CA^\pi B) & ((CA^\pi)^{j+1})
0 & 0
\end{bmatrix},
\]

\[
(QP)^n = \begin{bmatrix}
0 & (CA^\pi)^{j+1}CA^\pi B \\
(BCA^\pi)^{j+1}CA^\pi B & 0
\end{bmatrix},
\]

\[
((QP)^d)^n = \begin{bmatrix}
I & 0 \\
-(CA^\pi)^{j+1}CA^\pi B & (CA^\pi)^{j+1}
\end{bmatrix}.
\]

After substituting this expressions and (3.2) into (3.1) we complete the proof.

\[\square\]

**Remark 1** Bu et al. offered formulas for \( M^d \) under conditions \( ABCA^\pi = 0 \), \( A^\pi ABC = 0 \) [15, Theorem 4.1] and under conditions \( ABCA^\pi = 0 \), \( CBCA^\pi = 0 \) [15, Theorem 4.3]. In [11, Theorem 3.3] the representation for \( M^d \) is given under conditions \( ABCA^\pi = 0 \), \( BCA^\pi \) is nilpotent. We remark that a special case of Theorem 3.1 is when blocks of matrix \( M \) satisfy the condition \( ABCA^\pi = 0 \). Therefore the conditions \( A^\pi ABC = 0 \) from [15, Theorem 4.4], \( CBCA^\pi = 0 \) from [15, Theorem 4.3] and \( BCA^\pi \) is nilpotent from [11, Theorem 3.3] are superfluous.

The next theorem is an extension of a case when \( CA^\pi BC = 0 \) and \( AA^\pi BC = 0 \) hold, which was studied by Yang and Liu [13].

**Theorem 3.2** Let \( M \) be a matrix defined by (1.1), such that \( S = 0 \). If \( AA^\pi BCA = 0 \) and \( CA^\pi BCA = 0 \), then

\[
M^d = M \left((P_d^i)^2\right) \begin{bmatrix}
(A^\pi BC)^\pi & -AA^\pi B(\langle 1 \rangle^d)^\pi \\
0 & (CA^\pi B)^\pi
\end{bmatrix}
\]

\[
+ \sum_{i=0}^{l-1}(P_d^i)^{2i+4} \begin{bmatrix}
(A^\pi BC)^{i+1}(A^\pi BC)^\pi & AA^\pi B(\langle 1 \rangle^{i+1})(CA^\pi B)^\pi \\
0 & (CA^\pi B)^{i+1}(CA^\pi B)^\pi
\end{bmatrix}
\]

\[
+ \sum_{i=0}^{l-1}P^\pi P_{2i} \begin{bmatrix}
(\langle 1 \rangle^{i+1}A^\pi BC)^{i+1} & AA^\pi B(\langle 1 \rangle^{i+1})(CA^\pi B)^{i+1} \\
0 & (CA^\pi B)^{i+1}(CA^\pi B)^{i+1}
\end{bmatrix},
\]
where

\[ P = \begin{bmatrix} A & AA^d B \\ C & CA^d B \end{bmatrix}, \]

\[ (P^d)^n = (P_1^d)^n \left( I + \sum_{j=0}^{n-1} (P_1^d)^j \begin{bmatrix} 0 & 0 \\ CA^d A^\pi & 0 \end{bmatrix} \right), \]

\[ (P_1^d)^n = \begin{bmatrix} I & CA^d \end{bmatrix} ((AW)^d)^{n+1} A \begin{bmatrix} I & A^d B \end{bmatrix}, \]

\[ W = AA^d + A^d BCA^d, \]

for every \( n \in \mathbb{N} \), and \( r = \text{ind}(P^2), l = \text{ind}(A), t = \text{max \{\text{ind}(CA^\pi B), \text{ind}(A^\pi BC) - 1\}} \)

**Proof.** We can split matrix \( M \) as

\[ M = \begin{bmatrix} A & B \\ C & CA^d B \end{bmatrix} = \begin{bmatrix} A & AA^d B \\ C & CA^d B \end{bmatrix} + \begin{bmatrix} 0 & A^\pi B \\ 0 & 0 \end{bmatrix}. \]

If we denote by \( P = \begin{bmatrix} A & AA^d B \\ C & CA^d B \end{bmatrix} \) and \( Q = \begin{bmatrix} 0 & A^\pi B \\ 0 & 0 \end{bmatrix} \), we have that matrices \( P \) and \( Q \) satisfy Corollary 2.2. Using similar method as in Theorem 3.1 we get that the statement of the theorem is true. \( \square \)

**Remark 2** Martínez-Serrano and Castro-González derived a formula for \( M^d \) under conditions \( A^\pi BCA = 0 \) and \( A^\pi BC \) is nilpotent [11, Corollary 3.4]. Notice that Theorem 3.2 is an extension of a case when \( A^\pi BCA = 0 \). Hence, the condition \( A^\pi BC \) is nilpotent from [11, Corollary 3.4] is superfluous.

## 4 Numerical example

In this section we give a numerical example to demonstrate the application of Theorem 3.1.

**Example** Consider the block matrix \( M \) of a form (1.1), where

\[ A = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}, \quad B = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \]

\[ C = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}, \quad D = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}. \]

By computing we get that generalized Schur complement \( S = D - CA^d B \) is equal to zero and \( ABCA^\pi = 0 \). Since \( A^\pi ABC \neq 0 \), \( CBCA^\pi \neq 0 \) and matrix \( BCA^\pi \) is not nilpotent, formulas for \( M^d \) from [15, Theorem 4.1], [15, Theorem 4.3] and [11, Theorem 3.3] fail to apply. However, the conditions of Theorem 3.1 are satisfied, so we can apply it.
We have that \( \text{ind}(A) = 2 \) and
\[
A^d = \begin{bmatrix}
1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0
\end{bmatrix}.
\]

Also, we get that \( \text{ind}(P) = 3 \) and
\[
P^d = \begin{bmatrix}
\frac{1}{4} & 0 & 0 & 0 & \frac{1}{4} & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\frac{1}{8} & 0 & 0 & 0 & \frac{1}{8} & 0 & 0 & 0 \\
\frac{1}{16} & 0 & 0 & 0 & \frac{1}{16} & 0 & 0 & 0 \\
\frac{1}{8} & 0 & 0 & 0 & \frac{1}{8} & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0
\end{bmatrix}.
\]

After applying Theorem 3.1, we get
\[
M^d = \begin{bmatrix}
\frac{1}{4} & 0 & 0 & 0 & \frac{1}{4} & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\frac{1}{8} & 0 & 0 & 0 & \frac{1}{8} & 0 & 0 & 0 \\
\frac{1}{16} & 0 & 0 & 0 & \frac{1}{16} & 0 & 0 & 0 \\
\frac{1}{4} & 0 & 0 & 0 & \frac{1}{4} & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 & 0 & 0
\end{bmatrix}.
\]

**Acknowledgements**

The author would like to thank the anonymous referees for their relevant and useful comments, which helped to improve the paper.

**References**


